IMA Journal of Numerical Analysis (2010) Page 1 of 28
doi:10.1093/imanum/drnxxx

Quasi-Optimal Cardinality of AFEM Driven by Nonresidual Estimators

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[Received on ; revised on ; accepted on]

We examine adaptive finite element methods (AFEM) with any polynomial degree satisfying rather general assumptions on the a posteriori error estimators. We show that several non-residual estimators satisfy these assumptions. We design an AFEM with single Dörfler marking for the sum of error estimator and oscillation, prove a contraction property for the so-called total error, namely the scaled sum of energy error and oscillation, and derive quasi-optimal decay rates for the total error. We also reexamine the definition and role of oscillation in the approximation class.

Keywords: Error reduction, convergence, optimal cardinality, adaptive algorithm.

1. Introduction

Let Ω be a bounded, polyhedral domain in $\mathbb{R}^d$, $d \geqslant 2$. We consider a homogeneous Dirichlet boundary value problem for a *general* second order elliptic partial differential equation (PDE)

$$\begin{aligned} \mathscr{L}u := -\operatorname{div}(\mathbf{A}\nabla u) + \mathbf{b}\cdot\nabla u + c\,u = f \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

The choice of boundary condition is made for ease of presentation, since similar results are valid for other boundary conditions. Precise conditions on given data $\mathbf{D} := (\mathbf{A}, \mathbf{b}, c)$ and f are stated in §2.1. Our interest is on diffusion-dominated problems, that is $\mathbf{A}$ dominates $\mathbf{b}$ but $\mathscr{L}$ is *non-symmetric*; we point out that, except for Mekchay & Nochetto (2005), all previous results are for the symmetric case (see Binev *et al.*, 2004; Bonito & Nochetto, 2010; Cascón *et al.*, 2008; Diening & Kreuzer, 2008; Dörfler, 1996; Kreuzer & Siebert, 2010; Morin *et al.*, 2000, 2003; Stevenson, 2007).

An AFEM is based on iterations of the loop

$$\textsf{SOLVE} \rightarrow \textsf{ESTIMATE} \rightarrow \textsf{MARK} \rightarrow \textsf{REFINE}. \tag{1.2}$$

Here SOLVE computes the discrete solution *exactly*. The procedure ESTIMATE calculates the error indicators, which are used by the procedure MARK to make a judicious selection of elements to be refined. The procedure REFINE finally refines the marked elements and creates a *conforming* refinement.

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Convergence of AFEM for elliptic PDE of the form $-\operatorname{div}\mathbf{A}\nabla u = f$, with $\mathbf{A}$ piecewise constant, has been the subject of intense research, mostly for residual estimators, starting with Morin, Nochetto and Siebert (Morin *et al.*, 2000, 2002, 2003). They uncovered the crucial role of data oscillation, thereby improving upon the seminal ideas of Dörfler (1996). Mekchay & Nochetto (2005) later extended the theory to elliptic operators (1.1) with variable coefficients. The key difficulty with (1.1) is that the energy error and oscillation no longer decouple, hence they cannot be handled separately.

The theory of optimal cardinality of AFEM started with Binev, Dahmen and DeVore (Binev *et al.*, 2004), who added a coarsening step to (1.2) for the Laplace equation and $d = 2$. Stevenson (2007) removed this additional step via a fundamental insight on the structure of Dörfler marking for $d \geqslant 2$, but still for the Laplace equation; this insight will be crucial for us as well. However, in Stevenson (2007) data oscillation is reduced within an inner loop, which is in general not viable when oscillation depends on the discrete solution, as in (1.1). Cascón, Kreuzer, Nochetto and Siebert (Cascón *et al.*, 2008) got rid of the inner loop for $f \in L^2(\Omega)$. This was possible upon examining a new combined quantity, the sum of energy error and scaled error estimator, and proving that AFEM contracts it between *two consecutive* adaptive loops. Cascón *et al.* (2008) built on Stevenson's insight to derive optimal cardinality of AFEM for the total error, namely the scaled sum of energy error and oscillation. To deal with $f \in H^{-1}(\Omega)$, and perform a convergence and cardinality study of AFEM, an inner loop to handle data reappears in Cohen, DeVore and Nochetto (Cohen *et al.*, 2010).

The results in Cascón *et al.* (2008) are for the simplest and most standard AFEM based on *residual-type* estimators, with any polynomial degree $n \geqslant 1$ and for symmetric problems (1.1) (i.e. $\mathbf{b} = 0$). As is customary in practice, this AFEM marks exclusively according to the error estimator and performs a minimal element refinement. However, it is well documented that residual-type estimators are the crudest ones in the literature even though they are widely used (see Ainsworth & Oden, 2000; Babuška & Strouboulis, 2001; Verfürth, 1996). They are both reliable and efficient. In fact, they provide an upper bound for the error (reliability) as well as a lower bound (efficiency) up to oscillation terms; the mesh geometry enters these bounds through unknown interpolation constants of moderate size.

Alternative estimators are abundant in the literature. Hierarchical estimators have been proposed by Bornemann *et al.* (1996) and further developed in Veeser (2002); Verfürth (1996). Estimators based on solving local problems have also been analyzed, starting with Neumann problems on elements by Bank & Weiser (1985). This was further improved by Ainsworth & Oden (2000) via the so-called flux equilibration, which yields a better effectivity index (ratio of estimator and error); local algebraic problems on stars, or patches, for flux weights have to be solved. The idea of working on stars goes back to Babuška & Miller (1987), who introduced Dirichlet problems. Carstensen & Funken (1999) and Morin *et al.* (2003) proposed solving local weighted problems on stars which yield rather good effectivity indices; a convergence proof of AFEM of the form (1.2) was also given in Morin *et al.* (2003) for the Poisson problem. This method has been slightly simplified in Parés *et al.* (2006); Prudhomme *et al.* (2004); Strouboulis *et al.* (2006).

On the other hand, gradient recovery techniques have proven to be extremely successful beginning with the seminal work of Zienkiewicz & Zhu (1987). This estimator is generically superior to the others even though counterexamples show that the effectivity index may not tend to one asymptotically (asymptotic exactness). More recently, Bank & Xu (2003) have proposed a multilevel averaging technique that performs averages in rings of higher order than stars and thereby yields asymptotic exactness even on irregular (but still quasi-uniform) meshes. Both techniques in Bank & Xu (2003); Zienkiewicz & Zhu (1987) hinge on reconstructing higher derivatives from discrete data, and thus may overestimate the error if ∇u jumps, e.g. when $\mathbf{A}$ is discontinuous. Finally, Braess & Schöberl (2008) combined $H(\mathrm{div})$ elements of Raviart-Thomas and flux equilibration to derive local a posteriori error estimators with good effectivity index.

In order to point out the essential difficulties in dealing with non-residual estimators, let $\{\mathscr{T}_j, \mathbb{V}_j, U_j, \eta_j, \mathrm{osc}_j\}_{j\geqslant 0}$ be the sequence of meshes, finite element spaces, discrete solutions, estimators, and oscillations produced by AFEM in the jth step. The following two key issues, crucial in Cascón *et al.* (2008), must be overcome:

- The estimator η_j is not reliable, namely it does not bound the energy error from above, and does not dominate the oscillation osc_j. This means that η_j cannot drive AFEM alone, especially in the preasymptotic regime in which data may be underresolved and osc_j may dominate η_j.
- The estimator η_j does not longer decrease whenever U_j does not change. The heuristic idea behind the contraction property of Cascón *et al.* (2008) for the sum $|||u-U_j|||_\Omega^2 + \gamma\eta_j^2$ is that when $|||u-U_j|||_\Omega$ is not reduced, because U_j does not change upon refinement, it is η_j that decreases because the mesh-size does. This means that $|||u-U_j|||_\Omega^2 + \gamma\eta_j^2$ is not the correct quantity to monitor in the present context.

It is thus intriguing why non-residual estimators yield a practical performance of AFEM similar to residual-type estimators (see Binev *et al.*, 2004; Cascón *et al.*, 2008; Mekchay & Nochetto, 2005; Morin *et al.*, 2000, 2002; Stevenson, 2007). Plain convergence is shown by Morin, Siebert and Veeser (Morin *et al.*, 2008) and Siebert (2010), but their approach is too general as to allow for an energy decrease property adequate for cardinality analysis. No such analysis is available in the literature for AFEM driven by non-residual a posteriori estimators, except for the very recent paper by Kreuzer & Siebert (2010). This paper and Kreuzer & Siebert (2010) were developed simultaneously but independently, and provide different answers to the main issues at stake. Therefore, it is worth comparing the philosophies invoked, thereby emphasizing differences and similarities:

- The guiding principle in Kreuzer & Siebert (2010) is to use the equivalence of several non-residual estimators to the residual ones to transfer the decay rates of Cascón *et al.* (2008) to non-residual estimators. This has the advantage that the four basic procedures of (1.2) remain unchanged.
- Our approach examines directly a class of non-residual estimators satisfying general assumptions, which are shown later to hold for each estimator, and leads to a contraction property between a fixed number $\mathfrak{J}$ of iterates for the so-called *total error*, namely the sum of energy error and scaled oscillation (see Cascón *et al.*, 2008; Mekchay & Nochetto, 2005). Avoiding comparison with residual estimators gives better constants.
- As in Mekchay & Nochetto (2005), we need a discrete lower bound in order to prove the contraction property, and thus an interior node in each marked element. However, in contrast to Mekchay & Nochetto (2005), we do not enforce this extra refinement between consecutive iterations but rather modify slightly MARK so that it takes place after $\mathfrak{J}$ iterations; this is easy to do, for instance within ALBERTA (Schmidt & Siebert, 2005). Note that $\mathfrak{J}$ is explicit (for instance, $\mathfrak{J}=3$ for $d=2$ and $\mathfrak{J}=6$ for $d=3$).
- The analysis in Kreuzer & Siebert (2010) is for the Laplace equation and piecewise linear elements, whereas we consider the general non-symmetric operator $\mathscr{L}$ of (1.1) with variable coefficients and any polynomial degree $n \geqslant 1$.
- We, as well as Kreuzer & Siebert (2010), consider a single *Dörfler marking* for the quantity $\eta_j + \mathrm{osc}_j$, which is an upper bound for the energy error and is thus reliable. It is shown in (Cascón *et al.*, 2008, Section 6) that separate marking for η_j and osc_j might yield suboptimal meshes.

- We (resp. Kreuzer & Siebert, 2010) prove quasi-optimal decay rates for AFEM provided the triple $(u,\mathbf{D},f)$ belongs to a suitable approximation class $\mathbb{A}_s$ with $0 < s \leqslant n/d$ (resp. $s \leqslant 1/d$ for $n = 1$). Besides, we discuss the equivalence of classes $\mathbb{A}_s$, for $n \geqslant 1$ and more practical definitions of oscillation, to those in Cascón *et al.* (2008).
- We and Kreuzer & Siebert (2010) assume exact linear algebra and integration, and limit the discussion to refinement based on *bisection* (see Bänsch, 1991; Kossaczký, 1994; Maubach, 1995; Mitchell, 1989; Stevenson, 2008; Traxler, 1997; Schmidt & Siebert, 2005). This leads to conforming shape regular meshes; see (Nochetto *et al.*, 2009, Section 4) for a complete description. The theory extends to *non-conforming meshes*, provided REFINE generates nested meshes using subdivision rules with specific properties (Bonito & Nochetto, 2010, Condition 7). This is the case of hexahedral meshes with quad-refinement and simplicial meshes with red refinement provided that the level of nonconformity is fixed (Bonito & Nochetto, 2010, Section 6).
- Our a posteriori estimators are sensitive to large discontinuities of $\mathbf{A}$ or disparate sizes of $\mathbf{A}$ and c. In contrast, the results of Kreuzer & Siebert (2010) are robust.

This paper is organized as follows. In §2 we introduce the weak formulation and its discretization. In §3 we present several non-residual a posteriori error estimators and discuss its main features. Motivated by the examples of §3, we enunciate in §4 abstract properties that both estimator, η_j, and oscillation, osc_j, must fulfill, along with assumptions on the adaptive procedure AFEM. In §5 we prove our first main result, namely the following contraction property for the total error:

If AFEM satisfies the abstract properties of §4*, then there exist constants* $\gamma > 0$, $\mathfrak{J} \in \mathbb{N}$ *and* $0 < \alpha < 1$, *so that the total error contracts after* $\mathfrak{J}$ *consecutive steps*

$$|||u - U_{j+\mathfrak{J}}|||_\Omega^2 + \gamma \operatorname{osc}_{j+\mathfrak{J}}^2 \leqslant \alpha^2 \left(|||u - U_j|||_\Omega^2 + \gamma \operatorname{osc}_j^2 \right). \tag{1.3}$$

This combines ideas from Cascón *et al.* (2008); Chen & Feng (2004); Diening & Kreuzer (2008); Mekchay & Nochetto (2005). In §6 we examine the decay rate of the total error. Since all decisions of AFEM in MARK are based on the sum $\eta_j + \operatorname{osc}_j$, a decay rate for the true error must rely on this quantity. We will see in §6 that

$$|||u - U_j|||_\Omega + \operatorname{osc}_j \;\approx\; \eta_j + \operatorname{osc}_j,$$

because of the upper and lower global bounds for $|||u - U_j|||_\Omega$. Therefore, the performance of AFEM is intrinsically linked to the total error, which measures both the approximability of u via $|||u - U_j|||_\Omega$ as well as of data, encoded in osc_j. This is expressed in the approximation class $\mathbb{A}_s$: $(u,f,\mathbf{D}) \in \mathbb{A}_s$ if the best possible decay rate of the total error is N^{-s} for conforming bisection refinements of a coarse mesh $\mathscr{T}_0$ with N degrees of freedom more than those of $\mathscr{T}_0$; we say $|u,f,\mathbf{D}|_s < \infty$. In §6 we study the class $\mathbb{A}_s$ for more practical definitions of osc_j than that in Cascón *et al.* (2008), which is significant in the present context. We conclude in §6 with our second main result:

If AFEM satisfies the abstract properties of §4*, and* $(u,f,\mathbf{D}) \in \mathbb{A}_s$*, then there exists a constant* C *solely depending on* $\mathscr{T}_0, d, s$*, and* $\mathfrak{J}$*, such that*

$$|||u - U_j|||_\Omega + \operatorname{osc}_j \leqslant C |u,f,\mathbf{D}|_s \left(\#\mathscr{T}_j - \#\mathscr{T}_0\right)^{-s}. \tag{1.4}$$

We stress that AFEM does not exploit any knowledge of $\mathbb{A}_s$ in its formulation and still delivers the optimal decay rate. The derivation of (1.4) hinges on the insight of Stevenson (2007) on the role of Dörfler marking for the Laplace equation.

2. Problem Setting

We first introduce the problem setting along with assumptions on the given data $(\mathbf{A},\mathbf{b},c,f)$. We then present its discretization.

