A Posteriori Error Estimates for Variable Time-Step Discretizations of Nonlinear Evolution Equations

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Dedicated to Enrico Magenes on the occasion of his 75th birthday

Abstract

We study the backward Euler method with variable time-steps for abstract evolution equations in Hilbert spaces. Exploiting convexity of the underlying potential or the angle-bounded condition, thereby assuming no further regularity, we derive novel *a posteriori* estimates of the discretization error in terms of computable quantities related to the amount of energy dissipation or monotonicity residual. These estimators solely depend on the discrete solution and data and impose no constraints between consecutive time-steps. We also prove that they converge to zero with an optimal rate with respect to the regularity of the solution. We apply the abstract results to a number of concrete strongly nonlinear problems of parabolic type with degenerate or singular character.

1 Introduction

For the sole purpose of motivating the new ideas in a simple setting, we start with ordinary differential equations in $\mathbb{R}^d$. We consider the Cauchy problem

$$\begin{cases} u'(t) + \mathfrak{F}(u(t)) = 0 \quad \forall\, t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $\mathfrak{F} : \mathbb{R}^d \to \mathbb{R}^d$ is a given (nonlinear) vector field and $u_0 \in \mathbb{R}^d$.

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We are interested in studying the error between the solution u of (1.1) and the approximate solution U defined by a variable step implicit Euler scheme. More precisely, we choose a partition of the time interval $[0,T]$

$$\mathcal{P} := \{0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T\}, \tag{1.2}$$

with variable step $\tau_n := t_n - t_{n-1}$, and we consider the sequence $\{U_n\}_{n=0}^N$ whose first term is given and the other ones are recursively defined for $1 \leq n \leq N$ by

$$\frac{U_n - U_{n-1}}{\tau_n} + \mathfrak{F}(U_n) = 0. \tag{1.3}$$

Hereafter, U is the piecewise linear interpolant of the values $\{U_n\}_{n=0}^N$ on the grid $\mathcal{P}$ and $\tau := \max_{1\leq n\leq N} \tau_n$ denotes the maximum of the time-step sizes.

When $\mathfrak{F}$ is continuous and satisfies the so-called *one-sided Lipschitz condition* [23, (1.18)], [36, (6.2)] with respect to (w.r.t.) the euclidean scalar product $\langle \cdot, \cdot \rangle$, i.e. there exists $\lambda \in \mathbb{R}$ such that

$$\langle \mathfrak{F}(v) - \mathfrak{F}(w), v - w \rangle \geq -\lambda |v - w|^2 \quad \forall\, v, w \in \mathbb{R}^d, \tag{1.4}$$

then both (1.1) and (1.3), at least for sufficiently small τ s.t. $\lambda\tau < 1$, are uniquely solvable. Moreover, an *a priori* linear order of convergence is well known for $E_n := |u(t_n) - U_n|$ provided $u'' \in L^1(0,T)$ [37], [22, § 7.3,a]; for $\lambda > 0$,

$$E_n \leq \mathrm{e}^{2\lambda t_n} \big(E_0 + 2\tau \|u''\|_{L^1(0,T)} \big) \quad \text{if } 2\lambda\tau \leq 1. \tag{1.5}$$

This estimate is derived upon evaluating local truncation errors and is optimal w.r.t. the time-stepping method. It is usually coupled with a time-step selection based either on the evaluation of the local truncation error [32] or on duality techniques [28], [38], which require at least Lipschitz continuity of $\mathfrak{F}$ and so $u'' \in L^\infty(0,T)$. The *monotone* case $\lambda = 0$

$$\langle \mathfrak{F}(v) - \mathfrak{F}(w), v - w \rangle \geq 0 \quad \forall\, v, w \in \mathbb{R}^d, \tag{1.6}$$

is perhaps the most important one in that a vector field $\mathfrak{F}$ satisfying (1.4) with $\lambda > 0$ could be reduced to a monotone operator by adding a linear perturbation. For $\lambda \leq 0$, the error becomes [22, § 7.3,a]

$$E_n \leq \mathrm{e}^{\lambda t_n/2} E_0 + \tau \|u''\|_{L^1(0,T)} \quad \text{if } \lambda\tau \geq -1. \tag{1.7}$$

So, the initial error E_0 is no longer exponentially amplified if $\lambda = 0$ and even exhibits an exponential decay provided $\mathfrak{F}$ is *strongly* monotone, i.e. $\lambda < 0$.

However, (1.5) and (1.7) are *nonoptimal* w.r.t. the regularity of the exact solution u, since they need an *a priori* estimate of $\|u''\|_{L^1(0,T)}$, whereas a linear rate of convergence could be expected for u' only bounded. This fact is undesirable not only when (1.1) is a dissipative ODE associated to an *irregular* vector field, for which u' may exhibit jump discontinuities, but also when (1.1) comes from a space discretization of a parabolic PDE. In this case the Lipschitz constant of the $\mathfrak{F}$'s grows to $+\infty$ as the mesh parameter tends to 0.

Our prime interest here is to obtain alternative error estimates, which avoid this drawback; more precisely, we want to address five fundamental issues:

(a) *A posteriori* estimates for $|u - U|$: computable quantities which depend solely on time-steps, discrete solution, and data.

(b) Optimal rates of convergence: *a posteriori* bounds converging to zero as $\tau \downarrow 0$ with an optimal rate w.r.t. the regularity of the solution u.

(c) Minimal regularity: $\mathfrak{F}$ is a (Lipschitz perturbation of a) monotone vector field which admits a lower semicontinuous (l.s.c.) *convex potential* ϕ or satisfies more general *angle-boundedness* conditions, but no other regularity assumptions (such as Lipschitz continuity) are made on $\mathfrak{F}$.

(d) Uniform stability and explicit constants: the stability and error constants are uniform with respect to possible space discretizations (in particular they do not depend on the space dimension d) and they are explicitly determined without need of solving any auxiliary, or dual, problem.

(e) Variable time-steps: no *a priori* constraints between consecutive time-steps, which could just be tailored to the *a posteriori* error estimators alone.

We refer to (a) and (b) collectively as *optimal a posteriori* error estimates.

A Priori Error Estimates in Hilbert Spaces. Most contributions dealing with minimal regularity do not employ the finite dimension of $\mathbb{R}^d$ but simply its Hilbert structure. This generality allows for direct applications to various (nonlinear) partial differential equations of (degenerate or singular) parabolic type as those considered later, and multi-valued $\mathfrak{F}$ defined in a proper subdomain $D(\mathfrak{F})$ of a (functional) Hilbert space $\mathcal{H}$ could also be accommodated within this approach. In many concrete cases $D(\mathfrak{F})$ is related to some extra growth and regularity properties, which are necessary to define $\mathfrak{F}$ on the functions of $\mathcal{H}$, whereas the possibility to deal with multiple values allows a unified treatment of nondifferentiable constraints. In this framework, a seminal discovery was that the continuity of $\mathfrak{F}$ can be replaced by the simple *solvability* of (1.3) for $\lambda\tau < 1$. Combining this assumption with the weak monotonicity (1.4) (suitably extended to Banach spaces), Crandall and Liggett [21] proved a fundamental and deep result, in the case of a uniform partition $\mathcal{P}$ (i.e. $\tau_n \equiv \tau$ for every n): U converges uniformly to u as $\tau \downarrow 0$. This was later extended to the case of a nonuniform $\mathcal{P}$ in [20]; see also [19] for an interesting review of these and other related results and their implications to the well-posedness of (1.1) in a general Banach space. Moreover, when $\mathcal{P}$ is uniform and $u_0 \in D(\mathfrak{F})$, an *a priori* estimate of the error $|u - U|$ of order $O(\sqrt{\tau})$ is derived in [21], and of order $o(\sqrt{\tau})$ in [53]; Rulla showed in [53] that the latter is tight for general monotone operators.

However, improved *a priori* error estimates, typical of linear problems [5], [60], [61], are available provided $\mathfrak{F}$ is the gradient of a potential function ϕ. Assuming (1.6), for simplicity, then the underlying potential ϕ must be *convex* and $\mathfrak{F}$ can be characterized by means of the system of inequalities

$$z = \mathfrak{F}(w) \quad \Leftrightarrow \quad \langle z, v - w\rangle \le \phi(v) - \phi(w) \quad \forall\, v \in \mathbb{R}^d.$$

The differential equations (1.1) and (1.3) are thus equivalent to the following evolution variational inequalities, where the explicit occurrence of $\mathfrak{F}$ disappears,

$$\langle u'(t), u(t) - v\rangle + \phi(u(t)) - \phi(v) \leq 0 \quad \forall\, v \in \mathbb{R}^d, \tag{1.8}$$

$$\tau_n^{-1}\langle U_n - U_{n-1}, U_n - v\rangle + \phi(U_n) - \phi(v) \leq 0 \quad \forall\, v \in \mathbb{R}^d. \tag{1.9}$$

The variational inequalities (1.8) and (1.9) are convenient to study nonlinear problems governed by discontinuous functionals, such as obstacle problems. The lack of regularity of $\mathfrak{F}$, and so of u, is compensated by the convexity of ϕ.

An important first result was obtained by Baiocchi [4], who assumed ϕ to be the sum of a positive l.s.c. quadratic form and the indicator function of a convex set in a Hilbert space $\mathcal{H}$ (obstacle problem). Baiocchi showed that if $u_0 \in D(\mathfrak{F})$ then the error $u - U$ in the natural energy norm is *a priori* of order $O(\tau)$. This results is optimal w.r.t. both the order and regularity of the solution u, in that u' is only bounded and, even in one dimension, may exhibit jump discontinuities; both the truncation error analysis and duality argument make no sense in this context. Other error estimates and their relationships with regularity properties of u are studied in [54].

Rulla [53] has recently extended Baiocchi's result to generic l.s.c. convex functions ϕ upon exploiting fine operator properties of the Yosida approximation of $\mathfrak{F}$. The inequality approach has been examined by Savaré [55], who has derived similar error estimates to those of [53] under minimal regularity on an additional source term.

To recover a linear, and thus optimal, rate of convergence, all the above contributions rely on the mesh $\mathcal{P}$ being uniform. Moreover, the resulting estimates are *not* really *a posteriori* and thus computable, because they involve the *a priori* Lipschitz regularity of the unknown solution u.

A Posteriori Error Estimation via Duality. Quasi-optimal *a posteriori* error estimates for variable time-step discretizations are proved only for linear dissipative ODE's [38], linear parabolic PDE's [29] or mildly nonlinear variants of them [30], under the constraint $\tau_{n-1} \leq a\tau_n$ between consecutive time-steps. Suboptimal *a posteriori* error estimates are available also for degenerate parabolic problems [50], [49]. Their proof relies on duality techniques, which now we briefly recall.

Let $\mathcal{R}$ be the residual, that is the negative of the amount by which U misses to be an exact solution

$$\mathcal{R} := -U' - \mathfrak{F}(U). \tag{1.10}$$

Adding (1.10) to (1.1) we obtain the error equation for $e := u - U$

$$e' + \mathfrak{A}\, e = \mathcal{R}, \tag{1.11}$$

where

$$\mathfrak{A} := \int_0^1 \nabla\mathfrak{F}(su + (1-s)U)\, ds.$$

We multiply (1.11) by $\varphi \in C^1(0,T)$ and integrate by parts over $(0,T)$ to get

$$\langle e(T), \varphi(T)\rangle = \langle e(0), \varphi(0)\rangle + \int_0^T \langle e, \varphi' - \mathfrak{A}^*\varphi\rangle + \int_0^T \langle \mathcal{R}, \varphi\rangle, \tag{1.12}$$

where $\mathfrak{A}^*$ is the transpose of $\mathfrak{A}$. The *a posteriori* error estimate follows by selecting φ in (1.12) as the solution of the backward dual problem

$$\varphi' - \mathfrak{A}^*\varphi = 0 \quad \text{in } (0,T), \quad \varphi(T) = e(T), \tag{1.13}$$

and using *strong* stability properties of φ, such as a bound for φ', for evaluating the initial error and the residual terms. In fact, we notice that, for a linear $\mathfrak{F}$, $\langle \mathcal{R}, \varphi\rangle = \langle \bar{U} - U, \varphi'\rangle$ where $\bar{U}$ is the piecewise constant interpolant of $\{U_n\}_{n=1}^N$. This program may fail to apply to the present low regularity setting because

(i) $\mathfrak{F}$ may not be Lipschitz, and so $\mathfrak{A}$ and $\mathfrak{A}^*$ may be singular or not even be well-defined;

(ii) (1.13) may make no sense or φ' may be unbounded;

(iii) φ is not computable because it depends on the unknown u (this implies no precise control of stability constants in the *a posteriori* error estimators);

(iv) if $\mathfrak{A}$ were replaced by $\nabla\mathfrak{F}(U)$ to determine φ, then an additional term would arise involving further derivatives of the possibly singular $\nabla\mathfrak{F}$.

Our techniques below are not based on duality as those of Johnson *et al.* [29], [38] and Nochetto *et al.* [50], [49], and thus circumvent (i)-(iv) altogether. They seem to capture the essence of dissipation at a more elementary level.

A Posteriori Error Estimation via Discrete Energy Dissipation. In the present paper, we resort to both the inequality and the operator approaches and we establish *a posteriori* error estimates for variable time-step discretizations (1.3) of (1.1), which are *optimal* w.r.t. both order and regularity, and impose no constraints between consecutive time-steps. We prove effective bounds for the local error arising at each step in terms of computable quantities, which are related to the amount of energy dissipation in the case $\mathfrak{F}$ admits a convex potential; these quantities are not residuals. This allows for simple *a posteriori* error estimates and adaptive procedures for the choice of the time-steps.

Besides the variational structure of (1.8) and (1.9), in the potential case our techniques exploit the Lyapunov properties of ϕ, which decreases along solution paths of both (1.1) and (1.3). In fact, it is known that ϕ satisfies the energy identity

$$|u'(t)|^2 + \tfrac{d}{dt}\phi(u(t)) = 0 \quad \text{a.e. } t \in (0,T), \tag{1.14}$$

and the discrete energy inequality

$$-\mathcal{E}_n := \Big|\frac{U_n - U_{n-1}}{\tau_n}\Big|^2 + \frac{\phi(U_n) - \phi(U_{n-1})}{\tau_n} \leq 0 \quad \forall\, 1 \leq n \leq N, \tag{1.15}$$

which follows directly from (1.9) upon choosing $v = U_{n-1}$. The discrete quantity $\mathcal{E}_n \geq 0$ in (1.15) is thus a measure of the deviation of $\{U_n\}_{n=0}^N$ from satisfaction of the conservation form (1.14). We now discuss the rather elementary derivation of the following *a posteriori* upper error bound for ODE's

$$\max_{t\in[0,T]} |u(t) - U(t)|^2 \leq |u_0 - U_0|^2 + \sum_{n=1}^N \tau_n^2 \mathcal{E}_n. \tag{1.16}$$

First of all, starting from (1.9) we write the *continuous* evolution variational inequality satisfied by U for all $t \in (t_{n-1}, t_n)$ and $v \in \mathbb{R}^d$

$$\langle U'(t), U(t) - v\rangle + \phi(U(t)) - \phi(v) \le \mathcal{R}(t), \tag{1.17}$$

where $\mathcal{R}(t)$ is a residual term, independent of the test function v, given by

$$\mathcal{R}(t) := \langle U'(t), U(t) - U_n\rangle + \phi(U(t)) - \phi(U_n).$$

The second step consists in estimating $\mathcal{R}(t)$. Here we use the elementary identity

$$U(t) - U_n = \frac{t - t_n}{\tau_n}(U_n - U_{n-1}) = (t - t_n)U'(t),$$

and the convexity of ϕ to get the bound

$$\phi(U(t)) \le \frac{t_n - t}{\tau_n}\phi(U_{n-1}) + \frac{t - t_{n-1}}{\tau_n}\phi(U_n).$$

Hence, in view of (1.15), $\mathcal{R}(t)$ can be bounded by

$$\mathcal{R}(t) \le (t_n - t)\mathcal{E}_n.$$

Finally, choosing $v = U(t)$ in (1.8), $v = u(t)$ in (1.17), and adding the two inequalities, we see that the terms involving ϕ cancel out and we end up with the error inequality

$$\tfrac{1}{2}\tfrac{d}{dt}|u(t) - U(t)|^2 \le (t_n - t)\mathcal{E}_n \quad \text{a.e. } t \in (t_{n-1}, t_n).$$

Upon integration in time, this upper bound readily implies (1.16).

A Posteriori Error Estimation via Residual Monotone Terms. The discrete counterpart of the dissipation energy inequality

$$\tfrac{d}{dt}|u'(t)|^2 \le 0 \quad \text{a.e. } t \in (0, T),$$

namely

$$\left|\frac{U_n - U_{n-1}}{\tau_n}\right|^2 \le \left|\frac{U_{n-1} - U_{n-2}}{\tau_{n-1}}\right|^2 \quad \forall\, 1 \le n \le N, \tag{1.18}$$

can be derived as follows. Taking $v = U_n$ in (1.9) at the time-step $n-1$, with $U_{-1} := U_0 - \tau_0 \mathfrak{F}(U_0)$, we find

$$\tau_{n-1}^{-1}\langle U_{n-1} - U_{n-2}, U_{n-1} - U_n\rangle + \phi(U_{n-1}) - \phi(U_n) \le 0. \tag{1.19}$$

Dividing (1.19) by τ_n and replacing it into (1.15), (1.18) follows from the elementary inequality $2\langle v - w, v\rangle \ge |v|^2 - |w|^2$ for $v, w \in \mathbb{R}^d$. Recalling (1.3), we also deduce a crucial bound of $\mathcal{E}_n$ in terms of a residual monotone term

$$\mathcal{E}_n \le \tau_n^{-1}\langle \mathfrak{F}(U_n) - \mathfrak{F}(U_{n-1}), U_n - U_{n-1}\rangle =: \mathcal{D}_n. \tag{1.20}$$

Inserting (1.20) into (1.16) yields another *a posteriori* error estimate

$$\max_{t \in [0,T]} |u(t) - U(t)|^2 \le |u_0 - U_0|^2 + \sum_{n=1}^{N} \tau_n^2 \mathcal{D}_n. \tag{1.21}$$

Moreover, we prove in § 3 an exponential decay of the error provided $\mathfrak{F}$ is also *strongly* coercive.

We stress that $\mathcal{D}_n$ does not depend on the existence of a potential ϕ for $\mathfrak{F}$. An amazing fact, fully developed in § 4, is that (1.21) holds for a wider class of monotone operators than that of (sub)gradients, the so called *angle-bounded* operators; they appear to possess the right structure for *a posteriori* error control. This generalization applies to 3-*cyclically*-monotone operators, to linear sectorial (unbounded) operators, and to the classical formulation of variational inequalities (e.g. of obstacle type) in a Hilbert triplet. On the other hand, it is hopeless to seek an estimate such as (1.21) for general monotone operators. In fact, we show below that (1.21) implies *optimal a priori* rates of convergence of order $O(\tau)$, which are excluded by the counter-example of Rulla [53, Ex. 3].

Regularity and Rate of Convergence. We prove now that estimates (1.16) and (1.21) give rise to upper bounds of the error with respect to the largest time-step τ. A simple first result is directly suggested by (1.15), which implies

$$0 \leq \tau_n \mathcal{E}_n \leq \phi(U_{n-1}) - \phi(U_n).$$

Therefore, we end up with a telescopic sum

$$\sum_{n=1}^{N} \tau_n^2 \mathcal{E}_n \leq \tau \sum_{n=1}^{N} \big(\phi(U_{n-1}) - \phi(U_n)\big) = \tau\big(\phi(U_0) - \phi(U_N)\big).$$

Without loss of generality we may assume $\phi \geq 0$, for otherwise we may add an affine function, and we choose $U_0 := u_0$. We then obtain a rate of convergence of order $O(\sqrt{\tau})$

$$\max_{t\in[0,T]} |u(t) - U(t)|^2 \leq \phi(u_0)\, \tau. \tag{1.22}$$

On the other hand, making use of (1.3) in conjunction with (1.20), we can rewrite the residual monotone term as follows

$$\mathcal{D}_n = \langle \mathfrak{F}(U_{n-1}) - \mathfrak{F}(U_n), \mathfrak{F}(U_n)\rangle.$$

By virtue of property $\mathcal{D}_n \geq 0$ and inequality $2\langle v - w, v\rangle \geq |v|^2 - |w|^2$, we then see that the sum on the right-hand side of (1.21) telescopes

$$2\sum_{n=1}^{N} \tau_n^2 \mathcal{D}_n \leq \tau^2 \sum_{n=1}^{N} \big(|\mathfrak{F}(U_{n-1})|^2 - |\mathfrak{F}(U_n)|^2\big) \leq \tau^2 |\mathfrak{F}(U_0)|^2.$$

The following linear rate of convergence thus holds, again assuming $U_0 := u_0$

$$\max_{t\in[0,T]} |u(t) - U(t)|^2 \leq \tfrac{1}{2} |\mathfrak{F}(u_0)|^2 \tau^2. \tag{1.23}$$

We observe that we have so far tacitly assumed that both $\mathfrak{F}$ and ϕ are defined in the whole $\mathbb{R}^d$. However, even for scalar ODE's, the domains of $\mathfrak{F}$ and ϕ satisfy $D(\mathfrak{F}) \subset D(\phi)$ and may not coincide, as the following example reveals

$$\phi(w) := \begin{cases} -\sqrt{w} & \text{if } w \geq 0 \\ +\infty & \text{otherwise} \end{cases} \qquad \Rightarrow \qquad D(\phi) = [0, +\infty), \quad D(\mathfrak{F}) = (0, +\infty).$$

In the case of Hilbert spaces, the difference between $D(\phi)$ and $D(\mathfrak{F})$ plays a relevant role in that it reflects different regularity assumptions on the initial datum u_0 and thereby on the solution u. It follows that the difference between the rates of convergence of (1.22) and (1.23), which make sense provided $u_0 \in D(\phi)$ and $u_0 \in D(\mathfrak{F})$ respectively, is strictly related to the regularity of the exact solution u and thus to the approximability of u to a given order.

We point out that no extra regularity of u or U has been used (point (c)), as well as no constraints between consecutive time-steps (point (e)). We also stress that the error estimates (1.16), (1.21), (1.22), and (1.23) can be applied directly to dissipative ODE's arising from the space discretization of (nonlinear) parabolic PDE's with mesh parameter $h > 0$ and initial datum $u_{0,h}$, because the monotonicity properties of the resulting discrete operators ϕ_h and $\mathfrak{F}_h$ can be preserved by an adequate space discretization. However, the growth of $\phi_h(u_{0,h})$ and $|\mathfrak{F}_h(u_{0,h})|$ is usually different depending on the smoothness of u_0 even if the domains $D(\phi_h)$ and $D(\mathfrak{F}_h)$ coincide. The regularity of u_0 also dictates the size of the initial error $|u_0 - u_{0,h}|$ due to space discretization.