2.1 *Weak Formulation*

Let Ω be a bounded, polyhedral domain in $\mathbb{R}^d$, $d \geqslant 2$, and let $\mathscr{T}_0$ be a conforming triangulation of Ω made of simplices. We assume that the data of (1.1) have the following properties:

(a) $\mathbf{A}\colon \Omega \mapsto \mathbb{R}^{d\times d}$ is piecewise Lipschitz over $\mathscr{T}_0$ and is symmetric positive definite with eigenvalues in $[a_*, a^*]$ with $0 < a_* \leqslant a^* < \infty$, i.e.,

$$a_*(x)\,|\xi|^2 \leqslant \mathbf{A}(x)\xi\cdot\xi \leqslant a^*(x)\,|\xi|^2, \quad \forall \xi \in \mathbb{R}^d,\ x \in \Omega;$$

(b) $\mathbf{b} \in [L^\infty(\Omega)]^d$ is divergence free, i. e. $\operatorname{div}\mathbf{b} = 0$ in Ω;

(c) $c \in L^\infty(\Omega)$ is nonnegative, i. e. $c \geqslant 0$ in Ω;

(d) $f \in L^2(\Omega)$.

Even though it is customary to assume $f \in L^2(\Omega)$, which goes back to the seminal work of Babuška & Miller (1987), we mention the recent work of Cohen *et al.* (2010) which deals with the weakest and most natural condition $f \in H^{-1}(\Omega)$.

Now we turn to the weak formulation of (1.1). For any set $\omega \subset \mathbb{R}^d$ with non-empty interior we denote by $H^1(\omega)$ the usual Sobolev space of functions in $L^2(\omega)$ whose first weak derivatives are also in $L^2(\omega)$, endowed with the norm

$$\|u\|_{H^1(\omega)} := \left(\|u\|^2_{L^2(\omega)} + \|\nabla u\|^2_{L^2(\omega)}\right)^{1/2}.$$

Moreover, we denote by $\langle\cdot,\cdot\rangle_\omega$ the $L^2(\omega)$ scalar product. Finally we let $\mathbb{V} := H^1_0(\Omega)$ be the space of functions in $H^1(\Omega)$ with vanishing trace on $\partial\Omega$. A weak solution of (1.1) is a function u satisfying

$$u \in \mathbb{V}: \qquad \mathscr{B}[u,v] = \langle f, v\rangle_\Omega \qquad \forall v \in \mathbb{V}, \tag{2.1}$$

where the bilinear form is defined to be

$$\mathscr{B}[u,v] := \langle \mathbf{A}\nabla u, \nabla v\rangle_\Omega + \langle \mathbf{b}\cdot\nabla u + c\,u, v\rangle_\Omega \qquad \forall u,v \in \mathbb{V}.$$

In view of Poincaré-Friedrichs inequality ($\|v\|_{L^2(\Omega)} \leqslant C_\Omega \|\nabla v\|_{L^2(\Omega)}$ for all $v \in H^1_0(\Omega)$) and the divergence free condition $\operatorname{div}\mathbf{b} = 0$, one has *coercivity* in $\mathbb{V}$

$$\mathscr{B}[v,v] \geqslant \int_\Omega a_*\,|\nabla v|^2 + c\,v^2 \geqslant c_B^2\,\|v\|^2_{H^1(\Omega)},$$

and c_B depends only on data and Ω. The bilinear form $\mathscr{B}$ induces the so-called *energy seminorm*:

$$|||v|||_\omega := \left(\langle \mathbf{A}\nabla v, \nabla v\rangle_\omega + \langle cv, v\rangle_\omega\right)^{1/2} \qquad \forall v \in H^1(\omega),$$

which is a norm for $H^1_0(\omega)$. Note that $\mathscr{B}$ also fulfills the *local continuity*

$$\mathscr{B}[v,w] \leqslant C_B\,|||v|||_\omega\,|||w|||_\omega \qquad \forall v,w \in H^1(\omega),\ \operatorname{supp}(w) \subset \omega \subset \Omega,$$

where C_B depends on $a^*, \|\mathbf{b}\|_{L^\infty(\Omega)}$, and $\|c\|_{L^\infty(\Omega)}$. This local continuity is essential in deriving *local lower bounds* in the a posteriori error analysis. Furthermore it implies continuity of $\mathscr{B}[\cdot,\cdot]$ on $H^1(\Omega)$ at once. Thanks to coercivity and continuity of $\mathscr{B}$, the norm $|||\cdot|||_\Omega$ is equivalent to $\|\cdot\|_{H^1(\Omega)}$ on $H^1_0(\Omega)$. Existence and uniqueness of (2.1) thus follows from Lax-Milgram theorem (Gilbarg & Trudinger, 1983).

2.2 *Discretization*

We first introduce some notations relative to triangulations. We only consider the class of all *conforming* meshes $\mathbb{T}$ created by successive bisections of the initial conforming triangulation $\mathscr{T}_0$ of Ω. Given $\mathscr{T}$, $\mathscr{T}_* \in \mathbb{T}$ we write $\mathscr{T}_* \geqslant \mathscr{T}$ if $\mathscr{T}_*$ is a refinement of $\mathscr{T}$, that is $\mathscr{T}_*$ can be obtained from $\mathscr{T}$ upon applying a finite number of bisections. We denote simplices by T, interior interelement boundaries (sides) by σ, and their collection by $\mathscr{S}$.

The *generation* $g(T)$ of $T \in \mathscr{T}$ is the number of bisections needed to create T from $\mathscr{T}_0$. Given $\mathscr{T}_* \geqslant \mathscr{T}$ and any $T \in \mathscr{T}$, we define the *relative generation* of descendants of T belonging to $\mathscr{T}_*$ with respect to T to be

$$\mathrm{gen}_{\mathscr{T}_*}(T) = \min\{g(T') - g(T) \mid T' \subset T \text{ and } T' \in \mathscr{T}_*\}.$$

Finally, for any $T' \in \mathscr{T}_*$ we denote by $T = \mathrm{anc}_{\mathscr{T}}(T') \in \mathscr{T}$ the *ancestor* of T' in $\mathscr{T}$, i. e. the only element $T \in \mathscr{T}$ verifying $T' \subset T$.

Given any conforming triangulation $\mathscr{T} \in \mathbb{T}$ we define the finite element space

$$\mathbb{V}(\mathscr{T}) := \{V \in \mathbb{V} \mid V_{|T} \in \mathbb{P}_n(T)\ \forall T \in \mathscr{T}\},$$

where $\mathbb{P}_n$ denotes the space of all polynomials of degree $\leqslant n$. Since continuity and coercivity of $\mathscr{B}$ are inherited by any subspace of $\mathbb{V}$ the Lax-Milgram theorem implies existence and uniqueness of the Ritz-Galerkin approximation in $\mathbb{V}(\mathscr{T})$ uniquely defined by

$$U \in \mathbb{V}(\mathscr{T}): \qquad \mathscr{B}[U,V] = \langle f, V\rangle_\Omega \qquad \forall\, V \in \mathbb{V}(\mathscr{T}). \tag{2.2}$$

We will always assume that $\mathscr{T}_* \geqslant \mathscr{T}$ is a conforming refinement of $\mathscr{T}$ and that $U_* \in \mathbb{V}(\mathscr{T}_*)$ is the corresponding Ritz-Galerkin solution.

If $\mathbf{b} \neq 0$ in (1.1), the bilinear form $\mathscr{B}$ is no longer symmetric, and thus is not a scalar product. Therefore, we do not have an orthogonality relation between discrete solutions on nested spaces, the so-called Pythagoras equality (Dörfler, 1996; Morin *et al.*, 2000). We have instead a perturbation result referred to as quasi-orthogonality, provided that the initial mesh $\mathscr{T}_0$ is sufficiently fine (Mekchay & Nochetto, 2005, Lemma 2.1,). This is not a severe restriction because we consider $\mathbf{b}$ dominated by $\mathbf{A}$ (small Péclet number). The proof resorts to a duality argument (Ciarlet, 1978), and uses the regularity $H^{1+r}(\Omega)$ of the dual solution (Mekchay & Nochetto, 2005).

LEMMA 2.1 (Quasi-orthogonality) There exists $C_0 > 0$, solely depending on $\mathscr{T}_0$, the coercivity constant c_B, and $0 < r \leqslant 1$ characterizing the regularity H^{1+r} of the dual solution, such that if the mesh-size h_0 of $\mathscr{T}_0$ satisfies $C_0 h_0^r \|\mathbf{b}\|_{L^\infty(\Omega)} < 1$, then

$$|||u - U_*|||^2_\Omega \leqslant \Lambda_0\, |||u - U|||^2_\Omega - |||U - U_*|||^2_\Omega \tag{2.3}$$

where $\Lambda_0 := \left(1 - C_0 h_0^r \|\mathbf{b}\|_{L^\infty(\Omega)}\right)^{-1}$. The inequality in (2.3) becomes equality with $\Lambda_0 = 1$ and without restrictions on h_0 provided $\mathbf{b} = 0$ in Ω.

3. Non-Residual A Posteriori Error Estimators: A Review

In this section we introduce general notation about a posteriori estimators, and review a number of *non-residual* a posteriori estimators which split into local indicators. We assume that each a posteriori indicator is associated with a closed set K, from now on called *K-element or patch*, which is made of elements $T \in \mathscr{T}$ or sides $\sigma \in \mathscr{S}$. The shape of K and the number of constituent elements depend on the type of estimator used. For example, for an estimator based on local problems, in general K is a *star*, the union of elements sharing a vertex.

We denote by $\mathscr{K}_\mathscr{T}$ the set of all K-elements on mesh $\mathscr{T}$. For $K \in \mathscr{K}_\mathscr{T}$ we denote by $h_K := |K|^{\frac{1}{\dim(K)}}$ the local mesh-size. The elements of $\mathscr{K}_\mathscr{T}$ may have a finite overlapping, but the number of them is equivalent to the cardinality of $\mathscr{T}$, that is

$$\#\mathscr{K}_\mathscr{T} \approx \#\mathscr{T} \tag{3.1}$$

In contrast to Cascón *et al.* (2008), to develop our theory we will now need to consider several levels of refinement (or subdivision depth) between two (not necessarily consecutive) meshes $\mathscr{T} \leqslant \mathscr{T}_*$. We thus define the *refined set of order* j to be

$$\mathscr{R}^j_{\mathscr{T}\to\mathscr{T}_*} := \left\{K \in \mathscr{K}_\mathscr{T} \mid \quad \mathrm{gen}_{\mathscr{T}_*}(T) \geqslant j \quad \forall T \subset K\right\}. \tag{3.2}$$

We point out that the usual refined set corresponds to $j = 1$, and that

> $j \geqslant \mathfrak{n} := 3, 6$ *for* $d = 2, 3$ *implies that all the constituent elements, as well as their sides, of K-elements in* $\mathscr{R}^j_{\mathscr{T}\to\mathscr{T}_*}$ *contain a node of* $\mathscr{T}_*$ *in their interior (interior node property) (Mekchay & Nochetto, 2005; Morin* et al.*, 2000, 2002).* (3.3)

However, in contrast with Mekchay & Nochetto (2005); Morin *et al.* (2000, 2002), we do not enforce this property between consecutive steps.

For $K \in \mathscr{K}_\mathscr{T}$ and $V \in \mathbb{V}(\mathscr{T})$ we denote by $\eta_\mathscr{T}(V,K)$ and $\mathrm{osc}_\mathscr{T}(V,K)$ the K-element indicator and oscillation, and refer to following sections for specific examples. The quantity

$$\zeta_\mathscr{T}(V,K) := \left(\eta^2_\mathscr{T}(V,K) + \mathrm{osc}^2_\mathscr{T}(V,K)\right)^{1/2} \quad \forall K \in \mathscr{K}_\mathscr{T}, \tag{3.4}$$

the so-called *total error indicator*, will be used to mark elements for refinement as opposed to just $\eta_\mathscr{T}(V,K)$. Finally, for any subset $\mathscr{K}'_\mathscr{T} \subset \mathscr{K}_\mathscr{T}$ we set

$$\zeta_\mathscr{T}(V,\mathscr{K}'_\mathscr{T}) := \Big(\sum_{K\in\mathscr{K}'_\mathscr{T}} \zeta^2_\mathscr{T}(V,K)\Big)^{1/2},$$

and similarly for $\eta_\mathscr{T}(V,\mathscr{K}'_\mathscr{T})$, and $\mathrm{osc}_\mathscr{T}(T,\mathscr{K}'_\mathscr{T})$.

In the rest of this section, we review the following popular estimators for the model problem (1.1) and polynomial degree $n = 1$: the residual estimator, the *hierarchical estimator*, the *Morin-Nochetto-Siebert estimator*, the *Parés-Díez-Huerta estimator*; the *Zienkiewicz-Zhu estimator*, and the *Braess-Schöberl* estimator. We discuss the last two examples for the Laplace equation upon comparing with residual estimators; this may give rise to somewhat pessimistic constants.

In doing this, we assume that we have two conforming meshes $\mathscr{T} \leqslant \mathscr{T}_*$ and K-elements satisfying the interior node property (3.3). We let $U \in \mathbb{V}(\mathscr{T}), U_* \in \mathbb{V}(\mathscr{T}_*)$ be the corresponding Galerkin solutions and set $E_* := U - U_* \in \mathbb{V}(\mathscr{T}_*)$. Moreover, we let Π^p_m be the L^p-best approximation operator in the space of discontinuous polynomials of degree $\leqslant m$ over $K \in \mathscr{K}_\mathscr{T}$, and $E^p_n = I - \Pi^p_n$ be the operator error.

3.1 *Residual Estimator*

Given $\mathscr{T} \in \mathbb{T}$ and $V \in \mathbb{V}(\mathscr{T})$, we define the *element* and *jump residuals* by

$$R(V,T) := (f - \mathscr{L}V)_{|T} \quad \forall T \in \mathscr{T}, \qquad J(V,\sigma) := ([\![\mathbf{A}\nabla V]\!] \cdot \boldsymbol{\nu})_{|\sigma} \quad \forall \sigma \in \mathscr{S};$$

$[\![q]\!]$ is the jump of q across an interior side σ in the direction of the unit normal $\boldsymbol{\nu}$ to σ, and is uniquely defined. The error-residuals relation reads

$$\mathscr{B}[u-U,v] = \langle f,v\rangle_\Omega - \mathscr{B}[U,v] = \sum_{T\in\mathscr{T}} \int_T R(U,T)v + \sum_{\sigma\in\mathscr{S}} \int_\sigma J(U,\sigma)v \quad \forall v \in \mathbb{V}. \tag{3.5}$$

The residual indicators and oscillation for $T \in \mathscr{T}$ read

$$\begin{aligned} \eta_{\mathscr{T}}(U,T)^2 &:= \|h_T R(U,T)\|^2_{L^2(T)} + \|h_T^{1/2} J(U,\partial T)\|^2_{L^2(\partial T)}, \\ \mathrm{osc}_{\mathscr{T}}(U,T)^2 &:= \left\|h_T E^2_{n-1} R(U,T)\right\|^2_{L^2(T)} + \|h_T^{1/2} E^2_{n-1} J(U,\partial T)\|^2_{L^2(\partial T)}, \end{aligned} \tag{3.6}$$

where $J(U,\partial T)$ is viewed as a piecewise function over ∂T. We refer to Ainsworth & Oden (2000); Cascón *et al.* (2008); Mekchay & Nochetto (2005); Morin *et al.* (2000, 2002); Stevenson (2007); Verfürth (1996) for analysis of residual estimators. If they drive AFEM, then the present formulation is a bit more complicated than that in Cascón *et al.* (2008) because we now compute the oscillation and enforce a refinement depth $\mathfrak{n}$ according to (3.3). We point out that $\mathrm{osc}_{\mathscr{T}}(U,T)$ is different from the oscillation of Cascón *et al.* (2008):

$$\mathrm{osc}_{\mathscr{T}}(U,T)^2 := \left\|h_T E^2_{2n-2} R(U,T)\right\|^2_{L^2(T)} + \|h_T^{1/2} E^2_{2n-1} J(U,\partial T)\|^2_{L^2(\partial T)}. \tag{3.7}$$

The choice of polynomial degrees $2n-2$ and $2n-1$ guarantees an oscillation decay as fast as the energy error. We now tackle the more traditional polynomial degree $n-1$, which is easier to implement but more difficult to analyze. Our results in §§4, 5, and 6 cover this case. We refer to §6.2 for a further discussion.