Adaptivity. The *a posteriori* error estimates (1.16) and (1.21) can be used in an adaptive strategy; note that the stability constants are 1 (point (d)). Given an error tolerance ε, the error control

$$\max_{t\in[0,T]} |u(t) - U(t)| \leq \varepsilon$$

is guaranteed upon *equidistributing* local discretization errors. In fact, assuming that U_0 satisfies $|u_0 - U_0|^2 \leq \varepsilon^2/2$, we then select the time-step size τ_n so that

$$2T\tau_n\mathcal{E}_n \leq \varepsilon^2 \quad \text{or} \quad 2T\tau_n\mathcal{D}_n \leq \varepsilon^2.$$

Plan of the Paper. In view of numerous applications, we will derive our results in a general Hilbert framework. In § 2 we briefly recall this abstract setting, along with some relevant examples and a novel discussion on additional coercivity properties of the convex potential ϕ and gradient field $\mathfrak{F}$. In § 3 we prove and discuss our main results about *a posteriori* error estimation and ensuing optimal rates of convergence for *subdifferential* operators, including error estimates in (semi)norms other than the ambient one via coercivity. In § 4 we show that *angle-bounded* operators are the natural class for *a posteriori* error estimation, and present several important examples. We derive optimal *a posteriori* and *a priori* error estimates, thereby extending the results of § 3. We also analyze Lipschitz perturbations and Hilbert triplets, which fit naturally within the setting of angle-bounded operators. In § 5 we obtain new error estimates for several parabolic PDEs, mostly strongly nonlinear, for which the abstract theory applies.

2 Abstract Setting and Examples

In this section we recall definitions and basic existence results for the continuous problem, introduce several relevant examples, and discuss coercivity properties of ϕ and $\mathfrak{F}$ for subdifferential operators. We present further examples and coercivity properties of angle-bounded operators in § 4.

2.1 Continuous Problem and Existence Results

Maximal Monotone Operators. Let $\mathcal{H}$ be a separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$, and let

$$\mathfrak{F} : \mathcal{H} \to 2^{\mathcal{H}}$$

be a (multi-valued) operator, with proper (and not empty!) domain

$$D(\mathfrak{F}) := \{v \in \mathcal{H} : \mathfrak{F}(v) \neq \varnothing\}.$$

We will always use the symbol $\mathfrak{F}(v)$ to indicate any selection $v_* \in \mathfrak{F}(v)$, the same at every occurence of $\mathfrak{F}(v)$. With this convention in mind, we will assume that $\mathfrak{F}$ is *monotone*

$$\langle \mathfrak{F}(v) - \mathfrak{F}(w), v - w\rangle \geq 0 \quad \forall\, v, w \in D(\mathfrak{F}),$$

and *maximal*, i.e. [9, Prop. 2.2] for every $\varepsilon > 0$ and $v \in \mathcal{H}$

$$\text{the equation} \quad w + \varepsilon\,\mathfrak{F}(w) \ni v \quad \text{admits a unique solution } w \in D(\mathfrak{F}). \tag{2.1}$$

These properties imply in particular that for every $w \in D(\mathfrak{F})$ the (not empty) *set* $\mathfrak{F}(w)$ is closed and convex in $\mathcal{H}$; its element of minimum norm is usually called the *minimal selection* of $\mathfrak{F}(w)$ and it is indicated by $\mathfrak{F}(w)^\circ$.

Subdifferentials. A particular important case arises when $\mathfrak{F}$ is the *subdifferential* of a proper lower semicontinuous (l.s.c.) convex function ϕ with (not empty, convex) domain $D(\phi)$

$$\phi : \mathcal{H} \to (-\infty, +\infty], \quad D(\phi) := \{v \in \mathcal{H} : \phi(v) < +\infty\};$$

we say that $\mathfrak{F} = \partial\phi$. In this case [9, Prop. 2.11] $D(\mathfrak{F})$ is densily included into $D(\phi)$, namely $\overline{D(\mathfrak{F})} = \overline{D(\phi)}$ and $\mathfrak{F}$ can be characterized by the inequalities

$$w_* \in \mathfrak{F}(w) \;\Leftrightarrow\; \quad \langle w_*, v - w\rangle + \phi(w) - \phi(v) \leq 0 \quad \forall\, v \in D(\phi). \tag{2.2}$$

Evolution Equations: Strong, Weak, and "Energy" Solutions. Given

$$\text{an } \textit{initial datum } u_0 \in \overline{D(\mathfrak{F})} \text{ and a } \textit{function } f \in L^1(0,T;\mathcal{H}),$$

we will study suitable approximations of the Cauchy problem for the differential inclusion

$$u'(t) + \mathfrak{F}(u(t)) \ni f(t) \quad \text{a.e. } t \in (0,T), \tag{2.3}$$

subject to $u(0) = u_0$. To be precise, we have to distinguish between *strong* and *weak* solutions [9, § 3, Def. 1].

Definition 2.1 *We say that $u \in C^0([0,T];\mathcal{H})$ is a* strong *solution of* (2.3) *if u is also locally absolutely continuous in $(0,T)$ and satisfies* (2.3) *at almost every point; in particular, $u(t) \in D(\mathfrak{F})$ for a.e. $t \in (0,T)$.*
We say that $u \in C^0([0,T];\mathcal{H})$ is a weak *solution if u can be* uniformly *approximated by a sequence $\{u_k\}_{k\in\mathbb{N}}$ of strong solutions which solve* (2.3) *w.r.t. a family of data $\{f_k\}_{k\in\mathbb{N}}$, approaching f in $L^1(0,T;\mathcal{H})$; note that $u_k(0) \to u_0$ in $\mathcal{H}$.* □

When $\mathfrak{F}$ is the subdifferential of ϕ defined by (2.2), then (2.3) can be equivalently written as an evolution variational inequality for a.e. $t \in (0,T)$

$$\langle u'(t) - f(t), u(t) - v\rangle + \phi(u(t)) - \phi(v) \leq 0 \quad \forall\, v \in D(\phi); \tag{2.4}$$

in this case we will refer to u as an *energy solution.*

Basic Existence Results and Energy Solutions. Since it will be used extensively, we now recall some results of the theory of existence, regularity, perturbations, and (asymptotic) stability, of solutions of (2.3) and (2.4) [9, §,3, Thms. 4, 5, 6, 7]. To this end, we introduce the Banach space $BV(0,T;\mathcal{H})$ of functions $g:[0,T]\to\mathcal{H}$ with *bounded total variation* [9, Def. A.2]

$$\operatorname{Var} g := \sup_{0=r_0<r_1<\ldots<r_J=T} \sum_{j=1}^{J} |g(r_j)-g(r_{j-1})| < +\infty;$$

we recall that if $g\in BV(0,T;\mathcal{H})$ then at every point $t_0\in[0,T)$ there exists the *right* limit $g_+(t_0) := \lim_{t\downarrow t_0} g(t)$.

(i) Strong Lipschitz Solutions. *If $u_0\in D(\mathfrak{F})$ and $f\in BV(0,T;\mathcal{H})$ then there exists a unique strong solution u of* (2.3). *Moreover, u is also Lipschitz continuous, its range is included in $D(\mathfrak{F})$, and its right derivative $u'_+(t)$ exists at every point of $[0,T)$ and coincides with the minimal selection of $f_+(t)-\mathfrak{F}(u(t))$; in particular, at the initial time we have*

$$u'_+(0) = (f_+(0)-\mathfrak{F}(u_0))^\circ. \tag{2.5}$$

(ii) Stability and Weak Solutions. *The mapping*

$$(u_0,f)\in D(\mathfrak{F})\times BV(0,T;\mathcal{H})\mapsto u$$

defined by the previous statement, is Lipschitz continuous w.r.t. the metric of $\mathcal{H}\times L^1(0,T;\mathcal{H})$ with values in $C^0([0,T];\mathcal{H})$. In particular, for every $u_0\in\overline{D(\mathfrak{F})}$ and $f\in L^1(0,T;\mathcal{H})$ there exists a unique weak solution of (2.3).

(iii) Lipschitz Perturbations. *The previous results hold even if $\mathfrak{F}$ is the sum of a maximal monotone operator and a Lipschitz perturbation defined in the same domain, or, equivalently, if ($\mathfrak{I}$ is the identity)*

$$\exists\, L\ge 0: \quad \mathfrak{F}+L\mathfrak{I} \quad \textit{is maximal monotone}.$$

(iv) Energy Solutions and Regularizing Effects. *If $\mathfrak{F}$ is the subdifferential of a l.s.c. convex function ϕ, and u is a weak solution, then the function $t\mapsto\phi(u(t))$ belongs to $L^1(0,T)$ and it depends with continuity on the data $(u_0,f)\in\overline{D(\phi)}\times L^1(0,T;\mathcal{H})$; also $u(t)\in D(\phi)$ for a.e. $t\in(0,T)$.*
If $f\in L^2(0,T;\mathcal{H})$, then every weak solution u belongs to $H^1_{loc}(0,T;\mathcal{H})$ and it is a strong solution. The mapping $t\mapsto\phi(u(t))$ is locally absolutely continuous in $(0,T]$ and the following energy identity *holds*

$$|u'(t)|^2+\tfrac{d}{dt}\phi(u(t)) = \langle f(t),u'(t)\rangle \quad \textit{a.e. } t\in(0,T). \tag{2.6}$$

In addition, $u\in H^1(0,T;\mathcal{H})$ if $u_0\in D(\phi)$.

(v) Dissipation inequality. *Finally, if f is absolutely continuous and u is a strong solution, we have*

$$\tfrac{1}{2}\tfrac{d}{dt}|u'(t)|^2 \le \langle f'(t),u'(t)\rangle \quad \textit{in the distribution sense.} \quad \square \tag{2.7}$$

Remark 2.2 Weak solutions can also be characterized as the solutions of a suitable integral formulation of (2.3) (see [9, Lemma 3.1]). On the other hand, the *approximation approach* chosen in Definition 2.1 and the stability result of the previous Theorem allow us to deal, at least initially, only with strong solutions and then to extend to the weak ones every estimate involving only quantities depending on u_0 and f which are *continuous* w.r.t. the $\mathcal{H} \times L^1(0,T;\mathcal{H})$-metric. When $\mathfrak{F}$ is a subdifferential, a different way to extend the notion of weak solutions, in order to deal with f in the Sobolev spaces $H^{-s}(0,T;\mathcal{H})$ of negative order $-s \in (-1/2, 0)$, has been exploited in [55]. $\square$

2.2 Basic Examples of Subdifferential Type

We have already discussed in the introduction the case of dissipative ODE's in $\mathbb{R}^d$. We consider now other applications of the abstract framework provided $\mathfrak{F}$ admits a convex potential ϕ; we discuss non-potential operators $\mathfrak{F}$ and corresponding examples in § 4. Let Ω be a Lipschitz open set of $\mathbb{R}^m$, with outward unit normal $\boldsymbol{n}$; $Q := \Omega \times (0,T)$ is the parabolic cylinder, with lateral boundary $\Sigma := \partial\Omega \times (0,T)$.

Example 2.1: ODE's with Obstacles. We take the double obstacle potential in $\mathcal{H} := \mathbb{R}$

$$\phi(w) := \begin{cases} 0 & \text{if } w \in [-1,1], \\ +\infty & \text{otherwise,} \end{cases} \qquad D(\phi) := [-1,1];$$

correspondingly, the subdifferential $\mathfrak{F}$ of ϕ is given by [9, p. 44]

$$\mathfrak{F}(w) := \begin{cases} (-\infty, 0] & \text{if } w = -1, \\ \{0\} & \text{if } -1 < w < 1, \\ [0, +\infty) & \text{if } w = 1, \end{cases} \qquad D(\mathfrak{F}) := D(\phi);$$

ϕ is the indicator function $I_{[-1,1]}$ of $[-1,1]$ and $\mathfrak{F}$ is the inverse of the *sign* graph. In this case, equation (2.3) becomes [9, Rem. 3.9]

$$\begin{cases} u'(t) = -f(t)^- & \text{if } u(t) = 1, \\ u'(t) = f(t) & \text{if } -1 < u(t) < 1, \\ u'(t) = f(t)^+ & \text{if } u(t) = -1. \end{cases}$$

The importance of this kind of ODE's in phase transitions has been pointed out by Visintin [62] (see also [40]). They also arise in geometric motion of fronts (see [16], [48]).

Example 2.2: Linear Parabolic PDE's. We present several examples.
Heat Equation. If

$$\mathcal{H} := L^2(\Omega), \quad \phi(w) := \tfrac{1}{2} \int_\Omega |\nabla w(x)|^2 \, dx, \quad D(\phi) := H^1_0(\Omega), \tag{2.8}$$

then the subdifferential $\mathfrak{F}$ of ϕ is given by

$$\mathfrak{F}(w) := -\Delta w, \quad D(\mathfrak{F}) := \{v \in D(\phi) : \Delta v \in L^2(\Omega)\}, \tag{2.9}$$

and (2.3) becomes the *heat* equation with homogeneous Dirichlet condition

$$\partial_t u - \Delta u = f \quad \text{a.e. in } Q, \quad u = 0 \quad \text{on } \Sigma.$$

Replacing $D(\phi)$ in (2.8) by $D(\phi) := H^1(\Omega)$ leads to

$$\mathfrak{F}(w) := -\Delta w, \quad D(\mathfrak{F}) := \{v \in D(\phi) : \Delta v \in L^2(\Omega), \ \partial_{\boldsymbol{n}} v = 0 \text{ on } \partial\Omega\}, \tag{2.10}$$

and to the heat equation with homogeneous Neumann condition

$$\partial_t u - \Delta u = f \quad \text{a.e. in } Q, \quad \partial_{\boldsymbol{n}} u = 0 \quad \text{on } \Sigma.$$

Bilaplacian Operator. If Ω is $C^{1,1}$ or convex, and

$$\mathcal{H} := L^2(\Omega), \quad \phi(w) := \tfrac{1}{2}\int_\Omega |\Delta w(x)|^2\,dx, \quad D(\phi) := H^2_0(\Omega), \tag{2.11}$$

then the subdifferential $\mathfrak{F}$ of ϕ is the *bilaplacian*

$$\mathfrak{F}(w) := \Delta^2 w, \quad D(\mathfrak{F}) := \{v \in D(\phi) : \Delta^2 v \in L^2(\Omega)\}. \tag{2.12}$$

Now (2.3) is a fourth order parabolic equation with zero essential conditions

$$\partial_t u + \Delta^2 u = f \quad \text{a.e. in } Q, \quad u = \partial_{\boldsymbol{n}} u = 0 \quad \text{on } \Sigma.$$

Alternative boundary conditions can be obtained with different choices of $D(\phi)$ in (2.11): e.g. $D(\phi) := H^2(\Omega) \cap H^1_0(\Omega)$ and $D(\phi) := \{v \in H^2(\Omega) : \partial_{\boldsymbol{n}} v = 0 \text{ on } \partial\Omega\}$ lead respectively to

$$D(\mathfrak{F}) := \{v \in D(\phi) : \Delta^2 v \in L^2(\Omega), \Delta v = 0 \text{ on } \partial\Omega\},$$
$$D(\mathfrak{F}) := \{v \in D(\phi) : \Delta^2 v \in L^2(\Omega), \partial_{\boldsymbol{n}}(\Delta v) = 0 \text{ on } \partial\Omega\}.$$

Abstract Linear Equations: the Symmetric Case. Let $\mathcal{V}$ be another Hilbert space

$$\mathcal{V} \subset \mathcal{H}, \quad \text{the inclusion being continuous and dense,} \tag{2.13}$$

let $\mathfrak{a} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ be a *positive, continuous*, and *weakly coercive* bilinear form

$$[v]^2 := \mathfrak{a}(v, v) \geq 0, \quad c_1\|v\|^2_{\mathcal{V}} \leq [v]^2 + |v|^2 \leq c_2\|v\|^2_{\mathcal{V}} \quad \forall\, v \in \mathcal{V}, \tag{2.14}$$

for some constants $c_1, c_2 > 0$, and let $\mathfrak{A} : D(\mathfrak{A}) \subset \mathcal{H} \to \mathcal{H}$ be the linear operator associated to $\mathfrak{a}$

$$\langle \mathfrak{A}w, v\rangle := \mathfrak{a}(w, v) \quad \forall\, v \in \mathcal{V}, \quad D(\mathfrak{A}) := \Big\{v \in \mathcal{V} : \sup_{z\in\mathcal{V}, |z|=1} \mathfrak{a}(v, z) < +\infty\Big\}. \tag{2.15}$$

We consider, as a particular case of (2.3), the abstract *linear* evolution equation

$$u'(t) + \mathfrak{A}u(t) = f(t) \quad \text{in } (0, T). \tag{2.16}$$

If $\mathfrak{a}$ is *symmetric*

$$\mathfrak{a}(v, w) = \mathfrak{a}(w, v) \quad \forall\, v, w \in \mathcal{V}, \tag{2.17}$$

then $\mathfrak{A}$ is the subdifferential of [9, Ex. 2.3.7]

$$\phi(w) := \begin{cases} \frac{1}{2}\mathfrak{a}(w,w) & \text{if } w \in \mathcal{V}, \\ +\infty & \text{otherwise,} \end{cases} \qquad D(\phi) := \mathcal{V}, \tag{2.18}$$

and (2.16) is equivalent to (2.4).

We point out that (2.9) (or (2.10)) fits within this setting upon choosing

$$\mathcal{V} := H^1_0(\Omega) \quad (\text{or } \mathcal{V} := H^1(\Omega)), \quad \mathfrak{a}(v,w) := \int_\Omega \nabla v \cdot \nabla w \, dx \tag{2.19}$$

along with $\mathcal{H} := L^2(\Omega)$, and so does (2.12) provided $\mathcal{H} := L^2(\Omega)$, $\mathcal{V} := H^2_0(\Omega)$, and $\mathfrak{a}(v,w) := \int_\Omega \Delta v \, \Delta w \, dx$.

Example 2.3: Parabolic Variational Inequalities. We discuss several examples in turn.

Parabolic obstacle problems. Given $\mathcal{H} := L^2(\Omega)$ and an obstacle g satisfying $g \in H^1(\Omega)$ with $g \le 0$ on $\partial\Omega$, let

$$\mathcal{K} := \{v \in H^1_0(\Omega) : v(x) \ge g(x) \text{ for a.e. } x \in \Omega\}$$

be the convex set of admissible functions and let $I_\mathcal{K}$ be its indicator function

$$I_\mathcal{K}(w) := \begin{cases} 0 & \text{if } w \in \mathcal{K}, \\ +\infty & \text{otherwise;} \end{cases} \tag{2.20}$$

note that $I_\mathcal{K}$ is a *non-smooth* l.s.c. convex function. Weaker assumptions on g in the framework of the capacity theory are allowed [41]. We consider

$$\phi(w) := \tfrac{1}{2} \int_\Omega |\nabla w(x)|^2 \, dx + I_\mathcal{K}(w), \quad D(\phi) := \mathcal{K}, \tag{2.21}$$

so that the solution $u \in \mathcal{K}$ of (2.4) solves the variational inequality for a.e. $t \in (0,T)$

$$\int_\Omega \big(\partial_t u\,(u-v) + \nabla u \cdot (\nabla u - \nabla v)\big) dx \le \int_\Omega f(u-v)\,dx \quad \forall\, v \in \mathcal{K}; \tag{2.22}$$

we refer to (2.49) and Corollary 1 of § 2.3 for this slightly stronger variational inequality including coercivity. By the regularity result of Brézis-Stampacchia [11, Cor. II.3], if Ω is $C^{1,1}$ or convex and $g \in H^2(\Omega)$ then the previous variational inequality is equivalent to (2.3), where $\mathfrak{F}$ has the simple expression

$$w^* \in \mathfrak{F}(w) \quad \Leftrightarrow \quad w^* \in -\Delta w + \partial I_\mathcal{K}(w), \quad D(\mathfrak{F}) := \mathcal{K} \cap H^2(\Omega).$$

Abstract Evolution Variational Inequalities. We observe that other kind of constraints can be accommodated and an abstract theory can be developed within the same framework as Example 2.2 (see [4], [6], [8], [34], [43], [44], [54]). In fact, let $\mathcal{V}, \mathfrak{a}, \mathfrak{A}$ satisfy (2.13), (2.14), (2.15), and (2.17), and let

$$\psi : \mathcal{V} \to (-\infty, +\infty] \quad \text{be convex and l.s.c. w.r.t. the topology of } \mathcal{V}, \tag{2.23}$$

with not empty domain $D(\psi)$. Note that ψ may fail to be l.s.c. w.r.t. $\mathcal{H}$. Let u be a solution of the evolution variational inequality

$$\langle u' - f, u - v\rangle + \mathfrak{a}(u, u - v) + \psi(u) - \psi(v) \le 0 \quad \forall\, v \in D(\psi). \tag{2.24}$$

Upon choosing

$$\phi(w) := \tfrac{1}{2}\mathfrak{a}(w, w) + \psi(w), \quad D(\phi) := D(\psi),$$

and using that $\mathfrak{a}$ is symmetric, we realize that (2.24) is a slightly stronger form than (2.4) in that it incorporates the coercivity term $\frac{1}{2}\mathfrak{a}(u - v, u - v)$; see § 2.3 for a thorough discussion of coercivity. In this case the associated $\mathfrak{F}$ is given by

$$w_* \in \mathfrak{F}(w) \;\Leftrightarrow\; \mathfrak{a}(w, w - v) + \psi(w) \le \psi(v) + \langle w_*, w - v\rangle \;\; \forall\, v \in D(\psi). \tag{2.25}$$

In particular, if $\mathcal{V}$ and $\mathfrak{a}$ are those in (2.19) and $\psi = I_{\mathcal{K}}$ is as in (2.20), then we are dealing with the previous obstacle problem (2.22).

A compatibility condition of ψ w.r.t. $\mathfrak{A}$, defined in (2.15), again simplifies the structure of $\mathfrak{F}$. If ψ is the restriction to $\mathcal{V}$ of a convex function (still denoted by ψ) defined and l.s.c. in $\mathcal{H}$ and there exist $z \in \mathcal{H}$, $c \ge 0$ such that

$$\forall\, v \in D(\psi), \quad w_\varepsilon + \varepsilon \mathfrak{A} w_\varepsilon = v + \varepsilon z \quad \Rightarrow \quad \psi(w_\varepsilon) \le \psi(v) + c\varepsilon, \tag{2.26}$$

then the sum $\mathfrak{A} + \partial\psi$ is *maximal* monotone [9, Prop. 2.17]. Since $\mathfrak{A}w + \partial\psi(w) \in \mathfrak{F}(w)$ for all $w \in D(\mathfrak{A}) \cap D(\partial\psi)$, we infer the following characterization of $\mathfrak{F}$

$$\mathfrak{F} := \mathfrak{A} + \partial\psi, \quad D(\mathfrak{F}) := D(\mathfrak{A}) \cap D(\partial\psi). \tag{2.27}$$

Reaction-Diffusion Equations. Let $\mathcal{H}, \mathcal{V}, \mathfrak{a}$ be as in (2.19) and let ϕ be

$$\phi(w) := \int_\Omega \left(\tfrac{1}{2}|\nabla w|^2 + g(w)\right) dx, \quad D(\phi) := \{v \in H_0^1(\Omega) : g(v) \in L^1(\Omega)\},$$

where g is a continuous convex real function. Then [7, Cor. 13], [12, Lemma 3]

$$\mathfrak{F}(w) := -\Delta w + g'(w), \quad D(\mathfrak{F}) := \{v \in D(\phi) : -\Delta v,\, g'(v) \in L^2(\Omega)\}, \tag{2.28}$$

since (2.26) is valid, and the PDE corresponding to (2.24) becomes

$$\partial_t u - \Delta u + g'(u) = f \quad \text{a.e. in } Q. \tag{2.29}$$

When g is not convex, but nevertheless $r \mapsto g'(r) + \lambda r$ with a constant $\lambda > 0$ is nondecreasing, then the λ-perturbation $w \mapsto -\Delta w + g'(w) + \lambda w$ of $\mathfrak{F}$ in (2.28) is a subdifferential in $L^2(\Omega)$; the Basic Existence Results is still applicable.

Allen-Cahn Equation. If g is the double well potential $g(r) := (1 - r^2)^2$ for $r \in \mathbb{R}$, then (2.29) is the Allen-Cahn equation with regular potential upon proper rescaling [25]. We can also consider constraints: if $g(r) := I_{[-1,1]}(r) - r^2/2$ for $r \in \mathbb{R}$, then we get the Allen-Cahn equation with double obstacles ± 1 [16], [48].

Other choices of $\mathcal{H}, \mathcal{V}, \mathfrak{a}$ and of the smooth potential function g lead to important applications of reaction-diffusion systems (see e.g. [18]).

Example 2.4: Quasi-Linear Parabolic PDE's. Let $\mathcal{H} := L^2(\Omega)$, $p > 1$,

$$\phi(w) := \int_\Omega G(x, \nabla w(x))\, dx, \quad D(\phi) := L^2(\Omega) \cap W_0^{1,p}(\Omega). \tag{2.30}$$

Here $G(x, \boldsymbol{\xi}) : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function [27, Chap. 8, Def. 1.2] which is convex and continuously differentiable w.r.t. $\boldsymbol{\xi} \in \mathbb{R}^m$, for a.e. $x \in \Omega$, and satisfies, together with its differential $\boldsymbol{a}(x, \boldsymbol{\xi}) := \nabla_{\boldsymbol{\xi}} G(x, \boldsymbol{\xi})$, the usual $p-$growth conditions (see [47, p. 29], [26, Chap. II, § 1])

$$G(x, \boldsymbol{\xi}) \geq \alpha_0 |\boldsymbol{\xi}|^p - \alpha_1, \quad |\boldsymbol{a}(x, \boldsymbol{\xi})| \leq \alpha_2 (1 + |\boldsymbol{\xi}|^{p-1}), \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^m, \text{ a.e. } x \in \Omega,$$

where $\alpha_0, \alpha_1, \alpha_2$ are given positive constants. Then ϕ is convex and l.s.c. in $\mathcal{H}$, and its subdifferential is

$$\begin{aligned} \mathfrak{F}(w(\cdot)) &:= -\operatorname{div} \boldsymbol{a}(\cdot, \nabla w(\cdot)), \\ D(\mathfrak{F}) &:= \{v \in D(\phi) : \operatorname{div} \boldsymbol{a}(\cdot, \nabla v(\cdot)) \in L^2(\Omega)\}. \end{aligned} \tag{2.31}$$

The ensuing PDE reads

$$\partial_t u - \operatorname{div} \boldsymbol{a}(\cdot, \nabla u) = f \quad \text{a.e. in } Q. \tag{2.32}$$

p-Laplacian. The choice

$$G(x, \boldsymbol{\xi}) := \tfrac{1}{p} |\boldsymbol{\xi}|^p, \quad \boldsymbol{a}(x, \boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2} \boldsymbol{\xi}, \tag{2.33}$$

gives rise to the p-Laplacian $\mathfrak{F}(w) := -\operatorname{div}(|\nabla w|^{p-2} \nabla w)$ and associated evolution equation with homogeneous Dirichlet condition [8], [43]

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f \quad \text{a.e. in } Q, \quad u = 0 \quad \text{on } \Sigma.$$

Mean Curvature Operator for Cartesian Surfaces. We consider the case of *linear growth* $p = 1$, occurring e.g. for the area functional of Cartesian surfaces

$$G(\boldsymbol{\xi}) := \sqrt{1 + |\boldsymbol{\xi}|^2}, \quad \boldsymbol{a}(\boldsymbol{\xi}) = -\frac{\boldsymbol{\xi}}{\sqrt{1 + |\boldsymbol{\xi}|^2}},$$

which, for $f \equiv 0$, leads to the vertical evolution of graphs by mean curvature

$$\partial_t u - \operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0 \quad \text{a.e. in } Q. \tag{2.34}$$

For $u_0 \in W^{1,\infty}(\Omega)$ given, we supply the parabolic boundary condition

$$u(x, t) = u_0(x) \quad \text{for } t = 0 \text{ or } x \in \partial\Omega.$$

This problem does not admit, in general, a strong or weak solution which also satisfies the lateral boundary condition in the sense of traces in Sobolev spaces, unless $\partial\Omega$ is a smooth hypersurface of nonnegative mean curvature; weaker conditions allowing less regular domains are also possible [33, §§ 15.6-9], [42]. There are two alternative ways to pose this problem. The integral of (2.30) in $W^{1,1}(\Omega)$ could be replaced by the corresponding *relaxed* version in $BV(\Omega)$ which includes the total variation of w (see [27], [33, Chap. 14]). Otherwise, it is possible to introduce the notion of *pseudo-solution* (cf. [42], [59]) by formally substituting the lateral Dirichlet-type condition with the following nonlinear one

$$\frac{\partial_{\boldsymbol{n}} u}{\sqrt{1+|\nabla u|^2}} + \operatorname{sign}(u-u_0) \ni 0 \quad \text{on } \Sigma,$$

where it is implicitly understood that

$$u \neq u_0 \quad \Rightarrow \quad |\partial_{\boldsymbol{n}} u| = +\infty, \quad \Big|\frac{\partial_{\boldsymbol{n}} u}{\sqrt{1+|\nabla u|^2}}\Big| = 1. \tag{2.35}$$

This relaxed boundary condition (2.35) implies the stronger one $u = u_0$ under the extra geometric assumptions on $\partial\Omega$ which guarantee existence of a strong solution of (2.34) [42]; this justifies the prefix *pseudo* attached to this more general kind of solution.

We adopt this second approach here, and thus view (2.34) as the evolution inequality (2.4) in $\mathcal{H} := L^2(\Omega)$ for the *convex* area-type functional

$$\begin{gathered}\phi(w) := \int_\Omega \sqrt{1+|\nabla w|^2}\, dx + \int_{\partial\Omega} |w-u_0|\, dS, \\ D(\phi) := L^2(\Omega) \cap W^{1,1}(\Omega).\end{gathered} \tag{2.36}$$

In § 5.3 we show how to overcome the lack of lower semicontinuity of ϕ with respect to the strong topology of $L^2(\Omega)$.

Example 2.5: Degenerate Parabolic PDE's. We first discuss a general setting which consists of *changing the pivot space* from $\mathcal{H}$ to $\mathcal{V} \subset \mathcal{H}$. Let $\mathcal{V}, \mathcal{H}, \mathfrak{a}, \mathfrak{A}$ be as in (2.13)-(2.17), and in addition assume that

$$\mathfrak{a} \text{ is } \textit{strongly} \text{ coercive on } \mathcal{V}; \tag{2.37}$$

thus $[\cdot]$ is an equivalent norm for $\mathcal{V}$. Moreover, let $\phi : \mathcal{H} \to]-\infty, +\infty]$ be a l.s.c. convex function with $D(\phi) \subset \mathcal{V}$, and let $\mathfrak{B} : \mathcal{V} \to 2^{\mathcal{V}}$ be its subdifferential in $\mathcal{V}$ w.r.t. the scalar product $\mathfrak{a}(\cdot,\cdot)$. Consequently

$$v_* \in \mathfrak{B}(v) \quad \Leftrightarrow \quad \mathfrak{a}(v_*, z-v) + \phi(v) - \phi(z) \le 0 \quad \forall\, z \in D(\phi), \tag{2.38}$$

defines $\mathfrak{B}$. To express the subdifferential $\mathfrak{F}$ of ϕ in $\mathcal{H}$ in terms of $\mathfrak{A}$ and $\mathfrak{B}$ we proceed as follows. We first note that $w_* \in \mathfrak{F}(w)$ iff $w \in D(\phi)$ minimizes the functional $z \in \mathcal{V} \mapsto \phi(z) - \langle w_*, z\rangle = \phi(z) - \mathfrak{a}(\mathfrak{A}^{-1}w_*, z)$, where the last equality is due to (2.15). In view of (2.38), this yields

$$w \in D(\mathfrak{B}), \quad \mathfrak{A}^{-1}w_* \in \mathfrak{B}(w).$$

We now choose the ambient space $\mathcal{H}$, pivot space $\mathcal{V}$, and convex functional ϕ that yield degenerate parabolic PDE. Other choices $\mathcal{H}, \mathcal{V}$, and ϕ, lead to a wide range of phase transition problems [17], [24], [40], [46], [57], [62]. Let

$$\mathcal{V} := L^2(\Omega), \quad \mathfrak{a}(v,w) := \int_\Omega v(x)\, w(x)\, dx, \quad \mathcal{H} := H^{-1}(\Omega),$$

where $\mathcal{H}$ is endowed with the norm and scalar product

$$|w| := \|\nabla G w\|_{L^2(\Omega)}, \quad \langle v, w\rangle := {}_{H^{-1}(\Omega)}\langle v, Gw\rangle_{H^1_0(\Omega)}.$$

Here $-G$ denotes the inverse of the Laplace operator subject to a homogeneous Dirichlet boundary condition; hence $\mathfrak{A} = G^{-1} = -\Delta$ and $D(\mathfrak{A}) = H_0^1(\Omega)$.

Let $\beta : \mathbb{R} \mapsto \mathbb{R}$ be a continuous monotone function (β could also be a surjective monotone graph), such that $0 \in \beta(0)$ and $\liminf_{|s|\to+\infty} \beta(s)/s > 0$; the latter could be weakened [7]. We next consider the convex primitive of β

$$j(r) := \int_0^r \beta(s)\, ds \quad \forall\, r \in \mathbb{R}\,.$$

We introduce the l.s.c. convex functional

$$\phi(w) := \int_\Omega j(w(x))\, dx, \quad D(\phi) := \{v \in L^2(\Omega) : j(v) \in L^1(\Omega)\},$$

whose (sub)differential operator in $\mathcal{H}$ is given by

$$\mathfrak{F}(w) := -\Delta\beta(w), \quad D(\mathfrak{F}) := \{v \in D(\phi) : \beta(v) \in H_0^1(\Omega)\}. \tag{2.39}$$

We thus end up with the following equation in the sense of distributions

$$\partial_t u - \Delta\beta(u) = f \quad \text{in } Q, \quad \beta(u) = 0 \quad \text{on } \Sigma. \tag{2.40}$$

We discuss now three specific choices of β.
Two-Phase Stefan Problem. The simplest choice problem corresponds to

$$\beta(r) := (r-1)^+ - r^-. \tag{2.41}$$

Porous Medium Equation. The typical choice for β is

$$\beta(r) := |r|^{p-2} r, \quad p > 2. \tag{2.42}$$

Hele-Shaw Problem. If $\beta := H^{-1}$ is the inverse of the Heaviside graph and $j := I_{[0,1]}$, $D(\phi) := L^\infty(\Omega; [0,1])$, then $\mathfrak{F}$ in (2.39) turns out to be

$$w_* \in \mathfrak{F}(w) \quad \Leftrightarrow \quad Gw^* \in \beta(w) \quad \text{a.e. in } \Omega.$$

In terms of $v \in H^{-1}(u)$, (2.40) formally reads $\partial_t H(v) - \Delta v \ni f$, which is an elliptic-parabolic PDE.

2.3 Coercivity of a Subdifferential $\mathfrak{F}$

Before embarking on the analysis of the time discretization of (2.4), we explore the possibility of extracting extra information from (2.3) than merely (2.4). We motivate the definitions with the linear symmetric case (2.16). After multiplying (2.16) by $u(t) - v$, for a generic $v \in \mathcal{V}$, and using the identity

$$2\,\mathfrak{a}(u, u-v) = \mathfrak{a}(u,u) - \mathfrak{a}(v,v) + \mathfrak{a}(u-v, u-v),$$

from (2.15) and (2.18) we arrive at

$$\langle u' - f, u - v\rangle + \phi(u) - \phi(v) + \phi(u-v) = 0. \tag{2.43}$$

If we compare (2.43) with (2.4), we gain control of the positive quadratic term

$$\phi(u-v) = \tfrac{1}{2}\,\mathfrak{a}(u-v, u-v).$$

We stress that this fact is not related to the linearity of this example but rather to the *coercivity* of $\mathfrak{F}$. We formalize this property in the following definitions.

Definition 2.3 *For every $w \in D(\mathfrak{F})$ and $v \in D(\phi)$, let $\sigma(w;v) \geq 0$ be*

$$\sigma(w;v) := \phi(v) - \phi(w) - \sup_{w_* \in \mathfrak{F}(w)} \langle w_*, v - w \rangle. \tag{2.44}$$

Then for every $w, v \in D(\mathfrak{F})$ we define

$$\varrho(w,v) := \sigma(w;v) + \sigma(v;w) = \inf_{w_* \in \mathfrak{F}(w),\, v_* \in \mathfrak{F}(v)} \langle w_* - v_*, w - v \rangle. \quad \square \tag{2.45}$$

Remark 2.4 Our purpose in dealing with σ is to obtain error estimates in (semi)norms other than that of $\mathcal{H}$ in a concise and possibly sharp manner. Fine properties of σ related to one-sided derivatives of ϕ can be found in [52, § 23], [15, Thm. I-27]. Here we only observe that σ is a Borel function in its domain, so that its composition with continuous (or Borel) time-dependent functions is still measurable, and the related integrals are well defined. In fact, the separability of $\mathcal{H}$ and the usual Yosida approximation technique ensure that $\mathfrak{F}$ admits a sequence of *Borel* selections $\mathfrak{F}_k : D(\mathfrak{F}) \to \mathcal{H}$ such that $\mathfrak{F}(w) = \overline{\cup_k \{\mathfrak{F}_k(w)\}}$; therefore σ can be defined in terms of the supremum of a countable family of Borel functions. $\square$

Remark 2.5 We observe that, when $\mathfrak{F}$ is single-valued, we have the *identities*

$$\sigma(w;v) = \phi(v) - \phi(w) - \langle \mathfrak{F}(w), v - w \rangle \quad \forall\, w \in D(\mathfrak{F}),\ \forall\, v \in D(\phi), \tag{2.46}$$

$$\varrho(w,v) = \langle \mathfrak{F}(w) - \mathfrak{F}(v), w - v \rangle \quad \forall\, w, v \in D(\mathfrak{F}). \quad \square \tag{2.47}$$

Remark 2.6 In the general multi-valued case, from (2.44) we can only say that

$$w_* \in \mathfrak{F}(w) \Leftrightarrow \langle w_*, v - w \rangle + \sigma(w;v) \leq \phi(v) - \phi(w) \quad \forall\, v \in D(\phi). \tag{2.48}$$

Hence, the solution u of both (2.3) and (2.4) satisfies the *stronger* inequality

$$\langle u'(t) - f(t), u(t) - v \rangle + \phi(u(t)) - \phi(v) + \sigma(u(t); v) \leq 0 \tag{2.49}$$

for all $v \in D(\phi)$. $\square$

The following lower bounds of ϱ can be easily deduced for the previous examples. For the choices (2.9), (2.10), (2.21), and (2.28) we have

$$\varrho(w,v) \geq \int_\Omega |\nabla w - \nabla v|^2\, dx \quad \forall\, w, v \in D(\mathfrak{F}), \tag{2.50}$$

whereas for (2.12) we get

$$\varrho(w,v) \geq \int_\Omega |\Delta w - \Delta v|^2\, dx \quad \forall\, w, v \in D(\mathfrak{F}). \tag{2.51}$$

In general, if $\mathfrak{F}$ is defined by (2.25) with $\mathcal{V}, \mathfrak{a}, \psi$ satisfying (2.13), (2.14), (2.17), and (2.23), then

$$\varrho(w,v) \geq \mathfrak{a}(w - v) = [w - v]^2. \tag{2.52}$$

The following (less immediate) property [26, Ch. I, 4-(iii)]

$$\forall\, p \geq 2,\ \exists\, c_p > 0: \quad \big(|r|^{p-2} r - |s|^{p-2} s\big)(r - s) \geq c_p |r - s|^p \quad \forall\, r, s \in \mathbb{R},$$

leads to estimates for the p-Laplacian $\mathfrak{F}(w) = -\operatorname{div}(|\nabla w|^{p-2}\nabla w)$ (see (2.33))

$$\varrho(w,v) \geq c_p \int_\Omega |\nabla w - \nabla v|^p \, dx \quad \forall\, w, v \in D(\mathfrak{F}), \tag{2.53}$$

and for the porous medium operator $\mathfrak{F}(w) = -\Delta(|w|^{p-2}w)$ (see (2.42))

$$\varrho(w;v) \geq c_p \int_\Omega |w - v|^p \, dx \quad \forall\, w, v \in D(\mathfrak{F}), \tag{2.54}$$

for $p \geq 2$. We formalize the above lower bounds of ϱ in the following definition.

Definition 2.7 *We say that a function $[\cdot] : \mathcal{H} \to [0, +\infty]$ is a (generalized) l.s.c. seminorm on $\mathcal{H}$ if it is proper, l.s.c., convex, and positively homogeneous. The (multi-valued) operator $\mathfrak{F}$ is* p-coercive *w.r.t. $[\cdot]$ for a given $p \geq 2$, if*

$$\varrho(w,v) \geq [w-v]^p \quad \forall\, w, v \in D(\mathfrak{F}). \quad \square \tag{2.55}$$

We stress that $\mathfrak{F}$ could be p-coercive for $1 < p < 2$ only w.r.t. $[\cdot] \equiv 0$. An interesting and crucial fact is that (2.55) implies an analogous property for σ (see [8, (2.30)] for $p = 2$).

Lemma 2.8 *Let $\mathfrak{F}$ be p-coercive as in* (2.55). *Then*

$$\sigma(w;v) \geq \tfrac{1}{p}[w-v]^p \quad \forall\, w \in D(\mathfrak{F}),\ \forall\, v \in D(\phi). \tag{2.56}$$

Proof. Let us first observe that, for every choice of $w_* \in \mathfrak{F}(w)$,

$$0 \leq \phi(v) - \phi(w) - \langle w_*, v - w\rangle = \big(\phi(v) - \langle w_*, v\rangle\big) - \big(\phi(w) - \langle w_*, w\rangle\big) \quad \forall\, v \in D(\phi),$$

whence w is a minimum of $\phi(\cdot) - \langle w_*, \cdot\rangle$. Since ϱ does not change upon adding a linear term to ϕ, for proving (2.56) it suffices to show that

$$\phi(v) - \phi(w) \geq \tfrac{1}{p}[w-v]^p \quad \forall\, v \in D(\phi), \tag{2.57}$$

whenever w is a minimum point of ϕ, i.e. $0 \in \mathfrak{F}(w)$. If ϕ is Gateaux-differentiable, and so $D(\mathfrak{F}) = D(\phi)$, (2.57) follows easily by evaluating the derivative of the function $s \mapsto \phi(w_s)$, with $w_s := w + s(v-w)$; in fact, the Mean Value Theorem combined with (2.47) and (2.55) yields

$$\phi(v) - \phi(w) = \int_0^1 \tfrac{1}{s}\langle \mathfrak{F}(w_s), w_s - w\rangle \, ds = \int_0^1 \tfrac{1}{s}\varrho(w_s, w) ds \geq [w-v]^p \int_0^1 s^{p-1} ds.$$

Otherwise, we employ a standard approximation argument. For $\varepsilon > 0$, let

$$\mathfrak{J}_\varepsilon := (I + \varepsilon\mathfrak{F})^{-1}, \quad \phi_\varepsilon := \phi \circ \mathfrak{J}_\varepsilon, \quad \mathfrak{F}_\varepsilon := \varepsilon^{-1}(I - \mathfrak{J}_\varepsilon) \tag{2.58}$$

be the resolvent of $\mathfrak{F}$ and the Yosida approximations of ϕ and $\mathfrak{F}$ respectively. They satisfy [9, p. 28, Thm. 2.2, and Prop. 2.11]

$$\forall\, v \in \overline{D(\phi)}: \quad \mathfrak{J}_\varepsilon(v) \to v, \quad \phi_\varepsilon(v) \uparrow \phi(v), \quad \text{as } \varepsilon \downarrow 0,$$

as well as, since $0 \in \mathfrak{F}(w)$, $\mathfrak{J}_\varepsilon(w) \equiv w$, $\mathfrak{F}_\varepsilon(w) \equiv 0 = \mathfrak{F}(w)^\circ$ for every $\varepsilon > 0$. Since $\mathfrak{F}_\varepsilon$ is continuous and is the (everywhere defined) Fréchet differential of ϕ_ε, we have as before

$$\phi_\varepsilon(v) - \phi_\varepsilon(w) = \int_0^1 \tfrac{1}{s}\langle \mathfrak{F}_\varepsilon(w_s), w_s - w\rangle \, ds. \tag{2.59}$$

Since $w_s = \mathfrak{J}_\varepsilon(w_s) + \varepsilon\mathfrak{F}_\varepsilon(w_s)$ and $\mathfrak{F}_\varepsilon(w_s) \in \mathfrak{F}(\mathfrak{J}_\varepsilon(w_s))$, (2.45) and (2.55) yield

$$\langle \mathfrak{F}_\varepsilon(w_s), w_s - w\rangle = \langle \mathfrak{F}_\varepsilon(w_s), \mathfrak{J}_\varepsilon(w_s) - w\rangle + \varepsilon|\mathfrak{F}_\varepsilon(w_s))|^2 \geq [\mathfrak{J}_\varepsilon(w_s) - w]^p.$$

We note that $w_s \in D(\phi)$ because $D(\phi)$ is convex, whence $\mathfrak{J}_\varepsilon(w_s) \to w_s$ as $\varepsilon \downarrow 0$. Inserting this inequality into (2.59), recalling the limit properties of ϕ_ε as $\varepsilon \downarrow 0$, and invoking the lower semicontinuity of $[\cdot]$ and Fatou's Lemma, we obtain

$$\begin{aligned}\phi(v) - \phi(w) &= \lim_{\varepsilon\downarrow 0} \big(\phi_\varepsilon(v) - \phi_\varepsilon(w)\big)\\ &\geq \int_0^1 \tfrac{1}{s} \liminf_{\varepsilon\downarrow 0} [\mathfrak{J}_\varepsilon(w_s) - w]^p\, ds \geq \int_0^1 \tfrac{1}{s}[s(v-w)]^p\, ds. \quad \blacksquare\end{aligned}$$

Corollary 1 *Up to the factor $1/p$, we can replace $\varrho(\cdot,\cdot)$ by the corresponding $\sigma(\cdot;\cdot)$ in the estimates* (2.50), (2.51), (2.52), (2.53), *and* (2.54). $\blacksquare$

Remark 2.9 The notion of p-coercivity as well as Lemma 2.8 can be *localized*: if we know that (2.55) holds for every w, v in $D(\mathfrak{F}) \cap K$, K being a convex set where ϕ is differentiable along segments, then (2.56) holds for every $w, v \in D(\mathfrak{F}) \cap K$. The same is true if K satisfies the *compatibility* condition

$$\mathfrak{J}_\varepsilon(K) \subset K \quad \forall\, \varepsilon > 0 \text{ sufficiently small},$$

where $\mathfrak{J}_\varepsilon$ is the resolvent of $\mathfrak{F}$ defined in (2.58). A simple example (which will turn out to be useful in the application to the p-Laplacian (2.33) with $1 < p < 2$) is provided by the sublevels of ϕ

$$K_R := \{v \in \mathcal{H} : \phi(v) \leq R\},$$

which are compatible with respect to $\mathfrak{F}$, since $\phi(\mathfrak{J}_\varepsilon(v)) = \phi_\varepsilon(v) \leq \phi(v)$. $\square$

The case of the Stefan problem (2.39) and (2.41) does not fit into Lemma 2.8. However, since $0 \leq \beta'(r) \leq 1$ and $\mathcal{H} = H^{-1}(\Omega)$, we easily get

$$\begin{aligned}\varrho(w, v) &= \langle -\Delta(\beta(v) - \beta(w)), v - w\rangle = \int_\Omega \big(\beta(w) - \beta(v)\big)(w - v)\, dx\\ &\geq \int_\Omega \big|\beta(w) - \beta(v)\big|^2\, dx = [\mathfrak{F}(w) - \mathfrak{F}(v)]^2 \quad \forall\, w, v \in D(\mathfrak{F}),\end{aligned}$$

provided $[w] := \|Gw\|_{L^2(\Omega)}$ for all $w \in H^{-1}(\Omega)$ and G is defined in Example 2.5. We would like to obtain an analogous bound for $\sigma(w; v)$, which will be important in deriving error estimates for temperature $\beta(u)$. This property can be recovered from the following general dual result.