We mention, for later use in §3.4, that Babuška & Miller (1987) showed that the residual estimator is equivalent to the following jump estimator $\widehat{\eta}_{\mathscr{T}}$ up to oscillation:

$$\widehat{\eta}_{\mathscr{T}}(U,\mathscr{T})^2 := \sum_{\sigma\in\mathscr{S}} \|h_\sigma^{1/2} J(U,\sigma)\|^2_{L^2(\sigma)}. \tag{3.8}$$

This has been further explored in Babuška & Strouboulis (2001); Carstensen & Verfürth (1999); Cohen *et al.* (2010); Rodríguez (1994).

3.2 *Hierarchical Estimator*

We follow Bornemann *et al.* (1996), Veeser (2002) and Verfürth (1996). We let $\mathscr{K}_{\mathscr{T}}$ be the set of all interelement sides and simplices, $\mathscr{K}_{\mathscr{T}} := \{\sigma : \sigma \in \mathscr{S}\} \cup \{T : T \in \mathscr{T}\}$. For each $K \in \mathscr{K}_{\mathscr{T}}$, let $\lambda_K \in \mathbb{V}(\mathscr{T}_*)$ be the hat function corresponding to the interior node $z_K \in K$ guaranteed by (3.3), let $\omega_K := \mathrm{supp}\,\lambda_K$ be its support, and let φ_K be the renormalized function $\varphi_K := \frac{\lambda_K}{|\!|\!|\lambda_K|\!|\!|_\Omega}$. For $V \in \mathbb{V}(\mathscr{T})$ and $K \in \mathscr{K}_{\mathscr{T}}$ we define the K-element estimator and oscillation as

$$\eta_{\mathscr{T}}(V,K) := |\langle f,\varphi_K\rangle_\Omega - \mathscr{B}[V,\varphi_K]| = |\mathscr{B}[u-V,\varphi_K]|$$

$$\mathrm{osc}_{\mathscr{T}}(V,K) := \begin{cases} h_K^{1/2} \|(I-\Pi_0^2)J(V,K)\|_{L^2(K)}, & \text{if } K = \sigma \in \mathscr{S}, \\ h_K \|(I-\Pi_0^2)R(V,K)\|_{L^2(K)} & \text{if } K = T \in \mathscr{T}. \end{cases}$$

We now prove local lower and global upper a posteriori error estimates. In contrast with the residual estimators, the former turn out to be easier to derive. In fact, if $\mathscr{T} \leqslant \mathscr{T}_*$ and $K \in \mathscr{K}_{\mathscr{T}}$ has an interior node in $\mathscr{T}_*$, then $\varphi_K \in \mathbb{V}(\mathscr{T})$ and

$$\eta_{\mathscr{T}}(U,K) = |\mathscr{B}[u-U,\varphi_K]| \leqslant C_B \, |\!|\!| u-U |\!|\!|_{\omega_K} \tag{3.9}$$

as well as

$$\eta_{\mathscr{T}}(U,K) = |\mathscr{B}[U_*-U,\varphi_K]| \leqslant C_B \, |\!|\!| U_*-U |\!|\!|_{\omega_K} . \tag{3.10}$$

These are *local lower bounds*. We now introduce a linear operator $P_{\mathscr{T}} : H^1(\Omega) \to \operatorname{span}_{K\in\mathscr{K}_{\mathscr{T}}} \{\varphi_K\}$ to prove a global upper bound. We define $P_{\mathscr{T}}$ to be

$$P_{\mathscr{T}} v = \sum_{K\in\mathscr{K}_{\mathscr{T}}} \beta_K \varphi_K \quad \Leftrightarrow \quad \int_K P_{\mathscr{T}} v = \int_K v, \quad \forall K \in \mathscr{K}_{\mathscr{T}}, \tag{3.11}$$

and realize that the coefficients β_K are determined uniquely and satisfy the following *local* stability properties for any $v \in \mathbb{V}$, $\sigma \in \mathscr{S}$, and $T \in \mathscr{T}$ (Veeser, 2002, Lemma 3.1, p. 754):

$$|\beta_\sigma| \preccurlyeq h_\sigma^{-1/2} \|v\|_{L^2(\sigma)}, \qquad |\beta_T| \preccurlyeq h_T^{-1} \|v\|_{L^2(T)} + h_T^{-1/2} \|v\|_{L^2(\partial T\cap\Omega)} . \tag{3.12}$$

Let $v \in \mathbb{V}$ be a test function and $I_{\mathscr{T}} v$ be its Scott-Zhang interpolant. Using Galerkin orthogonality we obtain with $z = v - I_{\mathscr{T}} v$

$$\mathscr{B}[u-U,v] = \mathscr{B}[u-U,v-I_{\mathscr{T}}v] = \mathscr{B}[u-U,P_{\mathscr{T}}z] + \mathscr{B}[u-U,(I-P_{\mathscr{T}})z].$$

Combining the stability properties (3.12) of $P_{\mathscr{T}}$ with the approximation properties of $I_{\mathscr{T}}$ (Scott & Zhang, 1990), and invoking the definition of $\eta_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}})$, the first term is bounded by

$$\mathscr{B}[u-U,P_{\mathscr{T}}z] = \sum_{K\in\mathscr{K}_{\mathscr{T}}} \beta_K \mathscr{B}[u-U,\varphi_K] \preccurlyeq \eta_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}) \|v\|_{H^1(\Omega)}.$$

For the second term, we use (3.5) and the definition (3.11) of $P_{\mathscr{T}}$ to arrive at

$$\begin{aligned}\mathscr{B}[u-U,(I-P_{\mathscr{T}})z] = &\sum_{T\in\mathscr{T}} \left\langle (I-\Pi_0^2)R(U,T),(I-P_{\mathscr{T}})z \right\rangle_T \\ &+ \sum_{\sigma\in\mathscr{S}} \left\langle (I-\Pi_0^2)J(U,\sigma),(I-P_{\mathscr{T}})z \right\rangle_\sigma \preccurlyeq \operatorname{osc}_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}) \|v\|_{H^1(\Omega)}.\end{aligned}$$

Invoking the coercivity of $\mathscr{B}$, we end up with the *global upper bound*

$$|\!|\!| u-U |\!|\!|_\Omega^2 \leqslant C_1 \Big(\eta_{\mathscr{T}}^2(U,\mathscr{K}_{\mathscr{T}}) + \operatorname{osc}_{\mathscr{T}}^2(U,\mathscr{K}_{\mathscr{T}}) \Big) = C_1 \zeta_{\mathscr{T}}^2(U,\mathscr{K}_{\mathscr{T}}), \tag{3.13}$$

where the constant C_1 depends solely on $\mathscr{T}_0$, c_B, C_B, and the dimension d. This estimate can be localized to the refined region $\mathscr{R} = \mathscr{R}^1_{\mathscr{T}\to\mathscr{T}_*}$ of order 1 provided u is replaced by U_*, namely,

$$|\!|\!| U_*-U |\!|\!|_\Omega^2 \leqslant C_1 \left\{ \eta_{\mathscr{T}}^2(U,\mathscr{R}) + \operatorname{osc}_{\mathscr{T}}^2(U,\mathscr{R}) \right\}. \tag{3.14}$$

We prove this along the lines of (Cascón *et al.*, 2008, Lemma 3.6), after noting that if $\sigma \in \mathscr{R}$, then the two elements of $\mathscr{T}$ sharing this side are also in $\mathscr{R}$. Let Ω_* be the union of elements of $\mathscr{T}$ which are refined in $\mathscr{T}_*$, and denote by Ω_k one of the connected components of its interior. Let $\mathscr{T}_k$ be the subset of $\mathscr{T}$ contained in Ω_k and let $\mathbb{V}(\mathscr{T}_k)$ be the restriction of $\mathbb{V}(\mathscr{T})$ to Ω_k. Let $I_k\colon H^1(\Omega_k) \to \mathbb{V}(\mathscr{T}_k)$ be the Scott-Zhang interpolation operator over the mesh $\mathscr{T}_k$, which preserves conforming boundary values. Let $V \in \mathbb{V}(\mathscr{T})$ be the following approximation of the error $E_* = U_* - U \in \mathbb{V}(\mathscr{T}_*)$:

$$V := I_k E_* \quad \text{in } \Omega_k, \quad \text{and} \quad V := E_* \quad \text{elsewhere.} \tag{3.15}$$

By construction, $V \in \mathbb{V}(\mathscr{T})$ is a H^1-stable approximation to E_* in Ω and satisfies the Galerkin orthogonality $\mathscr{B}[E_*, V] = 0$ because $\mathbb{V}(\mathscr{T}) \subset \mathbb{V}(\mathscr{T}_*)$. Since $E_* = V$ in $\Omega \setminus \Omega_*$, and $P_{\mathscr{T}}$ is local, we proceed as with (3.13) to deduce (3.14), namely

$$|||E_*|||_\Omega^2 = \mathscr{B}[E_*, P_{\mathscr{T}}(E_* - V)] + \mathscr{B}[E_*, (I - P_{\mathscr{T}})(E_* - V)] \leqslant \zeta_{\mathscr{T}}(U, \mathscr{R})\, |||E_*|||_\Omega .$$

3.3 *Estimators based on Solving Local Problems*

We consider two a posteriori estimators which rely on the solution of small problems on stars. The first one is the estimator introduced by Morin, Nochetto and Siebert (Morin *et al.*, 2003), which organizes the information by stars. The second one is due to Parés, Díez and Huerta (Parés *et al.*, 2006), which slightly simplifies the estimator in Morin *et al.* (2003) and arranges it by triangles. We modify the formulations in Morin *et al.* (2003); Parés *et al.* (2006) to account for the general nature of operator $\mathscr{L}$ and derive a few bounds that are instrumental in the present theory. Both estimators were developed for $n = 1$ and $d = 2$; the results below are valid for $d \geqslant 2$.

We first introduce some notation common to both estimators. We indicate with $\mathscr{N}_{\mathscr{T}} := \{\mathbf{x}_i\}_{i=1}^{N_{\mathscr{T}}}$ the set of all nodes of triangulation $\mathscr{T}$. For each node $\mathbf{x}_i$, $\lambda_i \in \mathbb{V}(\mathscr{T})$ is the canonical piecewise linear function corresponding to $\mathbf{x}_i$, and ω_i is the *star* associated to $\mathbf{x}_i$, i. e. the support of λ_i. We denote by γ_i the union of the sides touching $\mathbf{x}_i$ that are contained in Ω. Finally, $\omega_T := \cup_{i \in T} \omega_i$ is the union of the $d+1$ stars containing T.

3.3.1 *Morin-Nochetto-Siebert Estimator (Morin* et al.*, 2003).*

Let $\mathscr{K}_{\mathscr{T}}$ be the set of all *stars*, i. e. the K-elements are the sets ω_i. We write ω_i instead of K to avoid confusion.

The local indicators hinge on the local weighted space $W(\omega_i)$ defined as

$$W(\omega_i) = \{v \in H^1_{\mathrm{loc}}(\omega_i) : \int_{\omega_i} v\lambda_i = 0 \text{ and } \int_K |\nabla v|^2 \lambda_i < \infty\}, \tag{3.16}$$

if $\mathbf{x}_i$ is an interior node, and

$$W(\omega_i) = \{v \in H^1_{\mathrm{loc}}(\omega_i) : v = 0 \text{ on } \partial\omega_i \cap \partial\Omega \text{ and } \int_K |\nabla v|^2 \lambda_i < \infty\}, \tag{3.17}$$

otherwise. The corresponding small problem is solved on subspace $\mathscr{P}_0^2(\omega_i) \subset W(\omega_i)$, of functions v which are piecewise quadratic on the star ω_i, vanish on $\partial\omega_i$, and satisfy $\int_{\omega_i} v\lambda_i = 0$. We also need to introduce the weighted bilinear form

$$\mathscr{B}_i[v_1, v_2] := \langle \mathbf{A}\nabla v_1, \nabla v_2 \lambda_i \rangle_{\omega_i} + \langle \mathbf{b} \cdot \nabla v_1 + c\, v_1, v_2 \lambda_i \rangle_{\omega_i}$$

For each star ω_i and $V \in \mathbb{V}(\mathscr{T})$, we define $\xi_i \in \mathscr{P}_0^2(\omega_i)$ to be the solution of

$$\xi_i \in \mathscr{P}_0^2(\omega_i) \quad : \quad \mathscr{B}_i[\xi_i, v] = \langle f, v\lambda_i \rangle_{\omega_i} - \mathscr{B}[V, v\lambda_i] \quad \text{for all } v \in \mathscr{P}_0^2(\omega_i), \tag{3.18}$$

then, the star error indicator and oscillation for $V \in \mathbb{V}(\mathscr{T})$ are defined as

$$\eta^2_{\mathscr{T}}(V,\omega_i) := |||\xi_i|||^2_{\lambda_i} := \mathscr{B}_i[\xi_i,\xi_i], \tag{3.19}$$

$$\operatorname{osc}^2_{\mathscr{T}}(V,\omega_i) := h_i^2 \| \left\{ (I-\Pi^2_{0,i})R(V,\omega_i) \right\} \lambda_i^{1/2} \|^2_{L^2(\omega_i)} + h_i \| \left\{ (I-\Pi^2_{0,i})J(V,\gamma_i) \right\} \lambda_i^{1/2} \|^2_{L^2(\gamma_i)}. \tag{3.20}$$

Here $\Pi^2_{0,i}$ denotes the projection on piecewise constants with the weighted scalar product $\int_\omega vw\lambda_i$, with either $\omega = \omega_i$ or $\omega = \sigma$ depending on the residual. We stress that the right-hand side of (3.18) is equivalent to the original one in Morin *et al.* (2003) but it does not involve the explicit computation of residuals (either jumps $J(U,\sigma)$ or $R(U,T)$).