Lemma 2.10 *Let $[\cdot]$ be a l.s.c. seminorm on $\mathcal{H}$ such that for a given $p \geq 2$*

$$\varrho(w, v) \geq [\mathfrak{F}(w) - \mathfrak{F}(v)]^p \quad \forall\, w, v \in D(\mathfrak{F}). \tag{2.60}$$

Then we have

$$\sigma(w; v) \geq \tfrac{1}{p}[\mathfrak{F}(w) - \mathfrak{F}(v)]^p \quad \forall\, w, v \in D(\mathfrak{F}). \tag{2.61}$$

PROOF. Let us denote by ϕ_* the l.s.c. conjugate convex function

$$\phi_*(w_*) := \sup_{v\in D(\phi)} \big(\langle w_*, v\rangle - \phi(v)\big).$$

Its subdifferential $\mathfrak{F}_*$ turns out to be the inverse of $\mathfrak{F}$ [9, p. 41], i.e.

$$\forall\, w, w_* \in \mathcal{H}: \quad w_* \in \mathfrak{F}(w) \quad \Leftrightarrow \quad w \in \mathfrak{F}_*(w_*).$$

Therefore, from (2.2) we have $\langle w_*, v\rangle - \phi(v) \le \langle w_*, w\rangle - \phi(w)$, whence

$$\phi_*(w_*) = \langle w_*, w\rangle - \phi(w) \quad \forall\, w_* \in D(\mathfrak{F}_*),\ \forall\, w \in \mathfrak{F}_*(w_*). \tag{2.62}$$

We note that (2.60) can be equivalently written as

$$\langle w_* - v_*, \mathfrak{F}_*(w_*) - \mathfrak{F}_*(v_*)\rangle \ge [w_* - v_*]^p \quad \forall\, w_*, v_* \in D(\mathfrak{F}_*),$$

and therefore the associated ϱ_* to $\mathfrak{F}_*$ satisfies

$$\varrho_*(w_*, v_*) \ge [w_* - v_*]^p \quad \forall\, w_*, v_* \in D(\mathfrak{F}_*).$$

By Lemma 2.8 and (2.44) we deduce

$$\phi_*(w_*) - \phi_*(v_*) - \langle v, w_* - v_*\rangle \ge \tfrac{1}{p}[w_* - v_*]^p \quad \forall\, w_*, v_* \in D(\mathfrak{F}_*),\ \forall\, v \in \mathfrak{F}_*(v_*).$$

Using (2.62) for v and w we get (2.61). ∎

Lemma 2.10 does not yet apply to the Stefan problem (2.41), because we need $v \in L^2(\Omega)$ instead of $v \in D(\mathfrak{F})$. Our next result assesses this issue.

Lemma 2.11 *Let β satisfy $0 \le \beta' \le 1$ and let $\mathfrak{F}(\cdot) := -\Delta\beta(\cdot)$. Then*

$$\sigma(w; v) \ge \tfrac{1}{2}\|\beta(w) - \beta(v)\|_{L^2(\Omega)} \quad \forall\, w \in D(\mathfrak{F}),\ \forall\, v \in L^2(\Omega).$$

PROOF. Applying Lemma 2.10 with $p = 2$ and $[w] := \|Gw\|_{L^2(\Omega)}$, we deduce

$$\sigma(w; v) \ge \tfrac{1}{2}[\mathfrak{F}(w) - \mathfrak{F}(v)]^2 = \tfrac{1}{2}\|\beta(w) - \beta(v)\|^2_{L^2(\Omega)} \quad \forall\, w, v \in D(\mathfrak{F}).$$

To show that this relation holds for all $v \in L^2(\Omega)$, we employ the Yosida approximation $v_\varepsilon := (I + \varepsilon\mathfrak{F})^{-1}v \in D(\mathfrak{F})$ of v, which satisfies $v_\varepsilon \to v$ in $H^{-1}(\Omega)$. On multiplying $v_\varepsilon - \varepsilon\Delta\beta(v_\varepsilon) = v$ by v_ε, we deduce $\|v_\varepsilon\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)}$ because $-\langle\Delta\beta(v_\varepsilon), v_\varepsilon\rangle \ge 0$ even for v_ε merely in $L^2(\Omega)$ [12, Lemma 2]. Since $\|v_\varepsilon - v\|^2_{L^2(\Omega)} = \|v_\varepsilon\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} - 2\langle v_\varepsilon, v\rangle$, we infer the strong convergence

$$v_\varepsilon \to v, \quad \beta(v_\varepsilon) \to \beta(v), \quad \text{in } L^2(\Omega),$$

the latter being a consequence of the former. Hence we conclude that

$$\begin{aligned}
\sigma(w; v) &= \phi(w) - \phi(v) - \int_\Omega \beta(w)(v - w)\,dx \\
&= \lim_{\varepsilon\to 0}\Big(\phi(w) - \phi(v_\varepsilon) - \int_\Omega \beta(w)(v_\varepsilon - w)\,dx\Big) = \lim_{\varepsilon\to 0}\sigma(w; v_\varepsilon) \\
&\ge \limsup_{\varepsilon\to 0} \tfrac{1}{2}\|\beta(w) - \beta(v_\varepsilon)\|^2_{L^2(\Omega)} = \tfrac{1}{2}\|\beta(w) - \beta(v)\|^2_{L^2(\Omega)}. \quad ∎
\end{aligned}$$

3 The Subdifferential Case

Notation. Let $\mathcal{P}$ be the partition (1.2) of the time interval $[0, T]$ and set

$$\tau_n := t_n - t_{n-1}, \quad \tau := \max_{1 \le n \le N} \tau_n.$$

For any sequence $\{W_n\}_{n=0}^N$ we define the piecewise constant functions $\bar{W}$, $\underline{W}$ and the piecewise linear interpolant W on the intervals $(t_{n-1}, t_n]$ as

$$\bar{W}(t) := W_n, \quad \underline{W}(t) := W_{n-1}, \quad W(t) := (1 - \ell(t))W_n + \ell(t)W_{n-1}, \qquad (3.1)$$

for $1 \le n \le N$, where

$$\ell(t) := \frac{t_n - t}{\tau_n}.$$

In particular $\bar{\tau}(t)$ takes the value τ_n on $(t_{n-1}, t_n]$. We also denote by $\{\delta W_n\}_{n=1}^N$ the discrete derivative of the sequence $\{W_n\}_{n=0}^N$

$$\delta W_n := \frac{W_n - W_{n-1}}{\tau_n} = \frac{\bar{W}(t) - \underline{W}(t)}{\bar{\tau}} = W'(t) \quad \forall\, t \in (t_{n-1}, t_n], \qquad (3.2)$$

and by $\delta\phi(W_n)$, $\delta\mathfrak{F}(W_n)$, and $\delta^2 W_n$ the discrete derivatives of the sequences $\{\phi(W_n)\}_{n=0}^N$, $\{\mathfrak{F}(W_n)\}_{n=0}^N$, and $\{\delta W_n\}_{n=0}^N$ for $1 \le n \le N$ respectively, the latter provided δW_0 is defined.

The Discretization Scheme. Let $U_0 \in \overline{D(\phi)}$ and $\{F_n\}_{n=1}^N \subset \mathcal{H}$ be suitable approximations of the initial datum u_0 and of the function f over $\{(t_{n-1}, t_n]\}_{n=1}^N$ respectively. The discrete solution $\{U_n\}_{n=1}^N \subset D(\phi)$ is defined recursively by solving at every step $1 \le n \le N$ the variational inequality

$$\langle \delta U_n - F_n, U_n - v\rangle + \phi(U_n) - \phi(v) \le 0 \quad \forall\, v \in D(\phi). \qquad (3.3)$$

Inequality (3.3) is related to the minimum problem for the functional

$$W \in D(\phi) \mapsto \Phi_n(W; U_{n-1}) := \tfrac{1}{2}\, \tau_n^{-1} |W - U_{n-1}|^2 + \phi(W) - \langle F_n, W\rangle.$$

In fact, once U_{n-1} is given, U_n is a solution of (3.3) if and only if it minimizes $\Phi_n(W; U_{n-1})$, i.e.

$$\Phi_n(U_n; U_{n-1}) = \min_{W \in \mathcal{H}} \Phi_n(W; U_{n-1}). \qquad (3.4)$$

Thanks to the lower semicontinuity and convexity of ϕ, (3.4) admits a unique solution U_n. In view of (2.2), we infer that (3.3) is equivalent to the variational inclusion

$$\delta U_n + \mathfrak{F}(U_n) \ni F_n, \qquad (3.5)$$

and thus $U_n \in D(\mathfrak{F})$ for all $1 \le n \le N$, irrespective of the choice of U_0. Moreover, (2.48) yields the following *stronger* variational inequality for U_n

$$\langle \delta U_n - F_n, U_n - v\rangle + \phi(U_n) - \phi(v) + \sigma(U_n; v) \le 0 \quad \forall\, v \in D(\phi). \qquad (3.6)$$

In particular, if $U_0 \in D(\mathfrak{F})$ and F_0 and τ_0 are defined, we set (see (2.5))

$$U_{-1} := U_0 - \tau_0 (F_0 - \mathfrak{F}(U_0))^\circ, \quad \delta U_0 := \frac{U_0 - U_{-1}}{\tau_0} = (F_0 - \mathfrak{F}(U_0))^\circ, \qquad (3.7)$$

so that (3.5) and (3.6) hold also for $n = 0$.

3.1 A Posteriori Error Analysis

Energy Dissipation and the Main A Posteriori Estimate. We now introduce a fundamental quantity, the amount of discrete *energy dissipation.*

Definition 3.1 *To every discrete solution* $\{U_n\}_{n=0}^N$ *of* (3.3) *we associate the error estimators*

$$\mathcal{E}_n := \langle F_n - \delta U_n, \delta U_n\rangle - \delta\phi(U_n) \quad \forall\, 2 \le n \le N; \tag{3.8}$$

$\mathcal{E}_1$ *is also defined provided* U_0 *belongs to* $D(\phi)$. □

These quantities are nonnegative (see Lemma 3.3 below), are computable, and will play a significant role in estimating the error $u - U$ *a posteriori.* Moreover, they are also strictly related to the variational formulation (3.4), because

$$\mathcal{E}_n = \tau_n^{-1}\big(\Phi_n(U_{n-1}; U_{n-1}) - \Phi_n(U_n; U_{n-1})\big) - \tfrac{1}{2}|\delta U_n|^2,$$

so that $\mathcal{E}_n$ measures the *discrete speed of decay* of the functional Φ_n.

Let $E := \max(E_{\mathcal{H}}, E_\sigma)$ be the error to be bounded and $E_{\mathcal{H}}, E_\sigma$ be given by

$$\begin{aligned} E_{\mathcal{H}} &:= \max_{t\in[0,T]} |u(t) - U(t)|,\\ E_\sigma &:= \Big(2\int_0^T \big(\sigma(u(t); U(t)) + \sigma(\bar U(t); u(t))\big)\,dt\Big)^{1/2}. \end{aligned} \tag{3.9}$$

Theorem 1 *Let* u *be the* strong *solution of* (2.4)*, let* $\{U_n\}_{n=0}^N$ *be the discrete solution of* (3.3) *with* $U_0 \in D(\phi)$*, and let* E *and* $\{\mathcal{E}_n\}_{n=1}^N$ *be defined in* (3.9) *and* (3.8)*. The following* a posteriori *error estimate holds*

$$E \le \Big(|u_0 - U_0|^2 + \sum_{n=1}^N \tau_n^2\mathcal{E}_n\Big)^{1/2} + \|f - \bar F\|_{L^1(0,T;\mathcal{H})}. \tag{3.10}$$

Remark 3.2 Theorem 1 is valid for *weak* solutions u provided σ in (3.9) is replaced by any function $\sigma_- : \overline{D(\phi)} \times \overline{D(\phi)} \to [0, +\infty]$ which is l.s.c. w.r.t. each argument separately and is dominated by σ on $D(\mathfrak{F}) \times D(\phi)$, i.e.

$$\sigma_-(w; v) \le \sigma(w, v) \quad \forall\, w \in D(\mathfrak{F}),\, \forall\, v \in D(\phi).$$

In fact, weak solutions are by definition uniform limits of the strong solutions u_k corresponding to approximations f_k of f in $L^1(0,T;\mathcal{H})$ (see Definition 2.1 and the Basic Existence Results of § 2). In addition, (3.10) is stable w.r.t. the uniform convergence of u_k and $L^1(0,T;\mathcal{H})$ convergence of f_k.
In particular, if $\mathfrak{F}$ is p-coercive w.r.t. a l.s.c. seminorm $[\cdot]$ as in Definition 2.7, by Lemma 2.8 it is possible to choose $\sigma_-(w; v) := [w - v]^p/p$. □

We start the proof of Theorem 1 by establishing a simple relation between $\mathcal{E}_n$ and $\sigma(U_n; U_{n-1})$, namely the discrete counterpart of (2.6).

Lemma 3.3 *Whenever it is defined, the energy dissipation quantity* $\mathcal{E}_n$ *satisfies*

$$\tau_n^{-1}\sigma(U_n; U_{n-1}) \le \mathcal{E}_n. \tag{3.11}$$

PROOF. Choosing $v = U_{n-1}$ in (3.6) we get

$$\tau_n\langle \delta U_n - F_n, \delta U_n\rangle + \tau_n \delta\phi(U_n) + \sigma(U_n; U_{n-1}) \leq 0, \tag{3.12}$$

which is (3.11) in disguise, namely $\sigma(U_n; U_{n-1}) \leq \tau_n \mathcal{E}_n$. ■

Now we derive the continuous evolution variational inequalities satisfied by the discrete solution U, $\bar{U}$ and by the error $u - U$.

Lemma 3.4 *Let $\{U_n\}_{n=0}^N$ be the discrete solution of (3.3) with $U_0 \in D(\phi)$. The interpolants U and $\bar{U}$ of $\{U_n\}_{n=0}^N$ defined in (3.1) satisfy a.e. in $(0,T)$*

$$\langle U' - \bar{F}, U - v\rangle + \phi(U) - \phi(v) + \sigma(\bar{U}; v) \leq \ell\bar{\tau}\bar{\mathcal{E}} \quad \forall\, v \in D(\phi). \tag{3.13}$$

PROOF. In view of (3.1) and (3.2) we can write the stronger discrete variational inequality (3.6) a.e. in $(0,T)$ in the form

$$\langle U' - \bar{F}, U - v\rangle + \phi(U) - \phi(v) + \sigma(\bar{U}; v) \leq \mathcal{R} \quad \forall\, v \in D(\phi),$$

where

$$\mathcal{R} := \langle U' - \bar{F}, U - \bar{U}\rangle + \phi(U) - \phi(\bar{U}).$$

Recalling again (3.1) and (3.2), we readily have the elementary identity

$$U - \bar{U} = -\ell(\bar{U} - \underline{U}) = -\ell\bar{\tau}U', \tag{3.14}$$

and, by the convexity of ϕ,

$$\phi(U) - \phi(\bar{U}) \leq (1-\ell)\phi(\bar{U}) + \ell\phi(\underline{U}) - \phi(\bar{U}) = -\ell\big(\phi(\bar{U}) - \phi(\underline{U})\big).$$

Therefore, we have $\mathcal{R} \leq -\ell\tau_n\big(\langle\delta U_n - F_n, \delta U_n\rangle + \delta\phi(U_n)\big)$ in every interval $(t_{n-1}, t_n]$ and the assertion follows from the definition (3.8) of $\mathcal{E}_n$. ■

The following error inequality is an immediate consequence of Lemma 3.4.

Lemma 3.5 *Let $U_0 \in D(\phi)$. The error $u - U$ satisfies a.e. in $(0,T)$*

$$\tfrac{1}{2}\tfrac{d}{dt}|u-U|^2 + \sigma(u; U) + \sigma(\bar{U}; u) \leq \ell\bar{\tau}\bar{\mathcal{E}} + \langle f - \bar{F}, u - U\rangle. \tag{3.15}$$

PROOF. We choose $v = U$ in (2.49) and $v = u$ in (3.13). Adding the two inequalities, we readily get (3.15). ■

Note that assuming $U_0 \in D(\phi)$ guarantees the validity of both Lemmas 3.4 and 3.5 all the way to $t = 0$; otherwise they would only hold on (τ_1, T), because $\mathcal{E}_n$ in (3.8) would be defined and (3.11) would hold only for $2 \leq n \leq N$.

Note also that $\phi(u)$ does not appear explicitly in the error equation (3.15), which is a key observation for arriving to *a posteriori* error estimates.

PROOF OF THEOREM 1. This is a consequence of (3.15) and the following Gronwall-like Lemma 3.6 (see also [4, Thm. 3.4]) with $\lambda := 0$ and

$$a := |u - U|, \;\; b^2 := 2\big(\sigma(u; U) + \sigma(\bar{U}; u)\big), \;\; c^2 := 2\ell\bar{\tau}\bar{\mathcal{E}}, \;\; d := |f - \bar{F}|. \;\; ■$$

Lemma 3.6 *Let $a, b, c, d : (0,T) \to [0, +\infty]$ be measurable functions, a^2 being also absolutely continuous on $[0,T]$. We assume they satisfy the differential inequality, depending on the paramenter $\lambda \in \mathbb{R}$,*

$$\tfrac{d}{dt} a^2(t) + b^2(t) + 2\lambda a^2(t) \le c^2(t) + 2d(t)a(t) \quad \textit{a.e. in } (0,T).$$

Then we have

$$\begin{aligned} \max\Big(\max_{t\in[0,T]} e^{\lambda t} a(t), \Big(\int_0^T e^{2\lambda s} b^2(s)\, ds\Big)^{1/2}\Big) \\ \le \Big(a^2(0) + \int_0^T e^{2\lambda t} c^2(t)\, dt\Big)^{1/2} + \int_0^T e^{\lambda t} d(t)\, dt. \end{aligned} \tag{3.16}$$

Proof. It is not restrictive to assume that c^2, d and therefore also b^2 are integrable on $(0,T)$. We introduce the absolutely continuous functions

$$e(t) := e^{2\lambda t} a^2(t) + \int_0^t e^{2\lambda s} b^2(s)\, ds,$$

$$r(t) := \Big(\Big(a^2(0) + \int_0^t e^{2\lambda s} c^2(s)\, ds\Big)^{1/2} + \int_0^t e^{\lambda s} d(s)\, ds\Big)^2,$$

and observe that they satisfy the opposite differential inequalities a.e. in $(0,T)$

$$e'(t) \le c^2(t) + 2d(t)\sqrt{e(t)}, \quad r'(t) \ge c^2(t) + 2d(t)\sqrt{r(t)}.$$

Since $e(0) = r(0)$, by a standard comparison argument for differential inequalities, it follows that $e(t) \le r(t)$ for all $t \in [0,T]$, which yields (3.16). ∎

Remark 3.7 The estimate (3.16) seems to be almost optimal: if $b = c \equiv 0$, $\lambda = 0$, and $0 \le d \in L^1(0,T)$, then (3.16) gives a sharp bound for the maximum of the function $a(t) := a_0 + \int_0^t d(s)\, ds$ with $a_0 \ge 0$, which obviously satisfies the ODE $\frac{d}{dt} a^2(t) = 2d(t)a(t)$ a.e. $t \in (0,T)$. If $d \equiv 0$, then (3.10) with $f \equiv 0$ coincides with the result (1.16), directly shown in the Introduction. □

Dissipation Inequalities and Residual Monotone Terms. When $\mathfrak{F}$ is single-valued we obtain a characterization of (3.10) in terms of $\sigma(U_n; U_{n-1})$ or, provided that $U_0 \in D(\mathfrak{F})$, in terms of $\varrho(U_n, U_{n-1})$. We recall from (2.47) that

$$\varrho(U_n, U_{n-1}) = \langle \mathfrak{F}(U_n) - \mathfrak{F}(U_{n-1}), U_n - U_{n-1}\rangle.$$

Corollary 2 *If $U_0 \in D(\phi)$ and $\mathfrak{F}$ is* single-valued *then $\mathcal{E}_n = \tau_n^{-1}\sigma(U_n; U_{n-1})$ and, if $U_0 \in D(\mathfrak{F})$, the error E defined in (3.9) is bounded by*

$$\begin{aligned} E \le \Big(|u_0 - U_0|^2 + \sum_{n=1}^N \tau_n \langle \mathfrak{F}(U_n) - \mathfrak{F}(U_{n-1}), U_n - U_{n-1}\rangle\Big)^{1/2} \\ + \|f - \bar F\|_{L^1(0,T;\mathcal{H})}. \end{aligned} \tag{3.17}$$

Proof. In view of (2.46) and (3.5), inequality (3.11) is in fact an identity provided $\mathfrak{F}$ is single-valued; therefore $U_0 \in D(\mathfrak{F})$ and (2.45) yield

$$\tau_n \mathcal{E}_n = \sigma(U_n; U_{n-1}) \le \varrho(U_n, U_{n-1}). \quad ∎$$

When $\mathfrak{F}$ is not single-valued, Corollary 2 cannot be applied, because it might happen $\tau_n\mathcal{E}_n > \sigma(U_n; U_{n-1})$. In addition, there seems to be an indeterminacy for the choice of a suitable element in $\mathfrak{F}(U_n)$ and $\mathfrak{F}(U_{n-1})$ in (3.17). However, (3.5) suggests a canonical way to perform such a choice, that is

$$F_n - \delta U_n \in \mathfrak{F}(U_n). \tag{3.18}$$

Therefore it is natural to replace the residual quantity

$$\langle \mathfrak{F}(U_n) - \mathfrak{F}(U_{n-1}), U_n - U_{n-1} \rangle \tag{3.19}$$

in (3.17) with

$$\langle (F_n - \delta U_n) - (F_{n-1} - \delta U_{n-1}), U_n - U_{n-1} \rangle = \tau_n^2 \langle \delta F_n - \delta^2 U_n, \delta U_n \rangle.$$

Due to the crucial role of this quantity in the subsequent estimates, we collect in the following definition the previous arguments.

Definition 3.8 *To every discrete solution* $\{U_n\}_{n=0}^N$ *of* (3.5) *we associate the error estimators*

$$\mathcal{D}_n := \tau_n \langle \delta F_n - \delta^2 U_n, \delta U_n \rangle \quad \forall\, 2 \le n \le N; \tag{3.20}$$

$\mathcal{D}_1$ *is also defined provided* $U_0 \in D(\mathfrak{F})$ *and* δU_0 *is given by* (3.7). $\square$

Since $U_n \in D(\mathfrak{F})$ for all $1 \le n \le N$, then $\mathcal{D}_n$ is well defined for $2 \le n \le N$, irrespective of the regularity of U_0.