The proofs of upper and lower a posteriori error bounds are technical and similar to those in (Morin *et al.*, 2003, Theorems 3.6 and 3.1), and thus not reported here. The first one requires a weighted Poincaré inequality (Morin *et al.*, 2003, Proposition 2.4), and the second one the definition of a suitable operator between $\mathscr{P}^2_0(\omega_i)$ and $W(\omega_i)$. They yield the existence of constants C_1, C_2, solely depending $\mathscr{T}_0$, C_B/c_B, and d, such that

$$|||u-U|||^2_\Omega \leqslant C_1 \left\{ \eta^2_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}) + \operatorname{osc}^2_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}) \right\}, \tag{3.21}$$

$$C_2 \eta^2_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}) \leqslant |||u-U|||^2_\Omega. \tag{3.22}$$

The following discrete counterpart of (3.21) is also based on (Morin *et al.*, 2003, Theorem 3.6): if $\mathscr{R} = \mathscr{R}^1_{\mathscr{T}\to\mathscr{T}_*} \subset \mathscr{K}_{\mathscr{T}}$ is the refined set of order 1, then

$$|||U_* - U|||^2_\Omega \leqslant C_1 \left\{ \eta^2_{\mathscr{T}}(U,\mathscr{R}) + \operatorname{osc}^2_{\mathscr{T}}(U,\mathscr{R}) \right\}. \tag{3.23}$$

Its proof employs the localization argument of (Cascón *et al.*, 2008, Lemma 3.6), with $V \in \mathbb{V}(\mathscr{T})$ defined as in (3.15). Properties of the partition of unity allow us to write

$$|||E_*|||^2_\Omega = \mathscr{B}[E_*, E_* - V] = \sum_{\omega_i \in \mathscr{R}} \langle R(U,\omega_i), (E_* - V)\lambda_i \rangle_{\omega_i} + \langle J(U,\gamma_i), (E_* - V)\lambda_i \rangle_{\gamma_i}.$$

The proof continues as in (Morin *et al.*, 2003, Theorem 3.6), but using the fact that $V = E_*$ for all $\omega_i \notin \mathscr{R}$. The following discrete counterpart of (3.22) is a slight variation of (Morin *et al.*, 2003, Lemma 5.2): if $\mathscr{R} = \mathscr{R}^j_{\mathscr{T}_j\to\mathscr{T}_*}$ is a refined set of order $j \geqslant \mathfrak{n}$, then

$$C_3 \eta^2_{\mathscr{T}}(\mathscr{R},U) \leqslant |||U_* - U|||^2_\Omega + \operatorname{osc}^2_{\mathscr{T}}(U,\mathscr{R}). \tag{3.24}$$

3.3.2 *Parés-Díez-Huerta Estimator (Parés* et al.*, 2006).* The K-elements are simplices and, to avoid confusion, we denote them by T: $\mathscr{K}_{\mathscr{T}} := \{T\}_{T\in\mathscr{K}_{\mathscr{T}}}$. Let $\mathscr{T}_*$ satisfy the interior node property (3.3) for all $T \in \mathscr{T}$, let $\mathbb{V}_* = \mathbb{V}(\mathscr{T}_*)$ and $\mathbb{V}_*(\omega_i)$ be the restriction of $\mathbb{V}_*$ to star ω_i. There is no boundary condition imposed on $\mathbb{V}_*(\omega_i)$.

For each ω_i and $V \in \mathbb{V}(\mathscr{T})$, we let the star indicator be

$$\xi_i \in \mathbb{V}_*(\omega_i) \quad : \quad \mathscr{B}_{\omega_i}[\xi_i, v] = \langle f, v\lambda_i \rangle_{\omega_i} - \mathscr{B}[V, v\lambda_i] \quad \text{for all } v \in \mathbb{V}_*(\omega_i), \tag{3.25}$$

which is similar to (3.18) and well-defined provided $c \geqslant c_0 > 0$; otherwise we demand ξ_i to have vanishing mean value. In contrast to (3.18), the local bilinear form $\mathscr{B}_{\omega_i}$ does not have the weight λ_i and the augmented space if made of piecewise linears instead of quadratics; these key differences make the estimator appealing for $d > 2$. We point out that it is sufficient for our purposes to restrict the local space $\mathbb{V}_*(\omega_i)$ to only the interior nodes in elements and interior sides within ω_i, as will become apparent in what follows. Even though this reduces the size of local problems considerably, especially for $d > 2$, we decided to stick to the original estimator for this discussion. The error indicator and oscillation are

$$\eta_{\mathscr{T}}(V,T) := \left|\!\left|\!\left| \sum_{\mathbf{x}_i \in T\cap\mathscr{N}_{\mathscr{T}}} \xi_i \right|\!\right|\!\right|_T, \tag{3.26}$$

$$\mathrm{osc}^2_{\mathscr{T}}(V,T) := h_T^2 \|(I-\Pi_0^2)R(V,T)\|^2_{L^2(T)} + h_T\|(I-\Pi_0^2)J(V,\partial T)\|^2_{L^2(\partial T)}. \tag{3.27}$$

for any element $T \in \mathscr{T}$ and function $V \in \mathbb{V}(\mathscr{T})$.

We present an alternative a posteriori analysis to Parés *et al.* (2006) and a few novel estimates; in particular we do not need a reference mesh. We first show a *global upper bound*: there exists a constant C_1, solely depending $\mathscr{T}_0$ and C_B/c_B, such that

$$|\!|\!| u-U |\!|\!|^2_\Omega \leqslant C_1\Big(\eta^2_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}) + \mathrm{osc}^2_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}})\Big). \tag{3.28}$$

We proceed as in §3.2 and invoke the local operator $P_{\mathscr{T}}$ of (3.11). Given a test function $v \in \mathbb{V}$, we let $I_{\mathscr{T}}v$ be its Scott-Zhang interpolant. Using the Galerkin orthogonality, and setting $z = v - I_{\mathscr{T}}v$, we obtain

$$\mathscr{B}[u-U,v] = \mathscr{B}[u-U,v-I_{\mathscr{T}}v] = \mathscr{B}[u-U,P_{\mathscr{T}}z)] + \mathscr{B}[u-U,(I-P_{\mathscr{T}})z]$$

On the one hand, the partition of unity $\{\lambda_i\}_{i=1}^{N_{\mathscr{T}}}$, in conjunction with (3.25), implies

$$\mathscr{B}[u-U,P_{\mathscr{T}}z] = \sum_{i=1}^{N_{\mathscr{T}}} \mathscr{B}[u-U,P_{\mathscr{T}}z\lambda_i] = \sum_{i=1}^{N_{\mathscr{T}}} \langle f, P_{\mathscr{T}}z\lambda_i\rangle_{\omega_i} - \mathscr{B}[U,P_{\mathscr{T}}z\lambda_i] = \sum_{i=1}^{N_{\mathscr{T}}} \mathscr{B}_{\omega_i}[\xi_i, P_{\mathscr{T}}z],$$

whence regrouping by elements T, results in

$$\sum_{i=1}^{N_{\mathscr{T}}} \mathscr{B}_{\omega_i}[\xi_i,P_{\mathscr{T}}z] = \sum_{i=1}^{N_{\mathscr{T}}}\sum_{T\subset\omega_i} \mathscr{B}_T[\xi_i,P_{\mathscr{T}}z] \leqslant \sum_{T\in\mathscr{T}} \mathscr{B}_T[\sum_{\mathbf{x}_i\in T}\xi_i, P_{\mathscr{T}}z] \leqslant C\eta^2_{\mathscr{T}}(U,\mathscr{T})\, |\!|\!| v |\!|\!|_\Omega .$$

Here we have used $\mathscr{B}_T$ to stand for the restriction of $\mathscr{B}$ to T, and the bound $|\!|\!| P_{\mathscr{T}}z |\!|\!|_\Omega \preccurlyeq |\!|\!| v |\!|\!|_\Omega$ from §3.2. On the other hand, the remaining term involving $(I-P_{\mathscr{T}})z$ can be estimated again as in §3.2, thereby concluding the proof of (3.28). This argument shows that the only relevant nodes, namely those defining the operator $P_{\mathscr{T}}$, are the interior nodes in ω_i. This observation can in turn be exploited to simplify the definition of ξ_i to precisely those nodes.

We now derive a *lower bound*, which is consistent with our theory, and local in nature, but different from the original one in Parés *et al.* (2006): there exists a constant C_2, solely depending on $\mathscr{T}_0$ and c_B/C_B, such that

$$C_2\eta^2_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}) \leqslant |\!|\!| u-U |\!|\!|^2_\Omega . \tag{3.29}$$

We first use the definition (3.25) of ξ_i and Galerkin orthogonality $\langle f,\lambda_i\rangle_{\omega_i} - \mathscr{B}[U,\lambda_i] = 0$, for interior nodes $\mathbf{x}_i$, to deduce

$$|\!|\!| \xi_i |\!|\!|^2_{\omega_i} = \mathscr{B}_{\omega_i}[\xi_i,\xi_i] = \langle f,\xi_i\lambda_i\rangle_{\omega_i} - \mathscr{B}[U,\xi_i\lambda_i] = \langle f,(\xi_i-c_i)\lambda_i\rangle_{\omega_i} - \mathscr{B}[U,(\xi_i-c_i)\lambda_i]$$

where $c_i = 0$ for boundary nodes, and $c_i = |\omega_i|^{-1} \int_{\omega_i} \xi_i$ otherwise. By continuity of $\mathscr{B}$ and Poincaré inequality $\|\xi_i - c_i\|_{L^2(\omega_i)} \preccurlyeq \|\!|\xi_i|\!\|_{\omega_i}$, we thus obtain

$$\|\!|\xi_i|\!\|^2_{\omega_i} = \mathscr{B}[u-U,(\xi_i-c_i)\lambda_i] \preccurlyeq \|\!|u-U|\!\|_{\omega_i} \|\!|\xi_i|\!\|_{\omega_i}.$$

This, together with (3.26), yields

$$\eta_{\mathscr{T}}(U,T) \leqslant \sum_{\mathbf{x}_i \in T \cap \mathscr{N}_{\mathscr{T}}} \|\!|\xi_i|\!\|_T \preccurlyeq \|\!|u-U|\!\|_{\omega_T},$$

and thus (3.29) upon summing over all $T \in \mathscr{T}$. We point out that we have argued by *stars*, instead of triangles as in Parés *et al.* (2006). However, in view of the definition of ξ_i by stars, this seems the only viable way to relate ξ_i with the error.

Our next task is to derive discrete versions of (3.28) and (3.29), which are in turn crucial for the subsequent development. We start with a *localized upper bound*: if $\mathscr{R} = \mathscr{R}^1_{\mathscr{T}\to\mathscr{T}_*} \subset \mathscr{K}_{\mathscr{T}}$ is the refined set of order 1, then

$$\|\!|U_* - U|\!\|^2_{\Omega} \leqslant C_1 \left(\eta^2_{\mathscr{T}}(U,\mathscr{R}) + \operatorname{osc}^2_{\mathscr{T}}(U,\mathscr{R})\right). \tag{3.30}$$

We proceed as with (3.14) and (3.23), namely we let $V = I_{\mathscr{T}}E^* \in \mathbb{V}(\mathscr{T})$ be the Scott-Zhang interpolant of $E_* = U - U_*$. Hence

$$\|\!|E_*|\!\|^2_{\Omega} = \mathscr{B}[E_*, P_{\mathscr{T}}(E_*-V)] + \mathscr{B}[E_*,(I-P_{\mathscr{T}})(E_*-V)]$$

which reduces to a sum over elements $T \in \mathscr{R}$ because $Z := E_* - V = P_{\mathscr{T}}Z = 0$ for all $T \notin \mathscr{R}$. Arguing as with (3.13), the second term leads to the oscillation over the refined set. For the first term, we denote by $\mathscr{N}(\mathscr{R})$ the set of nodes of $\mathscr{R}$ and recall the definition (3.25) of ξ_i to obtain

$$\begin{aligned}\mathscr{B}[E_*,P_{\mathscr{T}}Z] &= \sum_{\mathbf{x}_i\in\mathscr{N}(\mathscr{R})} \mathscr{B}[E_*,P_{\mathscr{T}}Z\lambda_i] = \sum_{\mathbf{x}_i\in\mathscr{N}(\mathscr{R})} \mathscr{B}_{\omega_i}[\xi_i,P_{\mathscr{T}}Z] \\ &= \sum_{T\subset\mathscr{R}} \mathscr{B}_T[\sum_{\mathbf{x}_i\in T\cap\mathscr{N}(\mathscr{R})} \xi_i, P_{\mathscr{T}}Z] \preccurlyeq \eta_{\mathscr{T}}(U,\mathscr{R})\,\|\!|E_*|\!\|_{\Omega},\end{aligned}$$

because $\|\!|P_{\mathscr{T}}Z|\!\|_{\Omega} \preccurlyeq \|\!|E_*|\!\|_{\Omega}$ according to §3.2. This shows (3.30).

To derive a discrete version of (3.29), we observe that the argument leading to (3.29) cannot be applied because $(\xi_i - c_i)\lambda_i \notin \mathbb{V}(\mathscr{T}_*)$ for any refinement $\mathscr{T}_*$ of $\mathscr{T}$. We resort again to the interpolation operator $P_{\mathscr{T}}$ of §3.2 to prove the following *discrete lower bound for $\mathscr{T} \leqslant \mathscr{T}_*$: if $T \in \mathscr{T}$ satisfies $\operatorname{gen}_{\mathscr{T}_*}(T') \geqslant \mathfrak{n}$ for all $T' \subset \omega_T$, then*

$$C_3\eta^2_{\mathscr{T}}(U,T) \leqslant \|\!|U_*-U|\!\|^2_{\omega_T} + \operatorname{osc}^2_{\mathscr{T}}(U,\omega_T). \tag{3.31}$$

Let ω_i be a star of $\mathscr{T}$ containing such $T \in \mathscr{T}$. We recall the representation formula

$$\|\!|\xi_i|\!\|^2_{\omega_i} = \langle f,(\xi_i-c_i)\lambda_i\rangle - \mathscr{B}[U,(\xi_i-c_i)\lambda_i]$$

used in dealing with (3.29). Since $\operatorname{gen}_{\mathscr{T}_*}(T') \geqslant \mathfrak{n}$ for all $T' \subset \omega_i$, they satisfy the interior node property (3.3) and $P_{\mathscr{T}}$ is well-defined in ω_i. If $z := (\xi_i - c_i)\lambda_i$, then we infer that $\operatorname{supp} P_{\mathscr{T}}z \subset \omega_i$ because of (3.11). We add and subtract $P_{\mathscr{T}}z$ to write

$$\|\!|\xi_i|\!\|^2_{\omega_i} = \langle f, P_{\mathscr{T}}z\rangle - \mathscr{B}[U,P_{\mathscr{T}}z] + \langle f,(I-P_{\mathscr{T}})z\rangle - \mathscr{B}[U,(I-P_{\mathscr{T}})z].$$

Since $P_{\mathscr{T}} z \in \mathbb{V}_*(\omega_i) \subset \mathbb{V}(\mathscr{T}_*)|_{\omega_i}$ we have for the first two terms

$$\langle f, P_{\mathscr{T}} z\rangle - \mathscr{B}[U, P_{\mathscr{T}} z] = \mathscr{B}[U^* - U, P_{\mathscr{T}} z] \leqslant C \, |||U^* - U|||_{\omega_i} \, |||\xi_i|||_{\omega_i} \,,$$

the last step being a consequence of the local stability (3.12) of $P_{\mathscr{T}}$ and Poincaré inequality. For the remaining terms, we proceed as with (3.13) to conclude

$$\langle f, (I - P_{\mathscr{T}})z\rangle_{\omega_i} - \mathscr{B}[U, (I - P_{\mathscr{T}})z] \preccurlyeq \operatorname{osc}_{\mathscr{T}}(U, \omega_i) \, |||\xi_i|||_{\omega_i} \,.$$

Summing over the three stars that contain $T \in \mathscr{M}$ we obtain (3.31), as asserted.