In view of (3.18), (3.19), and (2.45), we note that $\mathcal{D}_n$ satisfies $\mathcal{D}_n \ge 0$ and

$$\tau_n^{-1} \varrho(U_n, U_{n-1}) \le \mathcal{D}_n, \tag{3.21}$$

whenever it is defined, and that (3.21) is an identity provided $\mathfrak{F}$ is single-valued. A discrete counterpart of the dissipation inequality (2.7) follows from $\mathcal{D}_n \ge 0$

$$\tfrac{1}{2}\delta |\delta U_n|^2 \le \langle \delta^2 U_n, \delta U_n \rangle \le \langle \delta F_n, \delta U_n \rangle.$$

The estimators $\mathcal{E}_n$ and $\mathcal{D}_n$ are related by an interesting property, which yields a generalization of Corollary 2.

Lemma 3.9 *If* $U_0 \in D(\mathfrak{F})$, *then*

$$\mathcal{E}_n \le \mathcal{D}_n \quad \forall\, 1 \le n \le N. \tag{3.22}$$

Proof. We write (3.6) for the unknown U_{n-1}, making use of (3.7) for $n = 1$, and choose $v = U_n$ as a test function to arrive at

$$-\tau_n \delta \phi(U_n) \le \tau_n \langle \delta U_{n-1} - F_{n-1}, \delta U_n \rangle - \sigma(U_{n-1}; U_n).$$

Replacing this inequality into (3.12) yields

$$\tau_n \mathcal{E}_n \le \tau_n^2 \langle \delta F_n - \delta^2 U_n, \delta U_n \rangle - \sigma(U_{n-1}; U_n) = \tau_n \mathcal{D}_n - \sigma(U_{n-1}; U_n). \quad \blacksquare$$

Notice that $U_0 \in D(\mathfrak{F})$ guarantees the validity of both inequalities (3.21) and (3.22) for all $1 \le n \le N$; otherwise they would only hold for $2 \le n \le N$.

Corollary 3 *If $U_0 \in D(\mathfrak{F})$, then the error E defined in* (3.9) *is bounded by*

$$E \le \Big(|u_0 - U_0|^2 + \sum_{n=1}^{N} \tau_n^2 \mathcal{D}_n \Big)^{1/2} + \|f - \bar{F}\|_{L^1(0,T;\mathcal{H})}. \qquad (3.23)$$

Remark 3.10 When $\mathfrak{F}$ is 2-coercive w.r.t. a seminorm $[\cdot]$ as in (2.55), then we can give an alternative *a posteriori* error bound, which allows for a more general functional setting for the source term f; similar results are valid for $p > 2$. We will explore this idea in § 4.5 within the context of a classical *Hilbert triplet.* □

Exponential Decay. If $\mathfrak{F}$ is also 2-*coercive* w.r.t. an equivalent norm of $\mathcal{H}$, i.e. $\mathfrak{F}$ satisfies

$$\exists\, \alpha > 0 : \quad \varrho(w, v) \ge 2\alpha |w - v|^2 \quad \forall\, w, v \in D(\mathfrak{F}), \qquad (3.24)$$

we gain an exponential decay of the various errors.

Theorem 2 *Let $\mathfrak{F}$ satisfy* (3.24) *and $U_0 \in D(\phi)$. The final error is bounded by*

$$\begin{aligned} |u(T) - U(T)| \le \Big(\mathrm{e}^{-2\alpha T} |u_0 - U_0|^2 + \sum_{n=1}^{N} \mathrm{e}^{-2\alpha(T - t_n)} \tau_n^2 \mathcal{E}_n \Big)^{1/2} \\ + \int_0^T \mathrm{e}^{-\alpha(T-t)} |f(t) - \bar{F}(t)|\, dt. \end{aligned} \qquad (3.25)$$

Proof. We first notice that (3.15) and Lemma 2.8 imply

$$\tfrac{d}{dt} |u - U|^2 + 2\alpha |u - U|^2 \le 2\ell \bar{\tau} \bar{\mathcal{E}} + 2|f - \bar{F}||u - U|.$$

The asserted estimate (3.25) then follows from Lemma 3.6 with

$$a := |u - U|, \quad b := 0, \quad c^2 := 2\ell\bar{\tau}\bar{\mathcal{E}}, \quad d := |f - \bar{F}|, \quad \lambda := \alpha. \quad ■$$

3.2 Rate of Convergence and Regularity

In this section we demonstrate that the *a posteriori* estimators introduced in § 3.1 converge to zero with an optimal rate, and involve minimal regularity. No constraint between consecutive time-steps is required.

Energy Solutions. First we show that the *a posteriori* estimator $\sum_{n=1}^N \tau_n^2 \mathcal{E}_n$ in (3.10) possesses an order $O(\sqrt{\tau})$ under minimal regularity assumptions on the discrete data U_0 and F. This, in turn, yields an optimal $O(\sqrt{\tau})$ *a priori* rate of convergence for the error E provided $u_0 \in D(\phi)$ and $f \in L^2(0, T; \mathcal{H})$, which ensures the existence of a global energy solution.

Theorem 3 *If $U_0 \in D(\phi)$ and $\phi \ge 0$, then the discrete estimators $\mathcal{E}_n$ defined in* (3.8) *satisfy*

$$\sum_{n=1}^{N} \tau_n \mathcal{E}_n \le \phi(U_0) + \tfrac{1}{4} \|\bar{F}\|^2_{L^2(0,T;\mathcal{H})}. \qquad (3.26)$$

Proof. Multiplying (3.8) by τ_n and adding from $n = 1$ to N, we readily get

$$\sum_{n=1}^{N} \tau_n \mathcal{E}_n \le \phi(U_0) - \phi(U_N) + \int_0^T \langle \bar{F}, U' \rangle \, dt - \|U'\|^2_{L^2(0,T;\mathcal{H})}. \tag{3.27}$$

The positivity of $\phi(U_N)$ and the Cauchy inequality yield the assertion. ■

Out next task is to establish a uniform estimate for the error E when $f \in L^2(0,T;\mathcal{H})$, even though such a regularity does not give any order of convergence for $\|f - \bar{F}\|_{L^1(0,T;\mathcal{H})}$ in (3.10). The stability constant may be further improved.

Theorem 4 *If $U_0 := u_0 \in D(\phi)$, $F_n := \tau_n^{-1} \int_{t_{n-1}}^{t_n} f(t)\, dt$ for $1 \le n \le N$ and $F_0 := 0$, with $f \in L^2(0,T;\mathcal{H})$, and $\phi \ge 0$, then the solutions u of (2.4) and $\{U_n\}_{n=0}^N$ of (3.3) satisfy the following uniform estimate*

$$E \le \left(3\tau\big(\phi(u_0) + \|f\|^2_{L^2(0,T;\mathcal{H})}\big)\right)^{1/2}, \tag{3.28}$$

where E is the global error defined in (3.9).

Proof. In view of (3.10), (3.26), and the positivity of $\mathcal{E}_n$, to prove (3.28) it would remain to bound $\|f - \bar{F}\|_{L^1(0,T;\mathcal{H})}$. Since the sole regularity $f \in L^2(0,T;\mathcal{H})$ does not give any order of convergence for this term, we thus resort to the error equation (3.15) and argue as follows, for all $1 \le n \le N$. Being the L^2 projection of f in time over the piecewise constants, $\bar{F}$ satisfies

$$\|f - \bar{F}\|^2_{L^2(0,t_n;\mathcal{H})} + \|\bar{F}\|^2_{L^2(0,t_n;\mathcal{H})} = \|f\|^2_{L^2(0,t_n;\mathcal{H})}. \tag{3.29}$$

Let $\bar{R}(w)$ be the piecewise average of w over $\mathcal{P}$, namely $R_n := \tau_n^{-1} \int_{t_{n-1}}^{t_n} w(t)\, dt$. A simple but tedious calculation yields

$$\begin{aligned}
\|u - U - \bar{R}(u-U)\|_{L^2(0,t_n;\mathcal{H})} &\le \|u - \bar{R}(u)\|_{L^2(0,t_n;\mathcal{H})} + \|U - \bar{R}(U)\|_{L^2(0,t_n;\mathcal{H})} \\
&\le \tfrac{1}{\sqrt{6}} \tau \|u'\|_{L^2(0,t_n;\mathcal{H})} + \tfrac{1}{2\sqrt{3}} \tau \|U'\|_{L^2(0,t_n;\mathcal{H})}.
\end{aligned}$$

On integrating the error equation (3.15) over $(0,t_n)$ and using L^2 orthogonality in time, that is $\int_{t_{n-1}}^{t_n} \langle f - \bar{F}, \bar{R}(u-U) \rangle = 0$, and the Cauchy inequality, we get from (3.27) and the positivity of $\mathcal{E}_n$

$$\begin{aligned}
e(t_n) &:= |u(t_n) - U_n|^2 + 2\int_0^{t_n} \big(\sigma(u;U) + \sigma(\bar{U};u)\big)\, dt \\
&\le \int_0^{t_n} 2\ell(t)\bar{\tau}(t)\bar{\mathcal{E}}(t)\, dt + 2\|f - \bar{F}\|_{L^2(0,t_n;\mathcal{H})} \|u - U - \bar{R}(u-U)\|_{L^2(0,t_n;\mathcal{H})} \\
&\le \tau \Big(\phi(u_0) - \phi(U_n) - \|U'\|^2_{L^2(0,t_n;\mathcal{H})} + \|\bar{F}\|_{L^2(0,t_n;\mathcal{H})} \|U'\|_{L^2(0,t_n;\mathcal{H})} \\
&\quad + \tfrac{\sqrt{2}}{\sqrt{3}} \|f - \bar{F}\|_{L^2(0,t_n;\mathcal{H})} \|u'\|_{L^2(0,t_n;\mathcal{H})} \\
&\quad + \tfrac{1}{\sqrt{3}} \|f - \bar{F}\|_{L^2(0,t_n;\mathcal{H})} \|U'\|_{L^2(0,t_n;\mathcal{H})} \Big).
\end{aligned}$$

On using (3.29) and the elementary inequality

$$2ab \le \varepsilon a^2 + b^2/\varepsilon \quad \forall\, a, b \in \mathbb{R},\ \varepsilon > 0, \tag{3.30}$$

with $\varepsilon = 1$ on the last three terms, we arrive at

$$e(t_n) \leq \tau\big(\phi(u_0) - \phi(U_n) + \tfrac{1}{2}\|f\|^2_{L^2(0,t_n;\mathcal{H})} + \tfrac{1}{2}\|u'\|^2_{L^2(0,t_n;\mathcal{H})}\big).$$

To prove the $H^1(0,T;\mathcal{H})$ stability of u we integrate (2.6) in time and use the Cauchy inequality and the positivity of ϕ to obtain

$$\|u'\|^2_{L^2(0,t_n;\mathcal{H})} \leq \|f\|^2_{L^2(0,t_n;\mathcal{H})} + 2\phi(u_0).$$

Therefore we get

$$e(t_n) \leq \tau\big(2\phi(u_0) - \phi(U_n) + \|f\|^2_{L^2(0,t_n;\mathcal{H})}\big). \tag{3.31}$$

To complete the proof of (3.28), it remains to estimate the error $|u(t) - U(t)|$ for all $t \in (t_{n-1}, t_n)$. To this end, we apply (3.10) from t_{n-1} to t_n, and use (3.27), (3.30) with $\varepsilon = 1/2$, and the Cauchy inequality to obtain

$$\begin{aligned}\max_{t\in(t_{n-1},t_n)} |u(t) - U(t)|^2 &\leq \Big(\big(e(t_{n-1}) + \tau_n^2\mathcal{E}_n\big)^{1/2} + \int_{t_{n-1}}^{t_n} |f - F_n|\,dt\Big)^2 \\ &\leq \tfrac{3}{2}\Big(e(t_{n-1}) + \tau\big(\phi(U_{n-1}) - \phi(U_n) + \tfrac{1}{4}\tau_n|F_n|^2\big)\Big) + 3\tau_n \int_{t_{n-1}}^{t_n} |f - F_n|^2\,dt.\end{aligned}$$

In view of (3.29) and (3.31) with t_{n-1} in place of t_n, we finally conclude

$$\max_{t\in(t_{n-1},t_n)} |u(t) - U(t)|^2 \leq 3\tau\big(\phi(u_0) - \tfrac{1}{2}\phi(U_n) + \|f\|^2_{L^2(0,t_n;\mathcal{H})}\big). \quad \blacksquare$$

Strong Lipschitz Solutions. The estimate (3.23) turns out to be useful for proving a linear rate of convergence $O(\tau)$ for the error E provided $u_0 \in D(\mathfrak{F})$ and $f \in BV(0,T;\mathcal{H})$. To this end, we need a discrete counterpart of Lemma 3.6, which yields precise stability constants for a discrete system of inequalitites (see also [4, Thm. 3.3]); its proof is a discrete version of that of Lemma 3.6.

Lemma 3.11 *Let $\{a_n\}_{n=0}^N$ and $\{b_n, c_n, d_n\}_{n=1}^N$ be nonnegative numbers. If*

$$2a_n(a_n - a_{n-1}) + b_n^2 \leq c_n^2 + 2a_n d_n \quad \forall\, 1 \leq n \leq N, \tag{3.32}$$

then

$$\max_{1\leq n\leq N} a_n \leq \Big(a_0^2 + \sum_{n=1}^N c_n^2\Big)^{1/2} + \sum_{n=1}^N d_n, \tag{3.33}$$

$$\Big(\sum_{n=1}^N b_n^2\Big)^{1/2} \leq \Big(a_0^2 + \sum_{n=1}^N c_n^2\Big)^{1/2} + \sqrt{2}\sum_{n=1}^N d_n. \tag{3.34}$$

Proof. We set $R_0 := a_0$ and introduce the quantities for $1 \leq n \leq N$

$$C_n := \Big(a_0^2 + \sum_{k=1}^n c_k^2\Big)^{1/2}, \quad D_n := \sum_{k=1}^n d_k, \quad R_n := C_n + D_n.$$

We first prove the inequality $a_n \leq R_n$ for all $1 \leq n \leq N$, which yields (3.33). From (3.32) we obviously deduce the following quadratic inequality for a_n

$$a_n^2 - (a_{n-1} + d_n)a_n \leq \tfrac{1}{2}c_n^2.$$

This, together with $a_n \geq 0$, implies

$$2a_n \leq a_{n-1} + d_n + \big((a_{n-1} + d_n)^2 + 2c_n^2\big)^{1/2}. \tag{3.35}$$

It is not difficult to check that R_n satisfies the quadratic inequality

$$R_n^2 - (R_{n-1} + d_n)R_n = R_n(C_n - C_{n-1}) \geq C_n(C_n - C_{n-1}) \geq \tfrac{1}{2}c_n^2,$$

which, in view of $R_n \geq 0$, leads to

$$2R_n \geq R_{n-1} + d_n + \big((R_{n-1} + d_n)^2 + 2c_n^2\big)^{1/2}. \tag{3.36}$$

With the aid of (3.35), (3.36), and $a_0 = R_0$, a standard induction argument gives rise to the desired inequality $a_n \leq R_n$ for $1 \leq n \leq N$. To prove (3.34), we notice that (3.32) together with $a_n \leq R_n \leq R_N$ implies

$$b_n^2 \leq a_{n-1}^2 - a_n^2 + c_n^2 + 2R_N d_n,$$

which, upon summation from $n = 1$ to N, yields

$$\sum_{n=1}^{N} b_n^2 \leq C_N^2 + 2R_N D_N \leq (C_N + \sqrt{2}\, D_N)^2. \quad \blacksquare$$

Theorem 5 below shows how to control the second estimator $\sum_{n=1}^{N} \tau_n^2 \mathcal{D}_n$ in (3.23), and Corollary 6 expresses such bounds in terms of data.

Theorem 5 *If $U_0 \in D(\mathfrak{F})$, the discrete estimators $\mathcal{D}_n$ defined in* (3.20) *satisfy*

$$\Big(\sum_{n=1}^{N} \mathcal{D}_n\Big)^{1/2} \leq \tfrac{1}{\sqrt{2}}|\delta U_0| + \operatorname{Var} \bar{F}. \tag{3.37}$$

Proof. From (3.20) and the Cauchy inequality we deduce

$$|\delta U_n|\,\big(|\delta U_n| - |\delta U_{n-1}|\big) + \mathcal{D}_n \leq \tau_n |\delta F_n|\,|\delta U_n|. \tag{3.38}$$

Since $\mathcal{D}_n \geq 0$ from (3.21), applying Lemma 3.11 with

$$a_n := |\delta U_n|, \quad b_n^2 := 2\mathcal{D}_n, \quad c_n := 0, \quad d_n := |F_n - F_{n-1}|,$$

the assertion (3.37) follows from the relation $\operatorname{Var} \bar{F} = \sum_{n=1}^{N} \tau_n |\delta F_n|$. $\blacksquare$

Remark 3.12 It is interesting to notice that the Definition 3.8 of $\mathcal{D}_n$ and the proof of Theorem 5 do not rely on the subdifferential structure of $\mathfrak{F}$, but only on its monotonicity. This observation will be crucial in § 4 in dealing with the more general class of angle-bounded operators. The proof of Theorem 6, instead, is based on Corollary 3 and thus on the existence of a convex potential. $\square$

Theorem 6 *If $U_0 := u_0 \in D(\mathfrak{F})$ and $F_n := f_+(t_n)$, with $f \in BV(0,T;\mathcal{H})$, the solutions u of* (2.4) *and $\{U_n\}_{n=0}^N$ of* (3.3) *satisfy the following uniform estimate*

$$E \leq \tau\big(\tfrac{1}{\sqrt{2}}|(f_+(0) - \mathfrak{F}(u_0))^\circ| + 2\operatorname{Var} f\big), \tag{3.39}$$

where E is the global error defined in (3.9).

Proof. In view of Corollary 3 and Theorem 5, the estimate (3.39) follows from (3.7) and the properties $\|f - \bar{F}\|_{L^1(0,T;\mathcal{H})} \leq \tau \operatorname{Var} f$, and $\operatorname{Var} \bar{F} \leq \operatorname{Var} f$. $\blacksquare$

4 The Non-Potential Case

In this section we study (maximal) monotone operators $\mathfrak{F}$ which are *not* the subgradient of a (convex) functional. We are driven by the validity of Theorem 5 irrespective of the existence of a potential ϕ (see Remark 3.12). On the other hand, the hyperbolic-like counterexample of Rulla [53, Ex. 3] rules out the possibility of deriving a linear order of convergence $O(\tau)$ for monotone operators $\mathfrak{F}$ without further restrictive properties.

In order to understand what kind of (hopefully minimal) abstract assumptions are reasonable on $\mathfrak{F}$, we rewrite the continuous and discrete problems (2.3) and (3.5) as equalities, for a suitable selection in $\mathfrak{F}(u)$ and $\mathfrak{F}(\bar{U})$, namely,

$$u' + \mathfrak{F}(u) = f, \quad U' + \mathfrak{F}(\bar{U}) = \bar{F}.$$

Since we are interested in the error between u and U, it is natural to multiply the difference of the two equations by $u - U$, and use (3.1) to obtain

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}|u-U|^2 &\le \langle \mathfrak{F}(\bar{U}) - \mathfrak{F}(u), u - U\rangle + \langle f - \bar{F}, u - U\rangle \\ &\le (1-\ell)\langle \mathfrak{F}(\bar{U}) - \mathfrak{F}(u), u - \bar{U}\rangle + \ell\langle \mathfrak{F}(\bar{U}) - \mathfrak{F}(u), u - \underline{U}\rangle \\ &\quad + \langle f - \bar{F}, u - U\rangle. \end{aligned} \tag{4.1}$$

Since the first term on the righ-hand side is ≤ 0 by virtue of the monotonicity of $\mathfrak{F}$, we are thus led to investigate the residual term

$$\langle \mathfrak{F}(\bar{U}) - \mathfrak{F}(u), u - \underline{U}\rangle.$$

4.1 Angle-bounded Operators

We recall here the definition of *angle-bounded* operators, which was introduced by Brézis and Browder [10, § 1] in a completely different context, but turns out to be intimately related to the question of optimal approximability.

Definition 4.1 *A (multi-valued) operator $\mathfrak{F}$ from $\mathcal{H}$ to $2^{\mathcal{H}}$ is said to be angle-bounded (or γ^2-angle-bounded) if there exists a positive constant γ such that*

$$\langle \mathfrak{F}(v) - \mathfrak{F}(w), w - z\rangle \le \gamma^2 \langle \mathfrak{F}(v) - \mathfrak{F}(z), v - z\rangle \quad \forall\, v, w, z \in D(\mathfrak{F}). \tag{4.2}$$

As usual, (4.2) must hold for every selection in $\mathfrak{F}(v), \mathfrak{F}(w), \mathfrak{F}(z)$, the same for $\mathfrak{F}(v)$ on both sides of the inequality. □

Note that angle-bounded operators are monotone: take $z = v$ in (4.2). In many respects, angle-boundedness seems to be the right intermediate condition between monotonicity and existence of a convex potential. We elaborate upon this point further via several examples.

Example 4.1: Subdifferentials. We first check that if $\mathfrak{F}$ is the subdifferential of a convex potential ϕ, then it is angle-bounded with $\gamma = 1$. Since

$$\begin{aligned} \langle \mathfrak{F}(v) - \mathfrak{F}(w), w - z\rangle &= \langle \mathfrak{F}(v) - \mathfrak{F}(z), v - z\rangle \\ &\quad + \langle \mathfrak{F}(v), w - v\rangle + \langle \mathfrak{F}(w), z - w\rangle + \langle \mathfrak{F}(z), v - z\rangle, \end{aligned} \tag{4.3}$$

from (2.48) we deduce that, for every $v, w, z \in D(\mathfrak{F})$,

$$\begin{aligned} \langle \mathfrak{F}(v) - \mathfrak{F}(w), w - z\rangle &\le \langle \mathfrak{F}(v) - \mathfrak{F}(z), v - z\rangle \\ &\quad - \sigma(v; w) - \sigma(w; z) - \sigma(z; v) \le \langle \mathfrak{F}(v) - \mathfrak{F}(z), v - z\rangle. \end{aligned} \tag{4.4}$$

The class of angle-bounded operators is thus larger than that of subdifferentials.

Example 4.2: k-Cyclically-Monotone Operators. From (4.3) we see that a necessary and sufficient condition for 1-angle-boundedness is the property

$$\langle \mathfrak{F}(v), w - v\rangle + \langle \mathfrak{F}(w), z - w\rangle + \langle \mathfrak{F}(z), v - z\rangle \le 0 \quad \forall\, v, w, z \in D(\mathfrak{F}).$$

This class of operators is the case $k = 3$ of the k-cyclically-monotone operators, introduced by Rockafellar [51], [9, Chap. II, § 7]. Following his definition, we say that $\mathfrak{F}$ is *k-cyclically-monotone* iff for every choice of $v_1, v_2, \ldots, v_k \in D(\mathfrak{F})$

$$\langle \mathfrak{F}(v_1), v_2 - v_1\rangle + \langle \mathfrak{F}(v_2), v_3 - v_2\rangle + \ldots + \langle \mathfrak{F}(v_k), v_1 - v_k\rangle \le 0,$$

and that $\mathfrak{F}$ is *cyclically-monotone* if it is k-cyclically-monotone for every $k \in \mathbb{N}$. Since every k-cyclically-monotone operator, with $k \ge 3$, is also 3-cyclically-monotone, it is also 1-angle-bounded. Monotone operators correspond to 2-ciclically-monotone operators, which are not necessarily γ-angle-bounded. The most important link with the theory of subdifferentials of (l.s.c.) convex functions lies in their precise characterization as the (maximal) cyclically-monotone operators [51].
The correspondence between angle-bounded and cyclically-monotone operators is more transparent and complete provided $\mathfrak{F} = \mathfrak{A}$ is linear. Asplund [2] [3] proved that $\mathfrak{A}$ is k-cyclically-monotone for $k \ge 3$ iff $\mathfrak{A}$ is γ_k^2-angle-bounded with $2\gamma_k \cos(\pi/k) = 1$ (the 3-cyclically-monotone maps thus correspond to an angle of $\pi/3$ and $\gamma = 1$). For nontrivial linear operators, $1/2$ is thus the least admissible value of γ, and corresponds to symmetric positive semidefinite operators.