3.4 *Gradient Recovery Estimators (Zienkiewicz & Zhu, 1987)*

These among the most popular estimators in computational engineering because of their simplicity and accuracy. The first and most successful is due to Zienkiewicz & Zhu (1987), is defined on stars, and is the one consider here because it is *local*. Other global recovery-type estimators entail a global projection (Bank & Xu, 2003; Carstensen, 2003) but do not fit within our theory below.

Given $\mathscr{T}$ and $V \in \mathbb{V}(\mathscr{T})$, we denote by $G_{\mathscr{T}} V$ the orthogonal projection of ∇V into the vectorial linear finite element space with respect to the $\mathbb{V}(\mathscr{T})$-lumped $L_2(\Omega)$-scalar product. The nodal values of the recovered gradient $G_{\mathscr{T}} V$ obtained in this way can be written explicitly as follows for each node $\mathbf{x}_i$

$$G_{\mathscr{T}} V(\mathbf{x}_i) = \sum_{T \subset \omega_i} \frac{|T|}{|\omega_i|} \nabla V_{|T}$$

where ω_i is the star associated to $\mathbf{x}_i$. The estimator and oscillation are given by

$$\eta^2_{\mathscr{T}}(V,T) := \int_T |G_{\mathscr{T}} V - \nabla V|^2 , \qquad \operatorname{osc}^2_{\mathscr{T}}(T) := \sum_{\mathbf{x}_i \in T} \|h(f - f_i)\|^2_{L^2(T)} ,$$

where $f_i = |\omega_i|^{-1} \int_{\omega_i} f$. Rodríguez showed the equivalence of this and jump indicators by stars (Rodríguez, 1994, Theorem 3.1, Remark 3.1):

$$\sum_{T \subset \omega_i} \eta^2_{\mathscr{T}}(U,T) \approx \sum_{\sigma \subset \gamma_i} \|h_\sigma^{1/2} J(U,\sigma)\|^2_\sigma, \tag{3.32}$$

with γ_i as in §3.3. *Upper* and *lower* a posteriori bounds, up to oscillation, follow from the equivalence of (3.8) and the energy error (Rodríguez, 1994). We are not aware of a direct proof of equivalence.

If $\mathscr{T} \leqslant \mathscr{T}_*$ and $U \in \mathbb{V}(\mathscr{T})$, $U_* \in \mathbb{V}(\mathscr{T}_*)$ are solutions of (2.2), then the following *localized upper bound* is a consequence of the corresponding one for the residual estimator (Cascón *et al.*, 2008, Lemma 3.6):

$$|||U_* - U|||^2_\Omega \leqslant C_1 \left\{ \eta^2_{\mathscr{T}}(U, \widehat{\mathscr{R}}) + \operatorname{osc}^2_{\mathscr{T}}(U, \widehat{\mathscr{R}}) \right\}.$$

However, as the equivalence (3.32) is written by stars, this bound requires a larger set $\widehat{\mathscr{R}}$ than $\mathscr{R} = \mathscr{R}^1_{\mathscr{T} \to \mathscr{T}_*}$, made of all elements of stars containing triangles of $\mathscr{R}$. This does not affect our theory because $\#\widehat{\mathscr{R}} \approx \#\mathscr{R}$. Finally, a *discrete lower bound* similar to (3.31) follows from (3.32) and (Morin *et al.*, 2000, Lemma 4.2).

3.5 *Braess-Schöberl Estimator (Braess & Schöberl, 2008)*

This estimator hinges on a theorem of Prange and Synge: if $\mathbf{q} \in H(\mathrm{div};\Omega)$ is such that $\mathrm{div}\,\mathbf{q} + f = 0$, and $u \in H_0^1(\Omega)$ is the solution of the Laplace equation, then

$$\|\nabla u - \nabla v\|_{L^2(\Omega)}^2 + \|\nabla u - \mathbf{q}\|_{L^2(\Omega)}^2 = \|\nabla v - \mathbf{q}\|_{L^2(\Omega)}^2, \qquad \forall v \in H_0^1(\Omega).$$

An error estimate follows upon replacing v by U in the above formula. If f is assumed piecewise constant the optimal choice $\mathbf{q} \in H(\mathrm{div},\Omega)$ is the solution of the original formulation by the mixed method with Raviart-Thomas element. Since this procedure is too expensive for computing an a posteriori error estimator, Braess & Schöberl (2008) propose an alternative construction by solving cheap local problems. In fact, let $\mathbf{q}_\Delta$ belong to the broken Raviart-Thomas space and be defined on stars ω_i as the solutions of local problems

$$\mathbf{q}_\Delta = \sum_{i\in\mathcal{N}_\mathcal{T}} \mathbf{q}_i \quad \Leftrightarrow \quad \begin{cases} \mathrm{div}\,\mathbf{q}_i = -|T|^{-1}\int_T f\lambda_i & \text{in each } T \in \omega_i \\ [\![\mathbf{q}_i \cdot \boldsymbol{\nu}]\!] = -1/2\,[\![\nabla U \cdot \boldsymbol{\nu}]\!] & \text{on each edge } \sigma \in \gamma_i \\ \mathbf{q}_i \cdot \boldsymbol{\nu} = 0 & \text{on } \partial\omega_i \end{cases}$$

The vector field $\mathbf{q}_\Delta$ compensates for the jumps of ∇U whence $\mathbf{q} := \nabla U + \mathbf{q}_\Delta \in H(\mathrm{div},\Omega)$. The local indicator and oscillation are defined as

$$\eta_\mathcal{T}^2(V,T) := \|\mathbf{q}_\Delta\|_{L^2(\Omega)}^2, \qquad \mathrm{osc}_\mathcal{T}^2(T) := \|h(f - f_T)\|_{L^2(T)}^2$$

where $f_T = |T|^{-1}\int_T f$. Braess & Schöberl (2008) propose a simple algorithm (Algorithm 4) for the construction of $\mathbf{q}_i$ provided f is piecewise constant over $\mathcal{T}$. In this case, it is not difficult to obtain the equivalence

$$\|\mathbf{q}_i\|_{L^2(\omega_i)}^2 \approx \|hf\|_{L^2(\omega_i)}^2 + \left\|h^{1/2}J\right\|_{L^2(\gamma_i)}^2.$$

Otherwise, data oscillation appears in this equivalence. Exploiting this relation with the residual estimator of §3.1, upper and lower bounds, as well as their discrete counterparts, can be derived as in §3.4.

4. AFEM: Abstract Formulation

Motivated by the examples of §3, we now enunciate abstract properties that both $\eta_\mathcal{T}$ and $\mathrm{osc}_\mathcal{T}$ must fulfill, along with assumptions on the adaptive procedure AFEM, that enable us to derive a contraction property in §5 and decay rates in §6.

4.1 *A Posteriori Error Estimators*

We formulate two assumptions, the first one on $\eta_\mathcal{T}$ and the second on $\mathrm{osc}_\mathcal{T}$; see Cascón *et al.* (2008) for the residual estimator.

ASSUMPTION 4.1 (A posteriori error estimates) Let $u \in H_0^1(\Omega)$ be the solution of (1.1), and let $U \in \mathbb{V}(\mathcal{T})$ and $U_* \in \mathbb{V}(\mathcal{T}_*)$ be Galerkin solutions of (2.2) over meshes $\mathcal{T} \leqslant \mathcal{T}_*$. There exist constants $\{C_i\}_{i=1}^3$ such that the following properties hold.

(a) *Global upper bound (reliability)*: this gives an estimate of the energy error in terms of the *total* error estimator $\zeta_\mathcal{T}(U,\mathcal{K}_\mathcal{T})$

$$|\!|\!| u - U |\!|\!|_\Omega^2 \leqslant C_1 \left\{\eta_\mathcal{T}^2(U,\mathcal{K}_\mathcal{T}) + \mathrm{osc}_\mathcal{T}^2(U,\mathcal{K}_\mathcal{T})\right\} = C_1\,\zeta_\mathcal{T}^2(U,\mathcal{K}_\mathcal{T}). \tag{4.1}$$

(b) *Global lower bound (efficiency)*: this is a measure of quality of $\eta_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}})$

$$C_2\,\eta^2_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}) \leqslant |\!|\!| u-U |\!|\!|^2_{\Omega} + \operatorname{osc}^2_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}). \tag{4.2}$$

(c) *Localized upper bound*: this measures $|\!|\!| U_*-U |\!|\!|_{\Omega}$ in terms of the total estimator restricted to the refined set $\mathscr{R}^1=\mathscr{R}^1_{\mathscr{T}\to\mathscr{T}_*}\subset\mathscr{K}_{\mathscr{T}}$ of *order 1*

$$|\!|\!| U_*-U |\!|\!|^2_{\Omega} \leqslant C_1\left\{\eta^2_{\mathscr{T}}(V,\mathscr{R}^1)+\operatorname{osc}^2_{\mathscr{T}}(V,\mathscr{R}^1)\right\} = C_1\,\zeta^2_{\mathscr{T}}(U,\mathscr{R}^1). \tag{4.3}$$

(d) *Discrete local lower bound*: the estimator $\eta_{\mathscr{T}}(U,\mathscr{R}^{\mathfrak{n}})$ on the refined set $\mathscr{R}^{\mathfrak{n}}=\mathscr{R}^{\mathfrak{n}}_{\mathscr{T}\to\mathscr{T}_*}$ of *order* $\mathfrak{n}$ is a lower bound for $|\!|\!| U_*-U |\!|\!|_{\Omega}$

$$C_3\,\eta^2_{\mathscr{T}}(U,\mathscr{R}^{\mathfrak{n}}) \leqslant |\!|\!| U_*-U |\!|\!|^2_{\Omega} + \operatorname{osc}^2_{\mathscr{T}}(U,\mathscr{R}^1). \tag{4.4}$$

We observe the relation $C_2/C_1\leqslant 1$ that results from (4.1) and (4.2) in the particular case $\operatorname{osc}_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}})=0$. The quality of an estimator can be measured by the deviation of this ratio from 1.

ASSUMPTION 4.2 (Oscillation) We denote by Π^p_n the L^p-best approximation operator in the space of discontinuous polynomials of degree $\leqslant n$ over $K\in\mathscr{K}_{\mathscr{T}}$. We further let $\Pi^p_{-1}=0$ and $E^p_n=I-\Pi^p_n$ be the operator error. Our definition of oscillation with polynomial degree $n\geqslant 1$ conforms with the original one Morin *et al.* (2000, 2002); Mekchay & Nochetto (2005) but it is at variance with that in Cascón *et al.* (2008). For the residual estimator we let

$$\operatorname{osc}_{\mathscr{T}}(U,T)^2 := h_T^2\left\|E^2_{n-1}R(U,T)\right\|^2_{L^2(T)} + h_T\left\|E^2_{n-1}J(U,\partial T)\right\|^2_{L^2(\partial T)} \tag{4.5}$$

and similar expressions are valid for the other estimators. It is important to observe that the polynomial degree in (4.5) is consistent with the interior node property (3.3). We next introduce the oscillation of the coefficient $\mathbf{D}:=(\mathbf{A},\mathbf{b},c)$ on $T\in\mathscr{T}$:

$$\begin{aligned}\operatorname{osc}_{\mathscr{T}}(\mathbf{D},T) := h_T\Big(&\|E^\infty_0\operatorname{div}\mathbf{A}\|^2_{L^\infty(T)} + h_T^{-2}\|E^\infty_1\mathbf{A}\|^2_{L^\infty(\omega_T)}\\ &+\|E^\infty_0\mathbf{b}\|^2_{L^\infty(T)} + h_T^2\left\|E^\infty_{-1}c\right\|^2_{L^\infty(T)} + \|E^\infty_0 c\|^2_{L^\infty(T)}\Big)^{1/2};\end{aligned}$$

and for any subset $\mathscr{T}'\subset\mathscr{T}$ we define

$$\operatorname{osc}_{\mathscr{T}}(\mathbf{D},\mathscr{T}') := \max_{T\in\mathscr{T}'}\operatorname{osc}_{\mathscr{T}}(\mathbf{D},T).$$

The assume the following properties to be valid for any discrete functions $V\in\mathbb{V}(\mathscr{T}),V_*\in\mathbb{V}(\mathscr{T}_*)$, with $\mathscr{T}\leqslant\mathscr{T}_*$.

(a) *Oscillation reduction*: there exists a constant $0<\lambda<1$ so that

$$\operatorname{osc}^2_{\mathscr{T}_*}(V,\mathscr{K}_{\mathscr{T}_*}) \leqslant \operatorname{osc}^2_{\mathscr{T}}(V,\mathscr{K}_{\mathscr{T}}) - \lambda\,\operatorname{osc}^2_{\mathscr{T}}(V,\mathscr{R}^1). \tag{4.6}$$

(b) *Lipschitz property*: there exists a constant $C_4>0$ depending on the shape regularity of $\mathscr{T}_0$ and the polynomial degree n so that for all $K\in\mathscr{K}_{\mathscr{T}_*}$

$$\left|\operatorname{osc}_{\mathscr{T}_*}(V_*,K)-\operatorname{osc}_{\mathscr{T}_*}(V,K)\right| \leqslant C_4\operatorname{osc}_{\mathscr{T}_*}(\mathbf{D},\omega_K)\,|\!|\!| V_*-V |\!|\!|_{\omega_K}, \tag{4.7}$$

where ω_K is a small discrete neighborhood of K.

The proof of these properties follows along the same lines as (Cascón *et al.*, 2008, Proposition 3.3) because $\operatorname{osc}_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}), \operatorname{osc}_{\mathscr{T}}(\mathbf{D},\mathscr{T})$ are similar regardless of the specific estimator; see §3. Hence, we do not insist on this point any longer.

Property (4.7), couple with the finite overlapping of sets ω_K and Young's inequality, yields the following *perturbation property* on the set of unchanged elements

$$\operatorname{osc}^2_{\mathscr{T}}(V,\mathscr{K}_{\mathscr{T}}\cap\mathscr{K}_{\mathscr{T}_*}) \leqslant 2\operatorname{osc}^2_{\mathscr{T}_*}(V_*,\mathscr{K}_{\mathscr{T}}\cap\mathscr{K}_{\mathscr{T}_*}) + 2\Lambda_1 \operatorname{osc}^2_{\mathscr{T}_0}(\mathbf{D},\mathscr{K}_0)\, |||V-V_*|||^2_{\Omega}\,, \tag{4.8}$$

where Λ_1 is proportional to C_4^2. Combining (4.6) and (4.7) with Young's inequality we derive the following *quasi-reduction property* for all $\delta > 0$

$$\begin{aligned}\operatorname{osc}^2_{\mathscr{T}_*}(V_*,\mathscr{K}_{\mathscr{T}_*}) \leqslant (1+\delta)\left\{\operatorname{osc}^2_{\mathscr{T}}(V,\mathscr{K}_{\mathscr{T}}) - \lambda\, \operatorname{osc}^2_{\mathscr{T}}(V,\mathscr{R}^1)\right\}\\ + (1+\delta^{-1})\Lambda_1 \operatorname{osc}^2_{\mathscr{T}_0}(\mathbf{D},\mathscr{T}_0)\, |||V_*-V|||^2_{\Omega}\,.\end{aligned} \tag{4.9}$$

4.2 *AFEM*

The adaptive method consists of iterations of the form (1.2) so that

- SOLVE computes the exact Ritz-Galerkin solution of (2.2):

  $$U = \mathsf{SOLVE}(\mathscr{T}).$$

  We assume exact linear algebra and integration, the former just for simplicity.

- ESTIMATE calculates the *total error indicator* $\zeta_{\mathscr{T}}(U,K)$ of (3.4):

  $$\{\zeta_{\mathscr{T}}(U,K)\}_{K\in\mathscr{K}_{\mathscr{T}}} = \mathsf{ESTIMATE}(U,\mathscr{T}).$$

- MARK uses Dörfler marking with parameter $0 < \theta \leqslant 1$,

  $$\zeta_{\mathscr{T}}(U,\mathscr{M}) \geqslant \theta\, \zeta_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}), \tag{4.10}$$

  to select a set $\mathscr{M}\subset\mathscr{K}_{\mathscr{T}}$ to be refined

  $$\mathscr{M} = \mathsf{MARK}(\{\zeta_{\mathscr{T}}(U,K)\}_{K\in\mathscr{K}_{\mathscr{T}}},\mathscr{T}).$$

- REFINE bisects all elements $T\in\mathscr{T}$ contained in $\mathscr{M}$. Since bisections are performed elementwise, we introduce the *element refinement flag* $\rho_{\mathscr{T}}(T)\in\mathbb{N}$ for $T\in\mathscr{T}$, and decide that T must be refined provided $\rho_{\mathscr{T}}(T) > 0$. This flag is initialized $\rho_{\mathscr{T}_0}(T) = 0$ for all $T\in\mathscr{T}_0$. REFINE first updates the value of this flag according to the marked set

  $$\rho_{\mathscr{T}}(T) := \begin{cases} \mathfrak{n} & \text{if } T\subset K, \text{ with } K\in\mathscr{M} \\ \rho_{\mathscr{T}}(T) & \text{otherwise}\end{cases}$$

  where $\mathfrak{n}\in\mathbb{N}$ is defined in (3.3). REFINE next bisects $\mathfrak{b}\geqslant 1$ times the elements $T\in\mathscr{T}$ with $\rho_{\mathscr{T}}(T) > 0$, and generates a conforming triangulation $\mathscr{T}_*\geqslant\mathscr{T}$. The flags are updated in $\mathscr{T}_*$ as follows:

  $$\rho_{\mathscr{T}_*}(T) := \max\left\{0, \rho_{\mathscr{T}}(\operatorname{anc}_{\mathscr{T}}(T)) - \operatorname{gen}_{\mathscr{T}_*}(\operatorname{anc}_{\mathscr{T}}(T))\right\} \quad \forall T\in\mathscr{T}_*;$$

thus the element flag decreases with refinement depth. In summary, the marked and refined set satisfy $\mathscr{M} \subset \mathscr{R} = \mathscr{R}^{\mathfrak{b}}_{\mathscr{T}\to\mathscr{T}_*}$ and

$$\left\{\mathscr{T}_*, \{\rho_{\mathscr{T}_*}(T)\}_{T\in\mathscr{T}_*}\right\} = \mathsf{REFINE}(\mathscr{T}, \mathscr{M}, \{\rho_{\mathscr{T}}(T)\}_{T\in\mathscr{T}}).$$

REFINE is a minor modification of standard refinement routines and is easy to implement within ALBERTA (Schmidt & Siebert, 2005).

In order to obtain convergence and quasi-optimal cardinality, we have to impose some additional conditions on the initial mesh, and a requirement on the marking strategy, which we now enunciate. The first two conditions are crucial to get the contraction property (Theorem 5.1), whereas the other three requirements imply quasi-optimal cardinality (Theorem 6.1).

ASSUMPTION 4.3 (AFEM) We assume the following properties of AFEM:

(a) *Initial Mesh - Quasi-orthogonality*. If $\mathbf{b} \neq 0$ in Ω, then the initial grid $\mathscr{T}_0$ has to be sufficiently fine with respect to θ and $\mathbf{D}$ in the sense that

$$h_0^r \|\mathbf{b}\|_{L^\infty(\Omega)} < \frac{\theta^4}{C_0[\theta^4 + \mu_1\theta^2 + \mu_2 \operatorname{osc}^2_{\mathscr{T}_0}(\mathbf{D}, \mathscr{T}_0)]} \tag{4.11}$$

with C_0 the constant in Lemma 2.1 and

$$\mu_1 := \frac{2C_1}{C_3}, \qquad \mu_2 := \frac{4C_1(1+C_3)^2\Lambda_1}{\lambda^2 C_3^2}$$

where $h_0 = \max_{T\in\mathscr{T}_0} h_T$. If $\mathbf{b} \equiv 0$, then there is no restriction on $\mathscr{T}_0$ besides its alignment to the jumps of $\mathbf{A}$. This assumption clearly gives $C_0 h_0^r \|\mathbf{b}\|_{L^\infty(\Omega)} < 1$, and implies that the constant Λ_0 in (2.3) satisfies

$$\Lambda_0 < 1 + \frac{\theta^4}{\mu_1\theta^2 + \mu_2 \operatorname{osc}^2_{\mathscr{T}_0}(\mathbf{D}, \mathscr{T}_0)}. \tag{4.12}$$

(b) *REFINE- Refinement depth*. All simplices contained in marked K-elements in step j are subdivided at least $\mathfrak{n}$ times after $\mathfrak{J} := \mathfrak{J}(\mathfrak{n}, \mathfrak{b})$ steps of AFEM, i. e.

$$\operatorname{gen}_{\mathscr{T}_{j+\mathfrak{J}}}(T) \geqslant \mathfrak{n} \quad \forall T \subseteq K \in \mathscr{M}_j.$$

This requirement is vital to obtain the discrete lower bound (4.4) with $\mathscr{M} \subset \mathscr{R}^{\mathfrak{n}}$.

(c) *Initial Mesh - Complexity of REFINE*. The labeling of refinement edges on $\mathscr{T}_0$ satisfies (Stevenson, 2008, Condition (b), Section 4) for $d > 2$. The condition is simpler for $d = 2$ and is due to Mitchell (1989) and Binev *et al.* (2004). See also the survey (Nochetto *et al.*, 2009, Section 4).

(d) *MARK- Parameter θ*. The marking parameter θ satisfies $\theta \in (0, \theta_*)$ with

$$\theta_* := \Big(\frac{C_2}{(1+C_2)(1+C_1(2+2\Lambda_1 \operatorname{osc}^2_{\mathscr{T}_0}(\mathbf{D}, \mathscr{T}_0)))}\Big)^{1/2}.$$

(e) *MARK- Minimal cardinality*. The cardinality of the marked set $\mathscr{M}$ is minimal.

We now present AFEM with the iteration counter j as a subscript instead of $\mathscr{T}_j$: *given the initial grid $\mathscr{T}_0$ and marking parameter $0 < \theta \leqslant 1$ set $j := 0$ and iterate*

(i) $U_j = \mathsf{SOLVE}(\mathscr{T}_j)$;
(ii) $\{\zeta_j(U_j,K)\}_{K\in\mathscr{K}_j} = \mathsf{ESTIMATE}(U_j,\, \mathscr{T}_j)$;
(iii) $\mathscr{M}_j = \mathsf{MARK}(\{\zeta_j(U_j,K)\}_{K\in\mathscr{K}_j},\, \mathscr{T}_j)$;
(iv) $\{\mathscr{T}_{j+1}, \{\rho_{j+1}(T)\}_{T\in\mathscr{T}_{j+1}}\} = \mathsf{REFINE}(\mathscr{M}_j,\, \mathscr{T}_j, \{\rho_j(T)\}_{T\in\mathscr{T}_j})$;
(v) $j := j+1$;

REMARK 4.1 (Marking) In contrast to Cascón *et al.* (2008), the proposed AFEM utilizes the oscillation for marking. This could be avoided if $\eta_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}) \geqslant C\,\mathrm{osc}_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}})$ for $C>0$. While this property is trivial for the residual estimator with $C=1$, it is in general false for other families of estimators such as those in §3. This happens for unresolved data typical of the preasymptotic regime. Therefore the oscillation cannot be removed for marking without further assumptions.

REMARK 4.2 (Interior node property) AFEM does not enforce an interior node property between consecutive refinements, as in Morin *et al.* (2000, 2002); Veeser (2002); Morin *et al.* (2003); Mekchay & Nochetto (2005); Stevenson (2007), but after $\mathfrak{J}$ steps. This is easy to implement within ALBERTA (Schmidt & Siebert, 2005) and has an insignificant impact in the refinement process. This property was circumvented altogether in Cascón *et al.* (2008); Diening & Kreuzer (2008) for the residual estimator, employing the crucial property $\eta_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}) \geqslant \mathrm{osc}_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}})$, and in Kreuzer & Siebert (2010) for non-residual estimators upon exploiting their equivalence with the residual estimator.

5. Contraction Property of AFEM

We now prove that AFEM satisfies a contraction property with respect to the sum of energy error plus scaled oscillation, the so-called *total error*. The total error is reduced by a fixed rate after $\mathfrak{J}$ steps. The proof is inspired in results of Cascón *et al.* (2008); Mekchay & Nochetto (2005).

THEOREM 5.1 (Contraction Property) Let Assumptions 4.1(a,d), 4.2(a,b), and 4.3 (a,b) be valid. Let $\theta \in (0,1]$ be the marking parameter and let $\{\mathscr{T}_j, \mathbb{V}_j, U_j\}_{j\geqslant 0}$ be the sequence of meshes, finite element spaces, and discrete solutions produced by AFEM.

Then, there exist constants $\gamma > 0$, $0<\alpha<1$, and $\mathfrak{J}\in\mathbb{N}$, depending solely on the shape-regularity of $\mathscr{T}_0$, $\mathfrak{n}$, $\mathfrak{b}$ and θ, such that

$$|||u-U_{j+\mathfrak{J}}|||_{\Omega}^2 + \gamma\,\mathrm{osc}_{j+\mathfrak{J}}^2(U_{j+\mathfrak{J}},\mathscr{K}_{j+\mathfrak{J}}) \leqslant \alpha^2\left(|||u-U_j|||_{\Omega}^2 + \gamma\,\mathrm{osc}_j^2(U_j,\mathscr{K}_j)\right).$$

Proof. For convenience, we use the notation

$$e_j := |||u-U_j|||_{\Omega}\,,\ E_j := |||U_{j+\mathfrak{J}}-U_j|||_{\Omega}\,,\ \eta_j(\mathscr{M}_j) := \eta_j(U_j,\mathscr{M}_j),\ \mathscr{R}_j^{\mathfrak{n}} := \mathscr{R}^{\mathfrak{n}}_{\mathscr{T}_j\to\mathscr{T}_{j+\mathfrak{J}}}$$
$$\mathrm{osc}_j := \mathrm{osc}_j(U_j,\mathscr{K}_j),\quad \mathrm{osc}_j(\mathscr{M}_j) := \mathrm{osc}_j(U_j,\mathscr{M}_j),\quad \mathrm{osc}_0(\mathbf{D}) := \mathrm{osc}_0(\mathbf{D},\mathscr{T}_0).$$

We observe that $\mathscr{M}_j \subset \mathscr{R}_j^1$ always and Assumption 4.3(b) guarantees that $\mathscr{M}_j \subset \mathscr{R}_j^{\mathfrak{n}}$, so that all the elements contained in $\mathscr{M}_j$ are refined at least $\mathfrak{n}$ times in $\mathscr{T}_{j+\mathfrak{J}}$. We combine the quasi-orthogonality (2.3) with oscillation reduction (4.9) to write

$$e_{j+\mathfrak{J}}^2 + \gamma\,\mathrm{osc}_{j+\mathfrak{J}}^2 \leqslant \Lambda_0 e_j^2 - E_j^2 + (1+\delta^{-1})\,\gamma\Lambda_1\,\mathrm{osc}_0^2(\mathbf{D})E_j^2 + (1+\delta)\,\gamma\big(\mathrm{osc}_j^2 - \lambda\,\mathrm{osc}_j^2(\mathscr{R}_j^1)\big)\,.$$

To remove the third term on the right-hand side, we first write $E_j^2 = \beta E_j^2 + (1-\beta)E_j^2$ with a constant

$\beta \in (0,1)$ to be selected later. We choose γ depending on δ to be

$$\gamma := \frac{(1-\beta)}{(1+\delta^{-1})\,\Lambda_1\, \mathrm{osc}_0^2(\mathbf{D})} \qquad \Leftrightarrow \qquad \gamma(1+\delta) = \frac{\delta(1-\beta)}{\Lambda_1\, \mathrm{osc}_0^2(\mathbf{D})} \tag{5.1}$$

whence

$$e_{j+\mathfrak{J}}^2 + \gamma\, \mathrm{osc}_{j+\mathfrak{J}}^2 \leqslant \Lambda_0 e_j^2 - \beta E_j^2 + (1+\delta)\,\gamma\, \mathrm{osc}_j^2 - (1+\delta)\,\lambda\,\gamma\, \mathrm{osc}_j^2(\mathscr{R}_j^1).$$

Since $\mathscr{M}_j \subset \mathscr{R}_j^{\mathfrak{n}}$, the discrete lower bound $E_j^2 \geqslant C_3 \eta_j^2(\mathscr{M}_j) - \mathrm{osc}_j^2(\mathscr{R}_j^1)$ in (4.4) is valid, thereby giving

$$e_{j+\mathfrak{J}}^2 + \gamma\, \mathrm{osc}_{j+\mathfrak{J}}^2 \leqslant \Lambda_0 e_j^2 + (1+\delta)\,\gamma\, \mathrm{osc}_j^2 - \beta C_3 \eta_j^2(\mathscr{M}_j) - [(1+\delta)\,\lambda\,\gamma\, - \beta]\, \mathrm{osc}_j^2(\mathscr{R}_j^1).$$

We can further replace $\mathrm{osc}_j(\mathscr{R}_j^1)$ by $\mathrm{osc}_j(\mathscr{M}_j)$, which is smaller, and equate its coefficient with that of $\eta_j^2(\mathscr{M}_j)$ to derive an expression for β

$$\beta = \frac{1}{1+C_3}(1+\delta)\lambda\gamma.$$

We next use the definition of γ to show that the ensuing β is admissible, namely,

$$0 < \beta := \frac{\lambda\,\delta}{\lambda\,\delta + (1+C_3)\Lambda_1\, \mathrm{osc}_0^2(\mathbf{D})} < 1, \tag{5.2}$$

whence

$$\gamma(1+\delta) = \frac{(1+C_3)\delta}{\lambda\,\delta + (1+C_3)\Lambda_1\, \mathrm{osc}_0^2(\mathbf{D})}. \tag{5.3}$$