Example 4.3: Linear Nonsymmetric Operators.

Sectorial operators. If $\mathfrak{F} = \mathfrak{A}$ is linear, then Definition 4.1 is equivalent to the *strong sector condition*

$$|\langle \mathfrak{A}v, w\rangle|^2 \le 4\gamma^2 \langle \mathfrak{A}v, v\rangle\langle \mathfrak{A}w, w\rangle \quad \forall\, v, w \in D(\mathfrak{A}). \tag{4.5}$$

To see this, we set $\tilde{v} = v - z$ and $\tilde{w} = w - z$ in (4.2) to get the equivalent formulation (we omit the tildas)

$$\langle \mathfrak{A}v, w\rangle \le \gamma^2 \langle \mathfrak{A}v, v\rangle + \langle \mathfrak{A}w, w\rangle \quad \forall\, v, w \in D(\mathfrak{A}). \tag{4.6}$$

Then we replace v by rv, $r \in \mathbb{R}$, and argue with the resulting quadratic inequality in r to realize that (4.5) is equivalent to (4.6). In addition, (4.5) can be rephrased in terms of the antisymmetric part of $\mathfrak{A}$ as follows (see [1])

$$|\langle \mathfrak{A}v, w\rangle - \langle \mathfrak{A}w, v\rangle| \le 2\mu \langle \mathfrak{A}v, v\rangle^{1/2} \langle \mathfrak{A}w, w\rangle^{1/2} \le \mu\big(\langle \mathfrak{A}v, v\rangle + \langle \mathfrak{A}w, w\rangle\big), \tag{4.7}$$

for a suitable $\mu \ge 0$: it turns out [10, Prop. 11] that both (4.5) and (4.7) are equivalent with $\gamma^2 = (\mu^2 + 1)/4$.

Bilinear forms. Every linear operator $\mathfrak{A}$, maximal monotone and strongly sectorial in the sense of (4.5) or (4.7), can be associated via (2.15) to a bilinear form $\mathfrak{a}$ defined in a Hilbert space $\mathcal{V}$ satisfying (2.14) and (2.13), respectively [45]. In this case, (4.7) becomes the natural generalization of (2.17)

$$\exists\, \mu \ge 0: \quad \mathfrak{a}_a(v, w) \le \mu\, \mathfrak{a}(v, v)^{1/2} \mathfrak{a}(w, w)^{1/2} \quad \forall\, v, w \in \mathcal{V}, \tag{4.8}$$

where $\mathfrak{a}_a$ is the antisymmetric part of $\mathfrak{a}$

$$\mathfrak{a}_a(v,w) := \tfrac{1}{2}\big(\mathfrak{a}(v,w) - \mathfrak{a}(w,v)\big) \quad \forall\, v,w \in \mathcal{V}.$$

Such kind of bilinear forms obviously comprises the *strongly coercive* ones

$$\exists\, \alpha > 0 : \quad [v]^2 = \mathfrak{a}(v,v) \geq \alpha \|v\|_{\mathcal{V}}^2 \quad \forall\, v \in \mathcal{V}.$$

Second Order Advection-Diffusion Problems. A significant example of sectorial bilinear form in $\mathcal{H} := L^2(\Omega)$ and $\mathcal{V} := H^1(\Omega)$ is given by

$$\mathfrak{a}(v,w) := \int_\Omega \big(\nu^2\, \nabla v \cdot \nabla w + \boldsymbol{b} \cdot \nabla v\, w + c\, vw\big)\, dx \quad \forall\, v,w \in \mathcal{V}, \tag{4.9}$$

where $\nu^2 > 0$, $\boldsymbol{b}$ is a Lipschitz vector field defined in $\bar{\Omega}$, $c \in L^\infty(\Omega)$, and

$$-\tfrac{1}{2}\,\mathrm{div}\, \boldsymbol{b}(x) + c(x) \geq 0 \quad \text{a.e. } x \in \Omega, \qquad \boldsymbol{b} \cdot \boldsymbol{n} \geq 0 \quad \text{on } \partial\Omega.$$

In this case

$$\begin{aligned} \mathfrak{A} w &:= -\nu^2\, \Delta w + \boldsymbol{b} \cdot \nabla w + c\, w, \\ D(\mathfrak{A}) &:= \{ v \in H^1(\Omega) : \Delta v \in L^2(\Omega),\ \partial_{\boldsymbol{n}} v = 0 \text{ on } \partial\Omega \}, \end{aligned}$$

and (2.16) reads

$$\partial_t u - \nu^2\, \Delta u + \boldsymbol{b} \cdot \nabla u + c\, u = f \quad \text{a.e. in } Q, \quad \partial_{\boldsymbol{n}} u = 0 \quad \text{on } \Sigma. \tag{4.10}$$

The case of a nonhomogeneous Neumann condition $\partial_{\boldsymbol{n}} u = g(x,t)$ is developed in § 5.2. This requires Hilbert triplets, which are studied in §4.5.

Example 4.4: Abstract Variational Inequalities. Let $\mathcal{V}, \mathfrak{a}, \psi$ satisfy (2.13), (2.14), and (2.23), along with (4.8) instead of (2.17). The evolution variational inequality (2.24) still makes sense as well as the associated maximal monotone operator $\mathfrak{F}$ defined by (2.25). We will prove in Lemma 4.11 that

$$\mathfrak{F} \text{ is } \gamma^2\text{-angle-bounded with } \gamma^2 := \max\big(1, \tfrac{1}{4}(1+\mu^2)\big), \tag{4.11}$$

as well as $\mathfrak{F} = \mathfrak{A} + \partial\psi$ provided $\mathfrak{A}$ in (2.15) and ψ in (2.23) satisfy the compatibility condition (2.26).

Example 4.5: Nonlinear Variational Boundary Conditions. We consider the PDE of (4.10), but with another lateral boundary condition

$$\partial_t u - \nu^2\, \Delta u + \boldsymbol{b} \cdot \nabla u + c\, u = f \quad \text{in } Q, \quad \partial_{\boldsymbol{n}} u + \beta(u) \ni 0 \quad \text{on } \Sigma, \tag{4.12}$$

where $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone real graph, which is thus [9, Ch. 2, § 8] the subdifferential of a proper l.s.c. convex function $j : \mathbb{R} \to (-\infty, +\infty]$. Let us observe that when β is a continuous function, or equivalently j is differentiable, the lateral boundary condition in (4.12) is no longer a differential inclusion, but a simpler nonlinear equation: in particular, $\beta(r) \equiv -g \in \mathbb{R}$, corresponds to the usual *Neumann* boundary condition

$$\partial_{\boldsymbol{n}} u = g \quad \text{on } \Sigma.$$

On the other hand the choice (see [35, § 3.2.2])

$$\beta(r) := \begin{cases} (-\infty, +\infty) & \text{if } r = g, \\ \varnothing & \text{if } r \neq g, \end{cases}$$

corresponding to $j := I_{\{g\}}$, is associated to the *Dirichlet* boundary condition

$$u = g \quad \text{on } \Sigma,$$

whereas

$$\beta(r) := \begin{cases} \varnothing & \text{if } r < g, \\ (-\infty, 0] & \text{if } r = g, \\ \{0\} & \text{if } r > g, \end{cases}$$

corresponding to $j := I_{[g,+\infty)}$, gives rise to the (ambiguous) *Signorini* boundary condition (see e.g. [6, Ex. 7.9], [43, Ch. 2, Ex. 9.4], [34, Ch. 6, § 2.1.1])

$$u \geq g, \quad \partial_{\boldsymbol{n}} u \geq 0, \quad \partial_{\boldsymbol{n}} u\,(u - g) = 0, \quad \text{on } \Sigma.$$

Upon introducing the convex functional

$$\psi(w) := \int_{\partial\Omega} j(w(x))\, dS, \quad D(\psi) := \{v \in H^1(\Omega) : j(v) \in L^1(\partial\Omega)\}, \tag{4.13}$$

which is l.s.c. w.r.t. $\mathcal{V} := H^1(\Omega)$, but *not* w.r.t. $\mathcal{H} := L^2(\Omega)$, (4.12) admits the variational formulation (2.24). The related nonlinear operator $\mathfrak{F}$, defined in

$$D(\mathfrak{F}) := \{v \in H^1(\Omega) \cap D(\psi) : \Delta v \in L^2(\Omega),\ \partial_{\boldsymbol{n}} v + \beta(v) \ni 0 \text{ on } \partial\Omega\},$$

cannot be directly expressed in terms of $\mathfrak{A}$ and the subdifferential of ψ. To do so, we need the Hilbert triplets of §4.5.

Example 4.6: Angle-Bounded Vector Fields. This example is a direct generalization of Example 2.4. We consider (2.32) again, where now the Carathéodory vector field $\boldsymbol{a} : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is not a gradient but is 2-coercive and angle-bounded instead: there exists positive constants $\alpha_0, \alpha_1, \alpha_2, \gamma$ such that for every choice of $x \in \Omega$, $\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathbb{R}^m$

$$\begin{gathered} \alpha_0|\boldsymbol{\xi}|^2 - \alpha_1 \leq \boldsymbol{a}(x, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \leq \alpha_2(1 + |\boldsymbol{\xi}|^2), \\ (\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\eta} - \boldsymbol{\zeta}) \leq \gamma^2 (\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\zeta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\zeta}). \end{gathered}$$

It is then easy to see that the operator (compare with (2.31))

$$\mathfrak{F}(w(\cdot)) := -\operatorname{div} \boldsymbol{a}(\cdot, \nabla w(\cdot)), \quad D(\mathfrak{F}) := \{v \in H^1_0(\Omega) : \operatorname{div} \boldsymbol{a}(\cdot, \nabla v(\cdot)) \in L^2(\Omega)\},$$

is γ^2-angle-bounded in $\mathcal{H} := L^2(\Omega)$.

Remark 4.2 The concept of angle-boundedness for a linear operator $\mathfrak{A}$ in a real Hilbert space is strictly related to the concept of *sectorial* operator introduced by Kato [39, Chap. VI] in the complex framework. This complex structure shows, in particular, that the contraction semigroup generated by a *linear* angle-bounded operator is analytic (for a "real" proof, see [3]), and therefore has a regularizing effect on the initial data as in the case of a symmetric operator (and of a subdifferential, in the nonlinear case). We do not know whether *nonlinear* angle-bounded operators exhibit a regularizing effect. □

4.2 Coercivity of an Angle-Bounded Operator $\mathfrak{F}$.

We now define and study a measure of the coercivity of an angle-bounded operator $\mathfrak{F}$, which plays a role analogous to σ for subdifferentials; see § 2.3. For every $v, w, z \in D(\mathfrak{F})$ and $v_* \in \mathfrak{F}(v), w_* \in \mathfrak{F}(w), z_* \in \mathfrak{F}(z)$, we set

$$\sigma_\gamma(v, w, z) := \inf_{v_*, w_*, z_*} \left(\gamma^2 \langle v_* - z_*, v - z\rangle - \langle v_* - w_*, w - z\rangle\right). \tag{4.14}$$

Note that, choosing $z = v$, we have $\sigma_\gamma(v, w, v) = \varrho(w, v)$ for all $w, v \in D(\mathfrak{F})$.

We observe that if $\mathfrak{F}$ is a subdifferential, then (4.4) yields

$$\sigma_1(v, w, z) \geq \sigma(v; w) + \sigma(w; z) + \sigma(z; v) \quad \forall\, v, w, z \in D(\mathfrak{F}).$$

Consequently, if $\mathfrak{F}$ is a subdifferential and is also 2-coercive w.r.t. a seminorm $[\cdot]$ as in (2.55), then Lemma 2.8 gives

$$\sigma_1(v, w, z) \geq \tfrac{1}{2}[v - w]^2 + \tfrac{1}{2}[w - z]^2 + \tfrac{1}{2}[z - v]^2.$$

A similar estimate holds for coercive operators which satisfy a suitable dual Lipschitz condition. To motivate this condition, let us consider a linear angle-bounded operator $\mathfrak{A}$ and use the characterization (4.5). If we denote by $[\cdot]$ the l.s.c. extension of the Hilbert seminorm on $D(\mathfrak{A})$ and by $[\cdot]_*$ its conjugate

$$[w] := \langle \mathfrak{A}w, w\rangle^{1/2} \quad \forall\, w \in D(\mathfrak{A}), \tag{4.15}$$

$$[w]_* := \sup_{v \in D(\mathfrak{A}):\, [v] \leq 1} \langle w, v\rangle \quad \forall\, w \in D(\mathfrak{A}), \tag{4.16}$$

then (4.5) is equivalent to $\mathfrak{A}$ being 2-coercive w.r.t. $[\cdot]$ (given by (4.15)) and Lipschitz w.r.t. the conjugate norm (4.16)

$$[\mathfrak{A}(v - w)]_* \leq 2\gamma[v - w] \quad \forall\, v, w \in D(\mathfrak{A}).$$

It is important to notice that a similar simple result holds also in the nonlinear framework; other interesting properties can be found in [3].

Lemma 4.3 *Let $\mathfrak{F} : D(\mathfrak{F}) \to \mathcal{H}$ be 2-coercive w.r.t. a l.s.c. seminorm $[\cdot]$ as in (2.55) and let $\mathfrak{F}$ satisfy the dual Lipschitz-type condition*

$$\exists\, L > 0 : \quad [\mathfrak{F}(v) - \mathfrak{F}(w)]_* \leq L[v - w] \quad \forall\, v, w \in D(\mathfrak{F}). \tag{4.17}$$

Then $\mathfrak{F}$ is γ^2-angle-bounded for $\gamma := L/2$ and, for every $\eta \geq L/\sqrt{2}$ $(= \sqrt{2}\gamma)$, we have the extra coercivity-type *estimate*

$$\sigma_\eta(v, w, z) \geq \tfrac{1}{2}\max\left([v - w]^2, [w - z]^2\right) \quad \forall\, v, w, z \in D(\mathfrak{F}). \tag{4.18}$$

Proof. We get from (2.55) and (4.17) with straightforward calculations

$$\begin{aligned}\langle \mathfrak{F}(v) - \mathfrak{F}(w), w - z\rangle &= \langle \mathfrak{F}(v) - \mathfrak{F}(w), w - v\rangle + \langle \mathfrak{F}(v) - \mathfrak{F}(w), v - z\rangle \\ &\leq -[v - w]^2 + L[v - w]\,[v - z] \\ &\leq \tfrac{L^2}{4}[v - z]^2 \leq \tfrac{L^2}{4}\langle \mathfrak{F}(v) - \mathfrak{F}(z), v - z\rangle.\end{aligned}$$

This shows that $\mathfrak{F}$ is γ^2-angle-bounded with $\gamma = L/2$. We can modify slightly this derivation to obtain

$$\begin{aligned}\langle \mathfrak{F}(v) - \mathfrak{F}(w), w - z\rangle &\le -\tfrac{1}{2}[v-w]^2 + \tfrac{L^2}{2}[v-z]^2 \\ &\le -\tfrac{1}{2}[v-w]^2 + \tfrac{L^2}{2}\langle \mathfrak{F}(v) - \mathfrak{F}(z), v - z\rangle.\end{aligned}$$

Likewise, a symmetric argument yields

$$\begin{aligned}\langle \mathfrak{F}(v) - \mathfrak{F}(w), w - z\rangle &= \langle \mathfrak{F}(z) - \mathfrak{F}(w), w - z\rangle + \langle \mathfrak{F}(v) - \mathfrak{F}(z), w - z\rangle \\ &\le -[w-z]^2 + L[v-z]\,[w-z] \\ &\le -\tfrac{1}{2}[w-z]^2 + \tfrac{L^2}{2}\langle \mathfrak{F}(v) - \mathfrak{F}(z), v - z\rangle.\end{aligned}$$

Combining the last two inequalities we get (4.18). ■

Remark 4.4 If $\mathfrak{A}$ is linear and γ^2-angle-bounded, we can apply Lemma 4.3 and deduce the following coercivity estimate with respect to the induced seminorm $[\cdot]$ in (4.15), for every $\eta \ge \sqrt{2}\gamma$,

$$\sigma_\eta(v, w, z) \ge \tfrac{1}{2}\max\big([v-w], [w-z]\big) \quad \forall\, v, w, z \in D(\mathfrak{A}). \quad \square$$

Remark 4.5 It is easy to see that the sum of two γ_i^2-angle-bounded operators is still angle-bounded with $\gamma^2 := \max(\gamma_1^2, \gamma_2^2)$, and the resulting coercivity function σ_γ defined in (4.14) satisfies $\sigma_\gamma \ge \sigma_{\gamma_1} + \sigma_{\gamma_1}$. The same holds true for the nonlinear operators associated to (nonsymmetric) variational inequalities such as those in Example 4.4, but the proof is as follows. If ψ satisfies (2.23), then $\partial\psi : \mathcal{V} \to 2^{\mathcal{V}^*}$ is maximal monotone. At the same time, if (2.14) and (4.8) are valid, then the associated (linear) operator $\mathfrak{A} : \mathcal{V} \to \mathcal{V}^*$ is Lipschitz and coercive. The sum $\mathfrak{F}_\mathcal{V} = \mathfrak{A} + \partial\psi : \mathcal{V} \to 2^{\mathcal{V}^*}$ is thus maximal monotone and given by

$$w_* \in \mathfrak{F}_\mathcal{V}(w) \quad \Leftrightarrow \quad \mathfrak{a}(w, w-v) + \psi(w) \le \psi(v) + \langle w_*, w-v\rangle \ \ \forall\, v \in D(\psi); \qquad (4.19)$$

compare with (2.25). We infer that $\mathfrak{A} + \partial\psi$ is γ^2-angle-bounded, with $\gamma^2 = \max(1, (1+\mu^2)/4)$, because so is each summand with $\gamma^2 = (1+\mu^2)/4$ and $\gamma^2 = 1$ respectively (see Examples 4.1 and 4.3). Likewise we can argue with the coercivity function. Finally, if $\mathfrak{F} : \mathcal{H} \to 2^\mathcal{H}$ is the restriction of $\mathfrak{F}_\mathcal{V}$ to $\mathcal{H}$, namely,

$$\mathfrak{F}(w) := \mathfrak{F}_\mathcal{V}(w) \cap \mathcal{H}, \quad D(\mathfrak{F}) \subset D(\mathfrak{F}_\mathcal{V}) \subset D(\psi) \subset \mathcal{V}. \qquad (4.20)$$

then $\mathfrak{F}$ is still γ^2-angle-bounded with the same γ and coercivity function as $\mathfrak{F}_\mathcal{V}$, as well as maximal; we just remove certain pairs (w, w_*) in the graph of $\mathfrak{F}_\mathcal{V}$. Consequently, passing to the duality $\mathcal{V} - \mathcal{V}^*$ renders the above calculation feasible, but does not imply that $\mathfrak{F}$ is the sum of the restrictions of $\mathfrak{A}$ and $\partial\psi$ to $\mathcal{H}$. The latter is true provided the compatibility condition (2.26) holds. These properties may be useful in studying Lipschitz and coercive perturbations of subdifferential operators in the sense of (4.17) and (2.55). □

Remark 4.6 All the examples and remarks of this subsection deal only with the *algebraic* aspect of monotonicity and angle-boundedness. Of course, the related applications to evolution equations also require the fundamental *maximality* assumption (2.1). It is interesting to observe that a *linear* angle-bounded operator is maximal iff it (its graph) is *closed* and its domain is *dense*. On the contrary, a deeper investigation is required to check the *maximality* (2.1) of the sum of two angle-bounded operators, which in general is not a direct consequence of the maximality of each addendum (see e.g. [9, Ch. II, §§ 6 and 9]). For instance, if $\mathfrak{A}$ and ψ satisfy (2.13), (2.14), (2.15), (2.23), and (4.8), then a sufficient condition for $\mathfrak{F} := \mathfrak{A} + \partial\psi$ to be maximal monotone is the compatibility condition (2.26) [9, Prop.2.17]. □

4.3 Error Estimates

We observed in Remark 3.12 that the discrete residual (3.19) still makes sense in the non-potential framework, provided we select (3.18) in $\mathfrak{F}(U_n)$ and $\mathfrak{F}(U_{n-1})$, and yields the estimator $\mathcal{D}_n$ defined in (3.20). Moreover, these estimators $\mathcal{D}_n$ satisfy Theorem 5. We explore now this idea and show both *a posteriori* and uniform *a priori* error estimates similar to those in Corollary 3 and Theorem 6.