Replacing β into the above expression for $e_{j+\mathfrak{J}}^2 + \gamma\, \mathrm{osc}_{j+\mathfrak{J}}^2$, and recalling that $\zeta_j^2(\mathscr{M}_j) = \eta_j^2(\mathscr{M}_j) + \mathrm{osc}_j^2(\mathscr{M}_j)$, we obtain

$$e_{j+\mathfrak{J}}^2 + \gamma\, \mathrm{osc}_{j+\mathfrak{J}}^2 \leqslant \Lambda_0 e_j^2 + (1+\delta)\,\gamma\, \mathrm{osc}_j^2 - \frac{C_3}{1+C_3}\gamma\lambda(1+\delta)\zeta_j^2(\mathscr{M}_j).$$

Invoking Dörfler marking (4.10), namely $\zeta_j(\mathscr{M}_j) \geqslant \theta\zeta_j$, we deduce

$$e_{j+\mathfrak{J}}^2 + \gamma\, \mathrm{osc}_{j+\mathfrak{J}}^2 \leqslant \Lambda_0 e_j^2 + (1+\delta)\,\gamma\, \mathrm{osc}_j^2 - \frac{C_3}{1+C_3}\gamma\lambda(1+\delta)\theta^2\zeta_j^2.$$

Since $\zeta_j \geqslant \mathrm{osc}_j$ by construction, we infer that

$$e_{j+\mathfrak{J}}^2 + \gamma\, \mathrm{osc}_{j+\mathfrak{J}}^2 \leqslant \Lambda_0 e_j^2 + (1+\delta)\,\gamma\, \mathrm{osc}_j^2 - \frac{C_3}{2(1+C_3)}\gamma\lambda(1+\delta)\theta^2\big(\zeta_j^2 + \mathrm{osc}_j^2\big),$$

and thus apply the upper bound (4.1) to obtain

$$e_{j+\mathfrak{J}}^2 + \gamma\, \mathrm{osc}_{j+\mathfrak{J}}^2 \leqslant \alpha_1^2(\delta)\, e_j^2 + \gamma\alpha_2^2(\delta)\, \mathrm{osc}_j^2$$

with

$$\alpha_1^2(\delta) := \Lambda_0 - \frac{C_3\lambda\theta^2}{2C_1(1+C_3)}\gamma(1+\delta), \qquad \alpha_2^2(\delta) := (1+\delta)\left(1-\frac{C_3}{2(1+C_3)}\lambda\theta^2\right).$$

It remains to prove that the parameter δ can be chosen so that

$$\alpha^2 := \max\{\alpha_1^2, \alpha_2^2\} < 1.$$

Eliminating γ from (5.3) leads to the following conditions on δ for $\alpha < 1$:

$$\alpha_1(\delta) < 1 \Rightarrow \delta > \delta_- := \frac{(1+C_3)\Lambda_1 \operatorname{osc}_0^2(\mathbf{D})}{\lambda\left(\frac{C_3\theta^2}{2C_1(\Lambda_0-1)}-1\right)}, \quad \alpha_2(\delta) < 1 \Rightarrow \delta < \delta_+ := \frac{C_3\lambda\,\theta^2}{2(1+C_3)-C_3\lambda\theta^2}.$$

Using the restriction (4.12) on Λ_0, the condition on δ_- can be rewritten as

$$\delta_- < \frac{C_3\,\lambda\,\theta^2}{2(1+C_3)},$$

thereby showing that it is possible to choose a compatible δ so that

$$\frac{C_3\,\lambda\,\theta^2}{2(1+C_3)} < \delta < \frac{C_3\lambda\,\theta^2}{2(1+C_3)-C_3\lambda\theta^2}.$$

This completes the proof. □

6. Quasi-Optimal Cardinality of AFEM

In this section we prove quasi-optimal cardinality of AFEM. We proceed as in Cascón *et al.* (2008), who improve and extend the results of Binev *et al.* (2004) and Stevenson (2007) for the Poisson equation. We only list the results and main differences, and refer to Cascón *et al.* (2008) for complete proofs.

6.1 *Approximation Class*

Since all decisions of AFEM are based on the estimator $\zeta(U,\mathscr{K}_\mathscr{T})$, a decay rate for the AFEM can only be characterized by its properties. Invoking the upper and lower bounds, (4.1) and (4.2), we realize that this quantity is equivalent to the *total error*

$$|||u-U|||_\Omega^2 + \operatorname{osc}_\mathscr{T}^2(U,\mathscr{K}_\mathscr{T}) \;\approx\; \eta_\mathscr{T}^2(U,\mathscr{K}_\mathscr{T}) + \operatorname{osc}_\mathscr{T}^2(U,\mathscr{K}_\mathscr{T}) = \zeta_\mathscr{T}^2(U,\mathscr{K}_\mathscr{T})$$

which is strictly reduced by the AFEM. Therefore, as in Cascón *et al.* (2008), the definition of a suitable approximation class must be based on the total error. We start this section recalling that the total error satisfies a Cea's Lemma. Its proof is similar to (Cascón *et al.*, 2008, Lemma 5.2).

LEMMA 6.1 (Quasi-Optimality of the Total Error) Let u be the solution of (2.1) and for $\mathscr{T}\in\mathbb{T}$ let $U\in\mathbb{V}(\mathscr{T})$ be the Ritz-Galerkin approximation of (2.2).

Then, the total error satisfies

$$|||u-U|||_\Omega^2 + \operatorname{osc}_\mathscr{T}^2(U,\mathscr{K}_\mathscr{T}) \leqslant \Lambda_2 \inf_{V\in\mathbb{V}(\mathscr{T})}\Big(|||u-V|||_\Omega^2 + \operatorname{osc}_\mathscr{T}^2(V,\mathscr{K}_\mathscr{T})\Big),$$

where $\Lambda_2 = \max\left\{2, \Lambda_0(1+2\Lambda_1 \operatorname{osc}_{\mathscr{T}_0}^2(\mathbf{D},\mathscr{T}_0))\right\}$, with Λ_0,Λ_1 defined in (2.3) and (4.9), depends on data $\mathbf{D}$, shape-regularity of $\mathscr{T}_0$, and polynomial degree n.

We now proceed as in Cascón *et al.* (2008) to define the corresponding approximation class, that we denote $\widehat{\mathbb{A}}_s$. Let $\mathbb{T}_N \subset \mathbb{T}$ be the set of all possible conforming triangulations generated by REFINE from $\mathscr{T}_0$ with at most N elements more than $\mathscr{T}_0$:

$$\mathbb{T}_N := \{\mathscr{T} \in \mathbb{T} \mid \#\mathscr{T} - \#\mathscr{T}_0 \leqslant N\}.$$

The quality of the best approximation to the total error in the set $\mathbb{T}_N$ is given by

$$\widehat{\sigma}(N; v, f, \mathbf{D}) := \inf_{\mathscr{T} \in \mathbb{T}_N} \inf_{V \in \mathbb{V}(\mathscr{T})} \Big(|||v - V|||_\Omega^2 + \operatorname{osc}^2_{\mathscr{T}}(V, \mathscr{K}_{\mathscr{T}}) \Big)^{1/2}.$$

where $(f, \mathbf{D})$ are hidden in $\operatorname{osc}_{\mathscr{T}}(V, \mathscr{T})$; we refer to §3 for examples. For $s > 0$ we define the nonlinear approximation class $\widehat{\mathbb{A}}_s$ to be

$$\widehat{\mathbb{A}}_s := \Big\{ (v, f, \mathbf{D}) \mid |v, f, \mathbf{D}|_s := \sup_{N>0} \big(N^s\, \widehat{\sigma}(N; v, f, \mathbf{D}) \big) < \infty \Big\}.$$

The range of decay rates s is dictated by the polynomial degree n and the dimension d since, except in degenerate cases, $s \leqslant n/d$; this upper bound corresponds to full regularity and quasi-uniform refinement. Thanks to Lemma 6.1, the solution u of (1.1) with data $(f, \mathbf{D})$ satisfies

$$\widehat{\sigma}(N; v, f, \mathbf{D}) \approx \inf_{\mathscr{T} \in \mathbb{T}_N} \big\{ \zeta_{\mathscr{T}}(U, \mathscr{K}_{\mathscr{T}}) : U = \mathsf{SOLVE}(\mathscr{T}) \big\}. \tag{6.1}$$

6.2 *Equivalence of Approximation Classes*

The definition of $\widehat{\mathbb{A}}_s$ seems to depend on the notion of oscillation (3.7), or similar for the other estimators, which is different from that in Cascón *et al.* (2008), namely (3.6). We may thus wonder about the relation between the classes $\widehat{\mathbb{A}}_s$ and $\mathbb{A}_s$, the latter being defined in Cascón *et al.* (2008). We now prove that they are identical.

We recall that in residual estimation the oscillation can be defined using an L^2-projection onto piecewise polynomials of any degree (Verfürth, 1996). However, this margin of freedom is not possible for other families of estimators, such as those is section §3, whose analysis requires the discrete lower bound (4.4) and so the interior node property (3.3). Such property can be enforced provided we project onto piecewise polynomials of degree $\leqslant n-1$ in the definition (4.5) of oscillation.

LEMMA 6.2 ($\widehat{\mathbb{A}}_s = \mathbb{A}_s$) The approximation classes $\widehat{\mathbb{A}}_s$ associated with the estimators of §3 are identical to the class $\mathbb{A}_s$ of Cascón *et al.* (2008).

Proof. We relabel the local residual indicator and oscillation of (3.6) as follows:

$$\eta^R_{\mathscr{T}}(U, T)^2 = h_T^2 \, \|R(U)\|^2_{L^2(T)} + h_T \, \|J(U)\|^2_{L^2(\partial T \cap \Omega)},$$
$$\operatorname{osc}^R_{\mathscr{T}}(U, T; 2n-2)^2 = h_T^2 \, \big\|E^2_{2n-2} R(U)\big\|^2_{L^2(T)} + h_T \, \big\|E^2_{2n-1} J(U)\big\|^2_{L^2(\partial T \cap \Omega)},$$

and proceed in three steps. We first show that the class $\mathbb{A}_s$ is independent of the polynomial degree built in the definition of oscillation, and next deduce $\widehat{\mathbb{A}}_s = \mathbb{A}_s$.

$\boxed{1}$ The proof of lower bound (4.2) for residual estimators is local and requires projection of the residuals onto piecewise polynomials of any degree m (Verfürth, 1996). Local L^2-stability is used and the ensuing constant depends on m. However we always have the equivalence for any $m \geqslant -1$

$$|||u - U|||_\Omega + \operatorname{osc}^R_{\mathscr{T}}(U, \mathscr{T}; 2n-2) \approx \eta^R_{\mathscr{T}}(U, \mathscr{T}) \approx |||u - U|||_\Omega + \operatorname{osc}^R_{\mathscr{T}}(U, \mathscr{T}; m),$$

where $\Pi_{-1}^2 := 0$ and $\eta_{\mathscr{T}}^R(U,\mathscr{T}) = \mathrm{osc}_{\mathscr{T}}^R(U,\mathscr{T};-1)$. Hence $\mathbb{A}_s$ is characterized by Cascón *et al.* (2008)

$$\sigma(N;v,f,\mathbf{D}) = \inf_{\mathscr{T}\in\mathbb{T}_N} \left\{\eta_{\mathscr{T}}^R(U,\mathscr{T}) : U = \mathsf{SOLVE}(\mathscr{T})\right\}$$

and we infer that $\mathbb{A}_s$ is independent of the polynomial degree m used in the definition of oscillation.
2 To prove that $\widehat{\mathbb{A}}_s = \mathbb{A}_s$, we show that they control equivalent quantities, i. e.

$$\eta_{\mathscr{T}}^R(U,\mathscr{T}) \approx |||u-U|||_\Omega + \mathrm{osc}_{\mathscr{T}}^R(U,\mathscr{T}) \approx |||u-U|||_\Omega + \mathrm{osc}_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}),$$

where $\mathrm{osc}_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}})$ is the oscillation term of any of the examples in §3 (or its extension for $n>1$). We first observe that any oscillation of §3 satisfies

$$\mathrm{osc}_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}) \preccurlyeq \eta_{\mathscr{T}}^R(U,\mathscr{T}),$$

because the mere concept of oscillation is pretty much independent of the specific form of estimator at hand. This implies $\mathbb{A}_s \subset \widehat{\mathbb{A}}_s$.
3 We next consider the Morin-Nochetto-Siebert star indicator (3.19) and corresponding star oscillation (3.20). The other examples of §3 are somewhat simpler and can be handled similarly. Let $T\in\mathscr{T}$ be an element contained in star ω_i, and let $\xi := \Pi_{0,i}^2 R(U)\lambda_i^{1/2} b_T$, with b_T a polynomial bubble function associated to T. Therefore, we have

$$\|\Pi_{0,i}^2 R(U)\lambda_i^{1/2}\|_{L^2(T)}^2 \approx \langle \Pi_{0,i}^2 R(U)\lambda_i^{1/2},\xi\rangle \approx \|\xi\|_{L^2(T)^2},$$

and invoking the error residual equation (3.5), the continuity of the bilinear form $\mathscr{B}$ and an inverse inequality, we obtain

$$\begin{aligned}\|\Pi_{0,i}^2 R(U)\lambda_i^{1/2}\|_{L^2(T)}^2 &\approx -\langle E_{0,i}^2 R(U),\lambda_i^{1/2}\xi\rangle + \mathscr{B}[U,\lambda_i^{1/2}\xi]\\ &\preccurlyeq \|E_{0,i}^2 R(U)\lambda_i^{1/2}\|_{L^2(T)}\|\xi\|_{L^2(T)} + |||u-U|||_T\, h_T^{-1}\|\xi\|_{L^2(T)}.\end{aligned}$$

with $E_{0,i}^2 = I - \Pi_{0,i}^2$. Simplifying $\|\xi\|_{L^2(T)}$ we get

$$h_T\|\Pi_{0,i}^2 R(U)\lambda_i^{1/2}\|_{L^2(T)} \preccurlyeq h_T\|E_{0,i}^2 R(U)\lambda_i^{1/2}\|_{L^2(T)} + |||u-U|||_T\,. \tag{6.2}$$

On the other hand, since $\sum_{\mathbf{x}_i\in T}\lambda_i = 1$ and $0\leqslant \lambda_i \leqslant \lambda_i^{1/2}$ on T, we deduce

$$\|R(U)\|_{L^2(T)} \leqslant \sum_{\mathbf{x}_i\in T}\|R(U)\lambda_i\|_{L^2(T)} \leqslant \sum_{\mathbf{x}_i\in T}\|R(U)\lambda_i^{1/2}\|_{L^2(T)} \tag{6.3}$$

Combining (6.2) and (6.3), and using the fact that $h_T\approx h_i$ for all $\mathbf{x}_i\in T$, we obtain

$$h_T^2\|R(U)\|_{L^2(T)}^2 \preccurlyeq \sum_{\mathbf{x}_i\in T} h_i^2\|E_{0,i}^2 R(U)\lambda_i^{1/2}\|_{L^2(\omega_i)}^2 + |||u-U|||_T^2\,. \tag{6.4}$$

Given a side $\sigma\in\mathscr{S}$ and corresponding patch ω_σ, we employ a similar argument for $\Pi_{0,i}^2 J(U)$ to deduce

$$h_\sigma\|J(U)\lambda_i^{1/2}\|_{L^2(\sigma)}^2 \preccurlyeq h_\sigma\|E_{0,i}^2 J(U)\lambda_i^{1/2}\|_{L^2(\sigma)}^2 + \sum_{\mathbf{x}_j\in\omega_\sigma} h_j^2\|E_{0,i}^2 R(U)\lambda_i^{1/2}\|_{L^2(\omega_j)}^2 + |||u-U|||_{\omega_\sigma}^2\,.$$

Summing over all $T \in \mathscr{T}$, and using the property that each element T overlaps with $2(d+1)$ stars ω_j, we finally arrive at

$$\eta^R_{\mathscr{T}}(U,\mathscr{T}) \preccurlyeq |||u-U|||^2_\Omega + \mathrm{osc}_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}})^2.$$

This implies that $\widehat{\mathbb{A}}_s \subset \mathbb{A}_s$, and completes the proof. □

An important pending issue is the characterization of $\mathbb{A}_s$, which is beyond the scope of this paper. We refer to Cascón *et al.* (2008) and (Nochetto *et al.*, 2009, Section 9) for a discussion of $\mathbb{A}_s$ and to Binev *et al.* (2004, 2002); Stevenson (2007) for a connection with Besov classes for the Laplacian. We stress that $\mathbb{A}_s$ is not a typical linear approximation class for functions because of the nonlinear interaction between data $\mathbf{D} = (\mathbf{A}, \mathbf{b}, c)$ and U through $\mathrm{osc}_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}})$.