We define the global error as in (3.9), by replacing E_σ with a corresponding term arising from σ_γ. Therefore we set $E_\gamma := \max(E_{\mathcal{H}}, E_{\sigma_\gamma})$, with $E_{\mathcal{H}}$ defined in (3.9) and

$$E_{\sigma_\gamma} := \Big(2\int_0^T \big((1-\ell)\varrho(\bar U, u) + \ell\sigma_\gamma(\bar U, u, \underline{U})\big)\, dt\Big)^{1/2}. \tag{4.21}$$

Theorem 7 *Let u be the strong solution of (2.3) and $\{U_n\}_{n=0}^N$ be the solution of (3.5) with $U_0 \in D(\mathfrak{F})$. If $\mathfrak{F}$ is γ^2-angle-bounded, then*

$$E_\gamma \le \Big(|u_0 - U_0|^2 + \gamma^2 \sum_{n=1}^N \tau_n^2 \mathcal{D}_n\Big)^{1/2} + \|f - \bar F\|_{L^1(0,T;\mathcal{H})}. \tag{4.22}$$

Moreover, Theorem 5 holds, i.e.

$$\sum_{n=1}^N \tau_n^2 \mathcal{D}_n \le \tau^2 \big(\tfrac{1}{\sqrt 2}|\delta U_0| + \operatorname{Var} \bar F\big)^2,$$

and, if $U_0 := u_0$ and $F_n := f_+(t_n)$, with $f \in BV(0,T;\mathcal{H})$, we have

$$E_\gamma \le \tau\big(\tfrac{\gamma}{\sqrt 2}|(f_+(0) - \mathfrak{F}(u_0))^\circ| + (1+\gamma)\operatorname{Var} f\big). \tag{4.23}$$

Proof. In view of (4.14) and the fact that $\mathfrak{F}$ is γ^2-angle-bounded, we have

$$\langle \mathfrak{F}(\bar U) - \mathfrak{F}(u), u - \underline{U}\rangle + \sigma_\gamma(\bar U, u, \underline{U}) \le \gamma^2 \langle \mathfrak{F}(\bar U) - \mathfrak{F}(\underline{U}), \bar U - \underline{U}\rangle.$$

On using (2.45), from (4.1) we deduce

$$\begin{aligned}
\tfrac12 \tfrac{d}{dt}|u - U|^2 &+ (1-\ell)\varrho(\bar U, u) + \ell\sigma_\gamma(\bar U, u, \underline{U}) \\
&\le |f - \bar F|\,|u - U| + \ell\gamma^2\langle \mathfrak{F}(\bar U) - \mathfrak{F}(\underline{U}), \bar U - \underline{U}\rangle \\
&= |f - \bar F|\,|u - U| + \gamma^2 \ell \bar\tau \bar{\mathcal{D}}.
\end{aligned} \tag{4.24}$$

Applying Lemma 3.6 with

$$\begin{gathered}
a := |u - U|, \quad b^2 := 2\big((1-\ell)\varrho(\bar U, u) + \ell\sigma_\gamma(\bar U, u, \underline{U})\big), \\
c^2 := \gamma^2 2\ell\bar\tau\bar{\mathcal{D}}, \quad d := |f - \bar F|, \quad \lambda := 0,
\end{gathered}$$

we prove (4.22). Finally, (4.23) follows from Theorem 5 as in Theorem 6. ■

Remark 4.7 The *a posteriori* estimate (4.22) is valid also for weak solutions (see Remark 3.2). $\square$

We conclude this section by showing that, if $\mathfrak{F}$ is a γ^2-angle-bounded operator satisfying a finer bound like (4.18), then we can obtain a more precise result. So we assume that

$$\exists\, \eta \geq \gamma : \quad \sigma_\eta(v,w,z) \geq \tfrac{1}{2} \max\left([v-w]^2, [w-z]^2\right) \quad \forall\, v,w,z \in D(\mathfrak{F}); \tag{4.25}$$

choosing $z = v$ implies $\varrho(w,v) \geq [v-w]^2/2$ (compare with the usual 2-coercivity (2.55)). It follows from (4.21) that

$$E_{[\,]} := \Big(\max\Big(\int_0^T [u(t) - U(t)]^2\, dt,\ \int_0^T [u(t) - \bar{U}(t)]^2\, dt \Big)\Big)^{1/2} \leq E_{\sigma_\eta}. \tag{4.26}$$

In fact, (4.25), (2.55), and the convexity of $[\cdot]$ in conjunction with (3.1) yield

$$\begin{aligned} 2(1-\ell)\varrho(\bar{U},u) &+ 2\ell\sigma_\eta(\bar{U},u,\underline{U}) \\ &\geq (1-\ell)[u-\bar{U}]^2 + \ell \max\left([u-\bar{U}]^2, [u-\underline{U}]^2\right) \\ &= \max\left([u-\bar{U}]^2, (1-\ell)[u-\bar{U}]^2 + \ell[u-\underline{U}]^2\right) \\ &\geq \max\left([u-\bar{U}]^2, [u-U]^2\right). \end{aligned} \tag{4.27}$$

Corollary 4 *Let u be a* weak *solution of* (2.3) *and $\{U_n\}_{n=0}^N$ be the solution of* (3.5) *with $U_0 \in D(\mathfrak{F})$. If $\mathfrak{F}$ is γ^2-angle-bounded and coercive in the sense* (4.25), *then the error $E_{[\,]}$ defined in* (4.26) *satisfies*

$$E_{[\,]} \leq \Big(|u_0 - U_0|^2 + \eta^2 \sum_{n=1}^N \tau_n^2 \mathcal{D}_n \Big)^{1/2} + \|f - \bar{F}\|_{L^1(0,T;\mathcal{H})}. \tag{4.28}$$

Proof. Since $\eta \geq \gamma$ and so $\mathfrak{F}$ is η^2-angle-bounded, we use (4.26) together with (4.22) to prove (4.28) for strong solutions. In view of the lower semicontinuity of the seminorm $[\cdot]$, (4.28) can also be extended to weak solutions. $\blacksquare$

Corollary 5 *Let $\mathfrak{A}$ be a* linear *closed and densily defined operator satisfying* (4.5), *and let $[\cdot]$ be the Hilbert seminorm* (4.15). *If u is the weak solution of* (2.3) *and $\{U_n\}_{n=0}^N$ is the solution of* (3.5) *with $U_0 \in D(\mathfrak{A})$, then*

$$E_{\mathcal{H}} \leq \Big(|u_0 - U_0|^2 + \gamma^2 \sum_{n=1}^N \tau_n [U_n - U_{n-1}]^2 \Big)^{1/2} + \|f - \bar{F}\|_{L^1(0,T;\mathcal{H})}, \tag{4.29}$$

$$E_{[\,]} \leq \Big(|u_0 - U_0|^2 + 2\gamma^2 \sum_{n=1}^N \tau_n [U_n - U_{n-1}]^2 \Big)^{1/2} + \|f - \bar{F}\|_{L^1(0,T;\mathcal{H})}. \tag{4.30}$$

Moreover, Theorem 5 holds, i.e.

$$\sum_{n=1}^N \tau_n [U_n - U_{n-1}]^2 \leq \tau^2 \big(\tfrac{1}{\sqrt{2}} |\delta U_0| + \operatorname{Var} \bar{F} \big)^2, \tag{4.31}$$

and, if $U_0 := u_0$ and $F_n := f_+(t_n)$, with $f \in BV(0,T;\mathcal{H})$, then (4.23) *holds and*

$$\begin{aligned} E_{\mathcal{H}} &\leq \tau \big(\tfrac{\gamma}{\sqrt{2}} |f_+(0) - \mathfrak{A}u_0| + (1+\gamma) \operatorname{Var} f \big), \\ E_{[\,]} &\leq \tau \big(\gamma |f_+(0) - \mathfrak{A}u_0| + (1+\sqrt{2}\gamma) \operatorname{Var} f \big). \end{aligned} \tag{4.32}$$

PROOF. Since $\mathfrak{A}$ is linear and single-valued, (3.21) and (4.15) yield

$$\mathcal{D}_n := \tau_n^{-1}\langle \mathfrak{A}U_n - \mathfrak{A}U_{n-1}, U_n - U_{n-1}\rangle = \tau_n^{-1}[U_n - U_{n-1}]^2.$$

We next apply Theorem 7 to deduce (4.29), and Lemma 4.3 (with $L = 2\gamma$ and $\eta = \sqrt{2}\gamma$) together with Corollary 4 to obtain (4.30). The remaining estimate (4.32) follows from (4.31) as in Theorem 6. ■

We remark that (4.30) requires the bigger coefficient $2\gamma^2$, instead of γ^2, because we employ also the coercivity (4.25) of $\mathfrak{A}$, which is not required to estimate the maximum of $|u(t) - U(t)|$ in (4.29).

Remark 4.8 In the case of a *symmetric* positive semidefinite linear operator $\mathfrak{F}$, for which $\gamma = 1/2$, we could also apply the results of § 3, because $\mathfrak{F}$ is the subdifferential of a quadratic positive functional; this fact allows us to employ the finer estimators $\mathcal{E}_n$ and the intermediate results of Theorems 3 and 4. However, since we can apply Corollary 5 with $\gamma = 1/2$, both (4.29) and (4.30) give a slightly better result than (3.23), because $2\sigma(w; v) \geq [v - w]^2$. □

4.4 Lipschitz Perturbations

Now we show how to deal with Lipschitz perturbations of an angle-bounded operator (in particular, of a subdifferential one).

We assume the existence of a Lipschitz operator $\mathfrak{G} : D(\mathfrak{F}) \to \mathcal{H}$, i.e.

$$\exists\, L \geq 0 : \quad |\mathfrak{G}(v) - \mathfrak{G}(w)| \leq L\,|v - w| \quad \forall\, v, w \in D(\mathfrak{F}), \tag{4.33}$$

such that the perturbed operator in $D(\mathfrak{F})$

$$\tilde{\mathfrak{F}} := \mathfrak{F} + \mathfrak{G} \quad \textit{is maximal monotone and } \gamma^2\textit{-angle-bounded.}$$

By property (iii) of the Basic Existence Results of § 2.1, the Cauchy problem (2.3) is well posed again, and the discrete scheme (3.5) can be recursively solved if every step size τ_n satisfies $L\tau_n < 1$. If we set

$$g(t) := f(t) + \mathfrak{G}(u(t)), \quad G_n := F_n + \mathfrak{G}(U_n),$$

then (2.3), (3.5), and (4.1) can be rewritten in terms of the new operator $\tilde{\mathfrak{F}}$ with g and $\bar{G}$ instead of f and $\bar{F}$, respectively. Moreover, if $U_0 \in D(\mathfrak{F})$, then the estimator $\mathcal{D}_n$ defined in (3.20) becomes

$$\tilde{\mathcal{D}}_n := \tau_n\langle \delta F_n + \delta\mathfrak{G}(U_n) - \delta^2 U_n, \delta U_n\rangle \quad \forall\, 1 \leq n \leq N, \tag{4.34}$$

with δU_0 defined as in (3.7), and satisfies $0 \leq \tau_n^{-1}\tilde{\varrho}(U_n, U_{n-1}) \leq \tilde{\mathcal{D}}_n$, where $\tilde{\varrho}$ is the coercivity function associated to $\tilde{\mathfrak{F}}$.

We need a discrete version of Lemma 3.6, namely a variant of Lemma 3.11.

Lemma 4.9 *Let* $\{a_n\}_{n=0}^N$ *and* $\{b_n, c_n, d_n\}_{n=1}^N$ *be nonnegative numbers, let* $\{-1 < \lambda_n \leq 0\}_{n=1}^N$ *be given coefficients, and set*

$$\lambda := \min_{1\leq n\leq N} \lambda_n, \quad s_n := \sum_{k=1}^n \lambda_k, \quad \bar{s}_n := \frac{s_n + s_{n-1}}{2}, \quad s_0 := 0.$$

If

$$2a_n(a_n - a_{n-1}) + b_n^2 + 2\lambda_n a_n^2 \le c_n^2 + 2a_n d_n \quad \forall\, 1 \le n \le N, \tag{4.35}$$

then we have

$$\begin{aligned} \max\Big(\max_{1\le n\le N} \mathrm{e}^{s_n/(1+\lambda)} a_n, \Big(\sum_{n=1}^N \mathrm{e}^{2\bar{s}_n/(1+\lambda)} b_n^2 \Big)^{1/2} \Big) \\ \le \Big(a_0^2 + \sum_{n=1}^N \mathrm{e}^{2\bar{s}_n} c_n^2 \Big)^{1/2} + \sqrt{2} \sum_{n=1}^N \mathrm{e}^{s_{n-1}} d_n. \end{aligned} \tag{4.36}$$

Proof. We introduce the positive quantities $p_0 := 1$, $p_n := \prod_{k=1}^n (1+\lambda_k)$, and

$$\tilde{a}_n := p_n a_n, \quad \tilde{b}_n^2 := p_n p_{n-1} b_n^2, \quad \tilde{c}_n^2 := p_n p_{n-1} c_n^2, \quad \tilde{d}_n := p_{n-1} d_n;$$

p_n is a discrete integrating factor. In fact, multiplying (4.35) by $p_n p_{n-1}$ converts (4.35) into

$$2\tilde{a}_n(\tilde{a}_n - \tilde{a}_{n-1}) + \tilde{b}_n^2 \le \tilde{c}_n^2 + 2\tilde{a}_n \tilde{d}_n \quad \forall\, 1 \le n \le N.$$

Then (4.36) readily follows from (3.33) and (3.34), upon realizing that

$$\mathrm{e}^{\lambda_k/(1+\lambda)} \le 1 + \lambda_k \le e^{\lambda_k}$$

yields

$$\tilde{a}_n \ge \mathrm{e}^{s_n/(1+\lambda)} a_n, \quad \tilde{b}_n \ge \mathrm{e}^{\bar{s}_n/(1+\lambda)}, \quad \tilde{c}_n \le \mathrm{e}^{\bar{s}_n} c_n, \quad \tilde{d}_n \le \mathrm{e}^{s_{n-1}} d_n. \quad \blacksquare$$

Theorem 8 *Let u be the weak solution of (2.3), $\{U_n\}_{n=0}^N$ be the solution of (3.5) with $U_0 \in D(\mathfrak{F})$, and let $L\tau_n < 1$ for all $1 \le n \le N$. If $\tilde{\mathfrak{F}}$ is maximal monotone and γ^2-angle-bounded, then*

$$\begin{aligned} E_{\mathcal{H}} \le \mathrm{e}^{LT} \Bigg(\Big(|u_0 - U_0|^2 + \gamma^2 \sum_{n=1}^N \tau_n^2 \tilde{\mathcal{D}}_n \Big)^{1/2} + \tfrac{L}{2} \sum_{n=1}^N \tau_n^2 |\delta U_n| \\ + \|f - \bar{F}\|_{L^1(0,T;\mathcal{H})} \Bigg). \end{aligned} \tag{4.37}$$

Moreover, if $U_0 := u_0$ and $F_n := f_+(t_n)$, with $f \in BV(0,T;\mathcal{H})$, then

$$E_{\mathcal{H}} \le \tau \mathrm{e}^{LT} \big(\tfrac{1}{\sqrt{2}} \tilde{C} |(f_+(0) - \mathfrak{F}(u_0))^\circ| + (1+\tilde{C}) \operatorname{Var} f \big), \tag{4.38}$$

where $C_{\tau,T} := \mathrm{e}^{L\tau(1+LT)/(1-L\tau)}$ and $\tilde{C} := C_{\tau,T}(\gamma + LT/\sqrt{2})$.

Proof. We argue as in the proof of Theorem 7 but omit the coercivity terms. Since $\tilde{\mathfrak{F}}$ is γ^2-angle-bounded, we get from (4.1), (4.33), and (3.14)

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}|u - U|^2 &\le \langle \tilde{\mathfrak{F}}(\bar{U}) - \tilde{\mathfrak{F}}(u), u - U \rangle + \langle g - \bar{G}, u - U \rangle \\ &\le \ell\gamma^2 \langle \tilde{\mathfrak{F}}(\bar{U}) - \tilde{\mathfrak{F}}(\underline{U}), \bar{U} - \underline{U} \rangle + \langle f - \bar{F}, u - U \rangle \\ &\quad + \langle \mathfrak{G}(U) - \mathfrak{G}(\bar{U}), u - U \rangle + \langle \mathfrak{G}(u) - \mathfrak{G}(U), u - U \rangle \\ &\le \gamma^2 \ell \bar{\tau} \bar{\tilde{\mathcal{D}}} + \big(|f - \bar{F}| + L\ell\bar{\tau}|U'| \big) |u - U| + L|u - U|^2. \end{aligned}$$

Applying Lemma 3.6 with

$$a := |u - U|, \quad b := 0, \quad c^2 := \gamma^2 2\ell\bar{\tau}\bar{\tilde{\mathcal{D}}}, \quad d := |f - \bar{F}| + L\ell\bar{\tau}|U'|, \quad \lambda := -L,$$

we deduce the estimate (4.37) or, more precisely,

$$\begin{aligned} \max_{t\in[0,T]} \mathrm{e}^{-Lt}|u(t) - U(t)| \le \Big(|u_0 - U_0|^2 + \gamma^2 \sum_{n=1}^N \tau_n^2 \mathrm{e}^{-2Lt_{n-1}}\tilde{\mathcal{D}}_n\Big)^{1/2} \\ + \int_0^T \mathrm{e}^{-Lt}|f - \bar{F}|\,dt + \tfrac{L}{2}\sum_{n=1}^N \tau_n^2 \mathrm{e}^{-Lt_{n-1}}|\delta U_n|. \end{aligned} \tag{4.39}$$

We proceed as in Theorems 5 and 6 to prove (4.38). In view of (4.33) and (4.34), we get

$$|\delta U_n|\,\big(|\delta U_n| - |\delta U_{n-1}|\big) + \tilde{\mathcal{D}}_n - L\tau_n|\delta U_n|^2 \le \tau_n|\delta F_n|\,|\delta U_n|.$$

Applying Lemma 4.9 with

$$a_n := |\delta U_n|, \quad b_n^2 := 2\tilde{\mathcal{D}}_n, \quad c_n := 0, \quad d_n := |F_n - F_{n-1}|, \quad \lambda_n := -L\tau_n,$$

and using that $t_n/(1 - L\tau) \le t_{n-1} + \tau(1 + LT)/(1 - L\tau)$, we readily deduce

$$\begin{aligned} \max\Big(\max_{1\le n\le N} \mathrm{e}^{-Lt_{n-1}}|\delta U_n|, \Big(2\sum_{n=1}^N \mathrm{e}^{-2Lt_{n-1}}\tilde{\mathcal{D}}_n\Big)^{1/2}\Big) \\ \le C_{\tau,T}\Big(|\delta U_0| + \sqrt{2}\sum_{n=1}^N \mathrm{e}^{-Lt_{n-1}}|F_n - F_{n-1}|\Big). \end{aligned} \tag{4.40}$$

The assertion (4.38) follows from (4.39) and (4.40). ■

4.5 Hilbert Triplets

Let $\mathcal{V}$ be a Hilbert space with norm $\|\cdot\|$, continuously and densely imbedded into $\mathcal{H}$, and let us adopt the usual convention of identifying $\mathcal{H}$ with its dual $\mathcal{H}^*$. We can thus imbed $\mathcal{H}$ into $\mathcal{V}^*$, this inclusion being also dense, and consider the Hilbert triplet

$$\mathcal{V} \subset \mathcal{H} \equiv \mathcal{H}^* \subset \mathcal{V}^*.$$

Since the duality pairing between $\mathcal{V}$ and $\mathcal{V}^*$ is the (unique) extension by continuity of the $\mathcal{H}$-scalar product, we will use the same symbol for both bilinear forms. As in Examples 2.2 and 4.3, we are given a *positive, continuous*, and *weakly coercive* bilinear form $\mathfrak{a} : \mathcal{V}\times\mathcal{V} \to \mathbb{R}$ satisfying (2.14) and (4.8), namely

$$[v]^2 := \mathfrak{a}(v, v) \ge 0, \tag{4.41a}$$

$$v \mapsto \big([v]^2 + |v|^2\big)^{1/2} \quad \text{is an equivalent norm of } \mathcal{V}, \tag{4.41b}$$

$$\exists\,\mu \ge 0: \quad \mathfrak{a}_a(v, w) := \tfrac{1}{2}\big(\mathfrak{a}(v, w) - \mathfrak{a}(w, v)\big) \le \mu\,[v]\,[w] \quad \forall\, v, w \in \mathcal{V}. \tag{4.41c}$$

As in Example 2.3, we are also given a proper convex functional $\psi : \mathcal{V} \to (-\infty, +\infty]$, with domain $D(\psi)$, l.s.c. w.r.t. the (weak or strong) topology of $\mathcal{V}$.

Remark 4.10 We recall that if $\mathfrak{a}$ is *continuous and strongly coercive*, namely

$$\exists\, M,\alpha>0: \quad \mathfrak{a}(v,w)\le M\|v\|\,\|w\|, \quad [v]^2\ge\alpha\|v\|^2, \quad \forall\, v,w\in\mathcal{V}, \tag{4.42}$$

then (4.41a,b,c) surely hold with $\mu:=M/\alpha$; (4.42) is indeed the usual assumption in the stability and error analyses of variational inequalities. However, conditions (4.41a,b,c), which are related to the strong sector condition (4.5) or (4.7) for the associated operator $\mathfrak{A}$ defined in (2.15), lead to finer error estimates provided a precise bound for μ is known (e.g. $\mu=0$ if $\mathfrak{a}$ is symmetric). $\square$

Given a function $f:(0,T)\to\mathcal{V}^*$ and an initial datum $u_0\in\overline{D(\psi)}^{\mathcal{H}}$, we study the evolution variational inequality (2.24), namely,

$$\langle u'-f,u-v\rangle+\mathfrak{a}(u,u-v)+\psi(u)-\psi(v)\le 0 \quad \forall\, v\in D(\psi), \tag{4.43}$$

with the initial condition $u(0)=u_0$, and its discretization via the scheme

$$\langle \delta U_n-F_n,U_n-v\rangle+\mathfrak{a}(U_n,U_n-v)+\psi(U_n)-\psi(v)\le 0 \quad \forall\, v\in D(\psi). \tag{4.44}$$

The latter depends on the choice of $U_0\in\overline{D(\psi)}^{\mathcal{H}}$ and $F_n\in\mathcal{V}^*$ for $1\le n\le N$.

It is convenient to recall the definitions (4.19) and (4.20) of $\mathfrak{F}_{\mathcal{V}}:\mathcal{V}\to 2^{\mathcal{V}^*}$ and its restriction $\mathfrak{F}:\mathcal{H}\to 2^{\mathcal{H}}$, namely,

$$\begin{gathered} w_*\in\mathfrak{F}_{\mathcal{V}}(w) \;\Leftrightarrow\; \mathfrak{a}(w,w-v)+\psi(w)\le\psi(v)+\langle w_*,w-v\rangle \;\; \forall\, v\in D(\psi), \\ \mathfrak{F}(w):=\mathfrak{F}_{\mathcal{V}}(w)\cap\mathcal{H}, \quad D(\mathfrak{F})\subset D(\mathfrak{F}_{\mathcal{V}})\subset D(\psi)\subset\mathcal{V}. \end{gathered} \tag{4.45}$$

The following result provides a crucial link with the previous theory.

Lemma 4.11 *Let $\mathfrak{a}$ be a bilinear form satisfying* (4.41), *let $\mathfrak{A}$ be the associated linear operator in* (2.15), *and let $\mathfrak{F}$ be the multivalued operator defined by* (4.45). *Then $\mathfrak{F}$ is maximal monotone, γ^2-angle-bounded, and η^2-coercive in the sense* (4.25), *with*

$$\gamma^2:=\max\left(1,\tfrac14(1+\mu^2)\right), \quad \eta^2:=\max\left(1,\tfrac12(1+\mu^2)\right).$$

In particular, if $\mathfrak{a}$ is symmetric, *i.e. $\mu=0$ in* (4.41c), *then $\gamma=\eta:=1$ and $\mathfrak{F}$ is the subdifferential of the functional*

$$\phi(w):=\tfrac12\mathfrak{a}(w,w)+\psi(w), \quad D(\phi):=D(\psi)\subset\mathcal{V}. \tag{4.46}$$

If $\mathfrak{A}$ and ψ satisfy the compatibility condition (2.26), *then*

$$\mathfrak{F}:=\mathfrak{A}+\partial\psi, \quad D(\mathfrak{F}):=D(\mathfrak{A})\cap D(\partial\psi). \tag{4.47}$$

Finally, if

$$u_0\in D(\mathfrak{F}), \quad f\in BV(0,T;\mathcal{H}), \quad F_n\in\mathcal{H} \quad \forall\, 1\le n\le N, \tag{4.48}$$

then u (resp. $\{U_n\}_{n=0}^N$) is a solution of (4.43) *(resp.* (4.44)*) if and only if it is a strong solution of* (2.3) *(resp.* (3.5)*).*

PROOF. Since (4.41c) is (4.7) in disguise, it readily follows from the discussion in Example 4.3 that the linear operator $\mathfrak{A}$ defined in (2.15) is γ_0^2-angle-bounded with $\gamma_0^2 := (1+\mu^2)/4$. Moreover, in view of Remark 4.4, (4.14), and (4.41a) we deduce that for $\eta_0^2 := (1+\mu^2)/2$

$$\eta_0^2\mathfrak{a}(v-z,v-z) - \mathfrak{a}(v-w,w-z) \geq \tfrac{1}{2}\max\left([v-w]^2,[w-z]^2\right).$$

According to Remark 4.5, the operator $\mathfrak{F}$ defined in (4.45) is γ^2-angle-bounded and η^2-coercive with the asserted γ and η. In order to prove that $\mathfrak{F}$ is maximal monotone, it only remains to verify (see (2.1) that for every $z \in \mathcal{H}$ there exists $w \in D(\psi)$ such that $z - w \in \varepsilon\mathfrak{F}(w)$, or equivalently, by (4.45),

$$\varepsilon\mathfrak{a}(w,w-v) + \langle w,w-v\rangle + \varepsilon\psi(w) - \varepsilon\psi(v) \leq \langle z,w-v\rangle \quad \forall\, v \in D(\psi).$$

This is a direct consequence of the celebrated Lions-Stampacchia Theorem [44].