6.3 *Cardinality of $\mathscr{M}_j$*

We now assume that $(u, f, \mathbf{D}) \in \mathbb{A}_s$ for some $0 < s \leqslant n/d$, and prove that the approximation U_j generated by AFEM converges to u with the same rate $(\#\mathscr{T}_j - \#\mathscr{T}_0)^{-s}$ as the best approximation described by $\mathbb{A}_s$ up to a multiplicative constant. The proof follows Cascón *et al.* (2008), and is inspired in Stevenson's insight for the Laplacian with vanishing oscillation (Stevenson, 2007):

> *Any conforming refinement $\mathscr{T}_*$ of $\mathscr{T}$ which reduces the total error by a suitable percentage verifies a Dörfler marking (Lemma 6.3). This allows comparison with the best mesh because our Dörfler marking is minimal (Lemma 6.4).*

This clever observation is not enough though to examine (1.1) with variable coefficients, and so with non-vanishing oscillation, which is our main objective below. We reiterate that Lemma 6.3 below establishes a fundamental relation between total error reduction and Dörfler marking.

LEMMA 6.3 (Optimal Marking) Let AFEM satisfy Assumptions 4.1 (b,c), 4.2 (a,b), and 4.3 (d), and set $\mu := \frac{1}{2}(1 - \frac{\theta^2}{\theta_*^2}) > 0$. Let $\mathscr{T} \in \mathbb{T}$ and $U \in \mathbb{V}(\mathscr{T})$ be the discrete solution of (2.2), and let $\mathscr{T}_* \in \mathbb{T}$ be any refinement of $\mathscr{T}$, i. e. $\mathscr{T} \leqslant \mathscr{T}_*$, such that the discrete solution $U_* \in \mathbb{V}(\mathscr{T}_*)$ satisfies

$$|||u-U_*|||^2_\Omega + \mathrm{osc}^2_{\mathscr{T}_*}(U_*,\mathscr{K}_{\mathscr{T}_*}) \leqslant \mu\Big(|||u-U|||^2_\Omega + \mathrm{osc}^2_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}})\Big). \tag{6.5}$$

Then the refined set $\mathscr{R}^1 := \mathscr{R}^1_{\mathscr{T}\to\mathscr{T}_*} \subset \mathscr{K}_{\mathscr{T}}$ of order 1 satisfies the Dörfler property

$$\zeta_{\mathscr{T}}(U,\mathscr{R}^1) \geqslant \theta\, \zeta_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}). \tag{6.6}$$

Proof. The proof hinges on the following key ingredients: the following consequence of the global lower bound (4.2)

$$\frac{C_2}{1+C_2}\zeta^2_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}) \leqslant |||u-U|||^2_\Omega + \mathrm{osc}^2_{\mathscr{T}}(U,\mathscr{K}_{\mathscr{T}}),$$

the localized upper bound (4.3), the dominance $\mathrm{osc}_{\mathscr{T}}(U,K) \leqslant \zeta_{\mathscr{T}}(U,K)$, the perturbation of oscillation (4.8), and the following consequence of Young's inequality

$$|||u-U|||^2_\Omega \leqslant 2\,|||u-U_*|||^2_\Omega + 2\,|||U_*-U|||^2_\Omega\,.$$

It is otherwise identical to (Cascón *et al.*, 2008, Lemma 5.9) and so omitted. □

To estimate the cardinality of $\mathscr{M}_j$ we need to invoke optimal meshes, which are in principle unrelated to $\mathscr{T}_j$. The key to unravel the relation between AFEM and the class $\mathbb{A}_s$ is the fact that MARK selects a *minimal* set $\mathscr{M}_j$ (Assumption 4.3 (e)).

LEMMA 6.4 (Cardinality of $\mathscr{M}_j$) Let AFEM satisfy Assumptions 4.1 (b,c), 4.2 (a,b), and 4.3 (d,e). Let u be the solution of (1.1), and let $\{\mathscr{T}_j, \mathbb{V}_j, U_j\}_{j\geqslant 0}$ be the sequence of meshes, finite element spaces, and discrete solutions produced by AFEM. If $(u, f, \mathbf{D}) \in \mathbb{A}_s$, then the following estimate is valid

$$\#\mathscr{M}_j \preccurlyeq \big(1 - \frac{\theta^2}{\theta_*^2}\big)^{-\frac{1}{2s}} |u, f, \mathbf{D}|_s^{\frac{1}{s}} \Lambda_2^{\frac{1}{2s}} \Big(|||u - U_j|||_\Omega^2 + \mathrm{osc}_{\mathscr{T}_j}^2(U_j, \mathscr{K}_j) \Big)^{-\frac{1}{2s}}, \tag{6.7}$$

where Λ_2 is the constant in Lemma 6.1.

Proof. We proceed along the lines of (Cascón *et al.*, 2008, Lemma 5.10). Let $\varepsilon^2 := \mu \Lambda_2^{-1} \big(|||u - U_j|||_\Omega^2 + \mathrm{osc}_{\mathscr{T}_j}^2(U_j, \mathscr{T}_j) \big)$ and let $\mathscr{T}_\varepsilon \in \mathbb{T}$ and $V_\varepsilon \in \mathbb{V}(\mathscr{T}_\varepsilon)$ satisfy

$$\#\mathscr{T}_\varepsilon - \#\mathscr{T}_0 \preccurlyeq |u, f, \mathbf{D}|_s^{1/s} \varepsilon^{-1/s}, \quad |||u - V_\varepsilon|||_\Omega^2 + \mathrm{osc}_{\mathscr{T}_\varepsilon}^2(V_\varepsilon, \mathscr{T}_\varepsilon) \leqslant \varepsilon^2.$$

Let $\mathscr{T}_* = \mathscr{T}_j \oplus \mathscr{T}_\varepsilon$ be the overlay of $\mathscr{T}_j$ and $\mathscr{T}_\varepsilon$. Apply Lemma 6.1 to $\mathscr{T}_* \geqslant \mathscr{T}_\varepsilon$ to get

$$|||u - U_*|||_\Omega^2 + \mathrm{osc}_{\mathscr{T}_*}^2(U_*, \mathscr{T}_*) \leqslant \mu \Big(|||u - U_j|||_\Omega^2 + \mathrm{osc}_{\mathscr{T}_j}^2(U_j, \mathscr{T}_j) \Big),$$

whence $\mathscr{R}_j^1 = \mathscr{R}_{\mathscr{T}_j \to \mathscr{T}_*}^1$ satisfies Dörfler property because of Lemma 6.3. Invoking Assumption 4.3 (e), we finally deduce

$$\#\mathscr{M}_j \leqslant \#\mathscr{R}_j^1 \leqslant \#\mathscr{T}_* - \#\mathscr{T}_k \leqslant \#\mathscr{T}_\varepsilon - \#\mathscr{T}_0 \leqslant \preccurlyeq |u, f, \mathbf{D}|_s^{\frac{1}{s}} \varepsilon^{-\frac{1}{s}},$$

which is the asserted estimate. □

6.4 *Quasi-Optimal Decay Rates*

REFINE usually refines more elements than those in $\mathscr{M}$ to enforce conformity of $\mathscr{T}_*$ (completion). The cardinality of those additional elements is not controlled by that of marked ones in one single step (Nochetto *et al.*, 2009, Section 4.5). Binev, Dahmen, and DeVore for $d = 2$ (Binev *et al.*, 2004, Theorem 2.4) and Stevenson for $d > 2$ (Stevenson, 2008, Theorem 6.1) showed that the *cumulative* number of elements added by conformity does not inflate the total number of marked elements provided the initial mesh $\mathscr{T}_0$ is suitably labeled; see the survey (Nochetto *et al.*, 2009, Section 4) for details.

LEMMA 6.5 (Complexity of REFINE) Let Assumption 4.3 (c) be valid. Let $\{\mathscr{T}_j\}_{j\geqslant 0}$ be any sequence of refinements of $\mathscr{T}_0$ where $\mathscr{T}_{j+1}$ is generated from $\mathscr{T}_j$ by REFINE. Then, there exists a constant C_0 solely depending on $\mathscr{T}_0$, $\mathfrak{b}$ and $\mathfrak{n}$ such that

$$\#\mathscr{T}_j - \#\mathscr{T}_0 \leqslant C_0 \sum_{i=0}^{j-1} \#\mathscr{M}_i \qquad \forall j \geqslant 1.$$

Even though the original results are written in terms of one bisection per simplex Binev *et al.* (2004); Stevenson (2008), they easily extend to account for $\mathfrak{b}$ bisections per step and a refinement depth $\mathfrak{n}$ after $\mathfrak{J}(\mathfrak{n}, \mathfrak{b})$ steps. Moreover, the cardinality is usually expressed in terms of number of simplices, but it is as well valid for K-elements.

The following decay rate is a consequence of Lemmas 6.4 and 6.5, as well as Theorem 5.1, which establishes a contraction property of AFEM for the total error after $\mathfrak{J}$ iterates. Compared with Stevenson (2007); Cascón *et al.* (2008) we now have to account for $\mathfrak{J}$. We give a complete proof below for the sake of completeness.

THEOREM 6.1 (Quasi-Optimal Decay Rates) Let AFEM satisfy Assumptions 4.1, 4.2, and 4.3. Let u be the solution of (1.1), and let $\{\mathscr{T}_j, \mathbb{V}_j, U_j\}_{j\geqslant 0}$ be the sequence of meshes, finite element spaces, and discrete solutions produced by AFEM.

Let $(u, f, \mathbf{D}) \in \mathbb{A}_s$ and $\Theta(\alpha,\theta,s) := (1-\alpha^{1/s})^{-s}\big(1-\frac{\theta^2}{\theta_*^2}\big)^{-1/2}$ describe the asymptotics of AFEM as $\alpha \to 1$, $\theta \to \theta_*$ or $s \to 0$. Then there exists a constant C_5, depending on data, $\mathfrak{J}$, $\mathfrak{b}$ and $\mathscr{T}_0$, but independent of s, such that

$$|\!|\!| u - U_j |\!|\!|_\Omega + \operatorname{osc}_j(U_j, \mathscr{K}_j) \leqslant C_5\, \Theta(\theta,\alpha,s) |u, f, \mathbf{D}|_s \big(\#\mathscr{T}_j - \#\mathscr{T}_0\big)^{-s}.$$

Proof. Combining Lemmas 6.4 and 6.5, we deduce

$$\#\mathscr{T}_j - \#\mathscr{T}_0 \preccurlyeq \sum_{i=0}^{j-1} \#\mathscr{M}_i \preccurlyeq M \sum_{i=0}^{j-1} \Big\{ |\!|\!| u - U_i |\!|\!|_\Omega^2 + \operatorname{osc}_i^2(U_i, \mathscr{K}_i) \Big\}^{-\frac{1}{2s}}, \tag{6.8}$$

with $M := \big(1-\frac{\theta^2}{\theta_*^2}\big)^{-\frac{1}{2s}} |u, f, \mathbf{D}|_s^{\frac{1}{s}} \Lambda_2^{\frac{1}{2s}}$. We use Lemma 6.1 to obtain for $i \leqslant j$

$$|\!|\!| u - U_j |\!|\!|_\Omega^2 + \operatorname{osc}_j^2(U_j, \mathscr{K}_j) \leqslant \Lambda_2 \Big\{ |\!|\!| u - U_i |\!|\!|_\Omega^2 + \operatorname{osc}_i^2(U_i, \mathscr{K}_i) \Big\}.$$

We exploit this to rewrite (6.8) in groups of $\mathfrak{J}$ consecutive terms as follows

$$\#\mathscr{T}_j - \#\mathscr{T}_0 \preccurlyeq M \Lambda_2^{\frac{1}{2s}} \mathfrak{J} \sum_{i=0}^{[j/\mathfrak{J}]} \Big\{ |\!|\!| u - U_{j-i\mathfrak{J}} |\!|\!|_\Omega^2 + \operatorname{osc}_{j-i\mathfrak{J}}^2(U_{j-i\mathfrak{J}}, \mathscr{K}_{j-i\mathfrak{J}}) \Big\}^{-\frac{1}{2s}},$$

where $[\cdot]$ denotes the integer part function. On the other hand

$$|\!|\!| u - U_i |\!|\!|_\Omega^2 + \gamma \operatorname{osc}_i^2(U_i, \mathscr{K}_i) \leqslant \max\{1,\gamma\} \Big\{ |\!|\!| u - U_i |\!|\!|_\Omega^2 + \operatorname{osc}_i^2(U_i, \mathscr{K}_i) \Big\}$$

and the contraction property of Theorem 5.1, for the sum of energy error and scaled oscillation, implies for $0 \leqslant i \leqslant [j/\mathfrak{J}]$

$$|\!|\!| u - U_j |\!|\!|_\Omega^2 + \gamma \operatorname{osc}_j^2(U_j, \mathscr{K}_j) \leqslant \alpha^{2i} \Big\{ |\!|\!| u - U_{j-i\mathfrak{J}} |\!|\!|_\Omega^2 + \gamma \operatorname{osc}_{j-i\mathfrak{J}}^2(U_{j-i\mathfrak{J}}, \mathscr{K}_{j-i\mathfrak{J}}) \Big\}.$$

Combining these estimates we infer that

$$\#\mathscr{T}_j - \#\mathscr{T}_0 \preccurlyeq M \Lambda_2^{\frac{1}{2s}} \mathfrak{J} \max\{1,\gamma\}^{\frac{1}{2s}} \Big\{ |\!|\!| u - U_j |\!|\!|_\Omega^2 + \gamma \operatorname{osc}_j^2(U_j, \mathscr{K}_j) \Big\}^{-\frac{1}{2s}} \sum_{i=0}^{[j/\mathfrak{J}]} \alpha^{\frac{i}{s}}.$$

Since $\alpha < 1$, the geometric series converges and completes the proof. □

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