If $\mathfrak{a}$ is symmetric, then from (4.45) we infer that $w_* \in \mathfrak{F}(w)$ implies

$$\begin{aligned}\langle w_*, v-w\rangle &\leq \mathfrak{a}(w,v-w) + \psi(v) - \psi(w)\\ &\leq \left(\tfrac{1}{2}\mathfrak{a}(v,v)+\psi(v)\right) - \left(\tfrac{1}{2}\mathfrak{a}(w,w)+\psi(w)\right) = \phi(v)-\phi(w).\end{aligned}$$

Therefore w_* belongs also to the subdifferential of the functional ϕ defined by (4.46). Since $\mathfrak{F}$ is maximal, we conclude that $\mathfrak{F}$ *coincides* with $\partial\phi$.

Finally, (4.47) is the same as (2.27), and the equivalence of the differential equations follows from straightforward calculations. ■

In the framework of variational inequalities, it is sometimes useful to handle $\mathcal{V}^*$-valued source terms, so it is natural to consider f given by the sum of two different contributions. On the other hand, since $\mathfrak{a}$ may not be strongly coercive w.r.t. $\mathcal{V}$, and thereby $[\cdot]$ only be a *seminorm* on $\mathcal{V}$, it is convenient to introduce the domain of the conjugate *norm* $[\cdot]_*$, which is the Hilbert space $\mathcal{V}_0^*$ defined by

$$\mathcal{V}_0^* := \left\{v \in \mathcal{V}^* : [v]_* := \sup\nolimits_{z\in\mathcal{V}:\,[z]\leq 1}\langle v,z\rangle < \infty\right\}, \quad \|w\|_{\mathcal{V}_0^*} := [w]_* \quad \forall\, w \in \mathcal{V}_0^*.$$

Remark 4.12 If (4.42) holds, then

$$\mathcal{V}_0^* = \mathcal{V}^*, \quad [w]_* \leq \tfrac{1}{\sqrt{\alpha}}\|w\|_{\mathcal{V}^*} \quad \forall\, w \in \mathcal{V}^*. \quad \square$$

We now consider the splitting

$$f := g + h, \quad g \in L^2(0,T;\mathcal{V}_0^*), \quad h \in L^1(0,T;\mathcal{H}), \tag{4.49}$$

along with its discrete counterpart

$$F_n := G_n + H_n, \quad G_n \in \mathcal{V}_0^*, \quad H_n \in \mathcal{H}, \quad 0 \leq n \leq N. \tag{4.50}$$

If $U_0 \in D(\mathfrak{F}_{\mathcal{V}})$ and F_0 satisfy the *initial compatibility* condition

$$(F_0 - \mathfrak{F}_{\mathcal{V}}(U_0)) \cap \mathcal{H} \neq \varnothing, \tag{4.51}$$

then we can define $\delta U_0 := (F_0 - \mathfrak{F}_{\mathcal{V}}(U_0))^\circ$ and the discrete estimators $\mathcal{D}_n$ for $1 \leq n \leq N$ as in (3.20). If $\mathfrak{a}$ is *symmetric*, then we can also introduce the discrete estimators $\mathcal{E}_n$ defined in (3.8) in terms of the functional ϕ of (4.46).

Theorem 9 *Let u be the* weak *solution of (4.43) with $u_0 \in \overline{D(\psi)}^{\mathcal{H}}$ and f satisfying (4.49). Let $\{U_n\}_{n=0}^N$ be the discrete solution of (4.44) with $U_0 \in D(\mathfrak{F}_{\mathcal{V}})$ and F_n satisfying (4.50) and (4.51). If $\mathcal{D}_n$ is defined in (3.20) for $1 \le n \le N$ and $\eta := \max(1, (1+\mu^2)/2)$, then the errors $E_{\mathcal{H}}$ and $E_{[]}$ defined in (3.9) and (4.26) satisfy*

$$\max(E_{\mathcal{H}}, \tfrac{1}{\sqrt{2}}E_{[]}) \le \Big(|u_0 - U_0|^2 + 2\|g - \bar{G}\|^2_{L^2(0,T;\mathcal{V}_0^*)} + \eta^2 \sum_{n=1}^N \tau_n^2 \mathcal{D}_n\Big)^{1/2} + \|h - \bar{H}\|_{L^1(0,T;\mathcal{H})}. \tag{4.52}$$

If $\mathfrak{a}$ is symmetric, *then the previous formula holds with $\eta := 1$ and $\mathcal{E}_n$ instead of $\mathcal{D}_n$, where $\mathcal{E}_n$ is defined by (3.8). Finally, if $U_0 := u_0 \in D(\mathfrak{F}_{\mathcal{V}})$, $(f_+(0) - \mathfrak{F}_{\mathcal{V}}(u_0)) \cap \mathcal{H} \ne \varnothing$, and $G_n := g(t_n)$, $H_n := h(t_n^+)$, with $g \in H^1(0,T;\mathcal{V}_0^*)$, $h \in BV(0,T;\mathcal{H})$, then u and $\{U_n\}_{n=0}^N$ satisfy the following uniform estimate*

$$\max(E_{\mathcal{H}}, \tfrac{1}{\sqrt{2}}E_{[]}) \le \tau\big(\eta|(f_+(0) - \mathfrak{F}_{\mathcal{V}}(u_0))^\circ| + (1+\eta)\|g'\|_{L^2(0,T;\mathcal{V}_0^*)} + (1+\sqrt{2}\eta)\operatorname{Var} h\big). \tag{4.53}$$

Proof. We argue as in Theorem 7. First we assume (4.48), so that u (resp. U_n) is a strong solution of (2.3) (resp. (3.5)), along with $U_0 \in D(\mathfrak{F})$. In view of Lemma 4.11 and (4.27), we thus write (4.24) as

$$\tfrac{d}{dt}|u - U|^2 + \max\big([u - \bar{U}]^2, [u - U]^2\big) \le \eta^2 2\ell\bar{\tau}\bar{\mathcal{D}} + 2\langle f - \bar{F}, u - U\rangle,$$

but need a novel estimate of the source term f to control a possibly $\mathcal{V}_0^*$-valued component g of f. We substitute f and $\bar{F}$ by their decompositions (4.49) and (4.50), and use (4.16) together with Cauchy inequality to arrive at

$$\tfrac{d}{dt}|u - U|^2 + \tfrac{1}{2}\max\big([u - \bar{U}]^2, [u - U]^2\big) \le \eta^2 2\ell\bar{\tau}\bar{\mathcal{D}} + 2|h - \bar{H}|\,|u - U| + 2[g - \bar{G}]_*^2.$$

Consequently, to obtain (4.52) we apply Lemma 3.6 with $\lambda := 0$ and

$$a := |u - U|, \quad b^2 := \tfrac{1}{2}\max\Big([u - \bar{U}]^2, [u - U]^2\Big),$$
$$c^2 := \eta^2 2\ell\bar{\tau}\bar{\mathcal{D}} + 2[g - \bar{G}]_*^2, \quad d := |h - \bar{H}|.$$

By a standard approximation argument, we note that (4.52) holds also if u is a weak solution and g, G_n are really $\mathcal{V}_0^*$-valued.

When $\mathfrak{a}$ is symmetric, Lemma 4.11 shows that $\mathfrak{F}$ is a subdifferential, so that we can repeat the previous calculations starting from (3.15) instead of (4.24).

Finally, to demonstrate the rate of convergence (4.53) we notice that

$$\mathcal{D}_n \ge \tau_n^{-1}\varrho(U_n, U_{n-1}) \ge \tau_n^{-1}\langle \mathfrak{A}(U_n - U_{n-1}), U_n - U_{n-1}\rangle = \tau_n[\delta U_n]^2,$$

and thus use (4.16) and Cauchy inequality to rewrite (3.38) as

$$2|\delta U_n|\,(|\delta U_n| - |\delta U_{n-1}|) + \tau_n[\delta U_n]^2 + \mathcal{D}_n \le 2\tau_n|\delta H_n|\,|\delta U_n| + \tau_n[\delta G_n]_*^2 + \tau_n[\delta U_n]^2.$$

We apply Lemma 3.11 with

$$a_n := |\delta U_n|, \quad b_n^2 := \mathcal{D}_n, \quad c_n^2 := \tau_n[\delta G_n]_*^2, \quad d_n := \tau_n|\delta H_n|,$$

thereby obtaining

$$\begin{aligned}\Big(\sum_{n=1}^N \mathcal{D}_n\Big)^{1/2} &\le \Big(|\delta U_0|^2 + \sum_{n=1}^N \tau_n[\delta G_n]_*^2\Big)^{1/2} + \sqrt{2}\,\mathrm{Var}\,\bar{H} \\ &\le \big(|(f_+(0) - \mathfrak{F}_{\mathcal{V}}(u_0))^\circ|^2 + \|g'\|^2_{L^2(0,T;\mathcal{V}_0^*)}\big)^{1/2} + \sqrt{2}\,\mathrm{Var}\,h.\end{aligned}$$

Using the properties

$$\sqrt{2}\|g - \bar{G}\|_{L^2(0,T;\mathcal{V}_0^*)} \le \tau\|g'\|_{L^2(0,T;\mathcal{V}_0^*)}, \quad \|h - \bar{H}\|_{L^1(0,T;\mathcal{H})} \le \tau\,\mathrm{Var}\,h,$$

and the elementary inequality $(a^2+b^2)^{1/2} \le |a| + |b|$, we easily deduce the last assertion and thus conclude the proof. ■

5 Applications

We apply the abstract theory of §§ 3 and 4 to the examples of §§ 2.2 and 4.1, and obtain several novel and concrete error estimates in each case. Our chief purpose is to show the flexibility and range of applicability of our results. We do not claim completeness, nor do we state the results under minimal regularity of the data. We stick to the notation of previous sections, and further recall

- Ω is a Lipschitz domain of $\mathbb{R}^m$, with outward unit normal $\boldsymbol{n}$; $Q := \Omega \times (0,T)$ is the parabolic cylinder and $\Sigma := \partial\Omega \times (0,T)$ its lateral boundary;
- p_D^2, p_N^2 are the first non-zero eigenvalues of the Laplacian in Ω with homogeneous Dirichlet and Neumann boundary conditions, respectively;
- the discrete partition of the time interval $[0,T]$ and the related piecewise constant and linear interpolants are defined at the beginning of § 3.

5.1 Second Order Linear Parabolic Equations

We consider, as in Examples 2.2 and 4.3, the parabolic problem

$$\partial_t u - \nu^2\Delta u + \boldsymbol{b}\cdot\nabla u + cu = f \quad \text{in } Q, \tag{5.1a}$$

$$u(\cdot,0) = u_0(\cdot) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Sigma, \tag{5.1b}$$

with elliptic operator $\mathfrak{A}$ defined on $D(\mathfrak{A}) := \{v \in H_0^1(\Omega) : \Delta v \in \mathcal{H} := L^2(\Omega)\}$

$$\mathfrak{A}w := -\nu^2\Delta w + \boldsymbol{b}\cdot\nabla w + cw. \tag{5.2}$$

The data $\nu^2 > 0$ constant, $\boldsymbol{b} \in W^{1,\infty}(\Omega)$, and $c \in L^\infty(\Omega)$ satisfy

$$|\boldsymbol{b}(x)| \le b_0, \quad d(x) := -\tfrac{1}{2}\,\mathrm{div}\,\boldsymbol{b}(x) + c(x) \ge d_0^2 \ge 0, \quad \text{a.e. } x \in \Omega. \tag{5.3}$$

If $U_0 \in L^2(\Omega)$ and $F_n \in L^2(\Omega)$, the time discretization of (5.1) on the grid (1.2) is equivalent to seeking $U_n \in H_0^1(\Omega)$ for $1 \le n \le N$ such that

$$\tau_n^{-1}(U_n - U_{n-1}) - \nu^2\Delta U_n + \boldsymbol{b}\cdot\nabla U_n + cU_n = F_n \quad \text{in } \Omega. \tag{5.4}$$

Lemma 5.1 *The operator* $\mathfrak{A}$ *in* (5.2) *with* (5.3) *is* γ^2*-angle-bounded with* $\gamma^2 := (1 + b_0^2/\nu^2(d_0^2 + \nu^2 p_D^2))/4$.

PROOF. First of all, we observe that $\mathfrak{A}$ satisfies

$$\langle \mathfrak{A}w, w\rangle = \int_\Omega \left(\nu^2|\nabla w|^2 + d(x)|w|^2\right) dx \ge (\nu^2 p_D^2 + d_0^2)\|w\|^2_{L^2(\Omega)} \quad \forall\, w \in D(\mathfrak{A}),$$

because p_D is the best constant in the Poincaré inequality with homogeneous Dirichlet boundary condition: $p_D\,\|w\|_{L^2(\Omega)} \le \|\nabla w\|_{L^2(\Omega)}$ for all $w \in H_0^1(\Omega)$. Moreover, the antisymmetric part of the bilinear form associated to $\mathfrak{A}$ satisfies

$$\begin{aligned} |\langle \mathfrak{A}v, w\rangle - \langle \mathfrak{A}w, v\rangle| &\le b_0\left(\|\nabla v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\right) \\ &\le \frac{2b_0}{\nu\sqrt{d_0^2 + \nu^2 p_D^2}}\langle \mathfrak{A}v, v\rangle^{1/2}\langle \mathfrak{A}w, w\rangle^{1/2} \quad \forall\, v, w \in D(\mathfrak{A}). \end{aligned}$$

Therefore, (4.7) is valid with $\mu := b_0/\nu\sqrt{d_0^2 + \nu^2 p_D^2}$ and $\mathfrak{A}$ is γ^2-angle-valued with $\gamma^2 := (1 + \mu^2)/4$ according to [10, Prop.11]. ∎

Theorem 10 *Let* u *be the* weak *solution of* (5.1) *corresponding to data*

$$u_0 \in L^2(\Omega), \quad f \in L^1(0, T; L^2(\Omega)),$$

and (5.3). *Let* $\{U_n\}_{n=0}^N$ *be the discrete solution of* (5.4) *with* $U_0 \in H_0^1(\Omega)$ *and* $F_n \in L^2(\Omega)$, *and let* $\bar{U}, U$ *be its piecewise constant and linear interpolants. Then the estimator* $\mathcal{D}_n$ *defined in* (3.20) *is given by*

$$\mathcal{D}_n = \int_\Omega \left(\nu^2|\nabla U_n - \nabla U_{n-1}|^2 + d(x)|U_n - U_{n-1}|^2\right) dx,$$

and the following a posteriori *error estimates hold*

$$\begin{aligned} &\max\left(\|u - U\|_{L^\infty(0,T;L^2(\Omega))},\ \nu\|\nabla u - \nabla U\|_{L^2(Q)},\ \nu\|\nabla u - \nabla\bar{U}\|_{L^2(Q)}\right) \\ &\quad \le \Big(\|u_0 - U_0\|^2_{L^2(\Omega)} + 2\gamma^2 \sum_{n=1}^N \tau_n^2 \mathcal{D}_n\Big)^{1/2} + \int_0^T \|f - \bar{F}\|_{L^2(\Omega)}\, dt. \end{aligned} \tag{5.5}$$

Moreover, if

$$U_0 := u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \quad \partial_t f \in L^1(0, T; L^2(\Omega)), \quad F_n(\cdot) := f(\cdot, t_n)),$$

then the right-hand side of (5.5) *can be bounded uniformly by*

$$\tau\Big(\gamma\|f(0) - \mathfrak{A}u_0\|_{L^2(\Omega)} + (1 + \sqrt{2}\gamma)\int_0^T \|\partial_t f\|_{L^2(\Omega)}\, dt\Big). \tag{5.6}$$

PROOF. Since $\mathfrak{A}$ is γ^2-angle-bounded in view of Lemma 5.1, we apply the theory developed in § 4, in particular Corollary 5, to conclude (5.5) and (5.6) even if (4.29) and (4.30) require $U_0 \in D(\mathfrak{A})$. In fact, since $\mathcal{D}_n$ is continuous with respect to the norm $H_0^1(\Omega)$, by a simple approximation technique we may extend those estimates of Corollary 5 to $U_0 \in H_0^1(\Omega)$. ∎

Remark 5.2 For a convection-dominated diffusion problem the constant γ in (5.5) and (5.6) is proportional to the advection/diffusion ratio

$$\gamma = O(b_0/\nu) \quad \text{if } d_0 > 0, \quad \gamma = O(b_0/\nu^2) \quad \text{if } d_0 = 0.$$

However there is no exponential factor depending on b as long as $d_0 \ge 0$. □

Remark 5.3 Since the linear operator $\mathfrak{A}$ in (5.2) generates an analytic semigroup, optimal a posteriori error estimates in $L^\infty(0,T;L^2(\Omega))$ are derived in [31] using the smoothing effect of the dual problem. Our a posteriori estimator $\sum_{n=1}^N \tau_n^2 \mathcal{D}_n$ is different from that in [31], namely $\max_{1\le n\le N} \|U_n - U_{n-1}\|_{L^2(\Omega)}$, but both exhibit optimal order and optimal regularity. To see this, differentiate (5.1a) and multiply by $\partial_t u$. In light of (5.3), we obtain

$$\begin{aligned}\tfrac{d}{dt}\|\partial_t u(t)\|^2_{L^2(\Omega)} + 2\nu\|\nabla \partial_t u(t)\|^2_{L^2(\Omega)} + 2\|\sqrt{d}\,\partial_t u(t)\|^2_{L^2(\Omega)}& \\ \le 2\|\partial_t u(t)\|_{L^2(\Omega)}\|\partial_t f(t)\|_{L^2(\Omega)}&.\end{aligned}$$

Making use of Lemma 3.6 we easily deduce the regularity bounds

$$\begin{aligned}\max\Big(\max_{t\in[0,T]}\|\partial_t u(t)\|^2_{L^2(\Omega)}, 2\int_0^T \big(\nu\|\nabla\partial_t u(t)\|^2_{L^2(\Omega)} + \|\sqrt{d}\,\partial_t u(t)\|^2_{L^2(\Omega)}\big)\,dt\Big)& \\ \le \Big(\|f(0) - \mathfrak{A}u_0\|_{L^2(\Omega)} + \int_0^T \|\partial_t f(t)\|_{L^2(\Omega)}\,dt\Big)^2&.\end{aligned}$$

This elucidates the equivalence of both estimators alluded to before: they are discrete forms of the left-hand side norms! In addition, we derive error estimates in $L^2(0,T;H_0^1(\Omega))$ via coercivity of $\mathfrak{A}$, which are not present in [31]. $\square$

5.2 Nonlinear Variational Boundary Conditions

We consider the following parabolic problem associated to the linear PDE (5.1a) with nonlinear boundary condition (see [7, §4, Ex. 6] for a similar problem)

$$\partial_t u - \nu^2\Delta u + \boldsymbol{b}\cdot\nabla u + cu = f \quad \text{in } Q, \tag{5.7a}$$

$$u(\cdot,0) = u_0(\cdot) \quad \text{in } \Omega, \quad \partial_{\boldsymbol{n}} u + \beta(u) \ni g \quad \text{on } \Sigma, \tag{5.7b}$$

where $\beta = \partial j : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone real graph (see Example 4.5) and $g = g(x,t)$. We assume for simplicity

$$|\boldsymbol{b}| \le b_0, \quad c = \operatorname{div}\boldsymbol{b} = 0, \quad \text{in } \Omega; \quad \boldsymbol{b}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \tag{5.8}$$

If $2c - \operatorname{div}\boldsymbol{b}$ (or $\boldsymbol{b}\cdot\boldsymbol{n}$) is strictly positive in a set of non-vanishing measure and Ω is connected, then $\mathfrak{a}$ is a coercive bilinear form on $H^1(\Omega)$, and some estimates can be obtained in a slightly easier way as shown in §5.1. To formalize this problem, we introduce the Hilbert spaces $\mathcal{H} := L^2(\Omega), \mathcal{V} := H^1(\Omega)$ as in Example 4.5, and the $\mathcal{V}$-l.s.c. convex functional (4.13): $\psi(w) := \int_{\partial\Omega} j(w(x))\,dS$ if $w \in H^1(\Omega)$, $j(w) \in L^1(\partial\Omega)$. The problem (5.7), with $c = 0$, admits the variational formulation (4.43) with bilinear form

$$\mathfrak{a}(v,w) := \int_\Omega \big(\nu^2\,\nabla u\cdot\nabla v + \boldsymbol{b}\cdot\nabla u\, v\big)\,dx \quad \forall\, v,w \in \mathcal{V}.$$

The symbol $\langle\cdot,\cdot\rangle$ denotes both the scalar product in $L^2(\Omega)$ and the duality between $\mathcal{V}^*$ and $\mathcal{V}$, and $f:(0,T)\to\mathcal{V}^*$ is defined by

$$\langle f(t), v\rangle := \int_{\partial\Omega} g(x,t)v(x)\,dS(x) \quad \forall\, v\in\mathcal{V}.$$

To discretize this problem, we choose

$$U_0 \in H^1(\Omega) \quad \text{with} \quad \partial_{\boldsymbol{n}} U_0 + \beta(U_0) \ni G_0 \quad \text{on } \partial\Omega, \quad G_n \in L^2(\partial\Omega), \tag{5.9}$$

and solve the elliptic equations for $1 \leq n \leq N$ on the grid (1.2)

$$\tau_n^{-1}(U_n - U_{n-1}) - \nu^2 \Delta U_n + \boldsymbol{b} \cdot \nabla U_n = 0 \quad \text{in } \Omega, \tag{5.10a}$$

$$\partial_{\boldsymbol{n}} U_n + \beta(U_n) \ni G_n \quad \text{on } \partial\Omega. \tag{5.10b}$$

The variational formulation of (5.10) can be written as (4.44) with

$$\langle F_n, v\rangle := \int_{\partial\Omega} G_n(x) v(x)\, dS(x) \quad \forall\, v \in \mathcal{V}.$$

Let κ^{-1} be the measure of Ω, $v_\Omega := \kappa \int_\Omega v$ be the average of v in Ω, and let i, I

$$i(t) := \int_{\partial\Omega} g(x,t)\, dS(x), \quad I(t) := \int_{\partial\Omega} G(x,t)\, dS(x).$$

The eigenvalue p_N is the best constant in Poincaré inequality $p_N \|w - w_\Omega\|_{L^2(\Omega)} \leq \|\nabla w\|_{L^2(\Omega)}$ for all $w \in \mathcal{V}$. Similarly, let $\theta > 0$ be the best constant in the inequality $\|w - w_\Omega\|_{L^2(\partial\Omega)} \leq \theta \|\nabla w\|_{L^2(\Omega)}$ for $w \in \mathcal{V}$.

Theorem 11 *Let u be the* weak *solution of* (5.7) *with* (5.8), *corresponding to*

$$u_0 \in L^2(\Omega), \quad g \in L^2(\Sigma).$$

Let $\{U_n\}_{n=0}^N$ be the discrete solution of (5.10) *with* (5.9) *and let $\bar U, U$ be its piecewise constant and linear interpolants If $\mathcal{D}_n$ is defined as in* (3.20), *then the following* a posteriori *error estimates hold*

$$\begin{aligned} \max\Big(\|u - U\|_{L^\infty(0,T;L^2(\Omega))},\ \tfrac{\nu}{\sqrt{2}}\|\nabla u - \nabla U\|_{L^2(Q)},\ \tfrac{\nu}{\sqrt{2}}\|\nabla u - \nabla \bar U\|_{L^2(Q)}\Big) \\ \leq \Big(\|u_0 - U_0\|^2_{L^2(\Omega)} + \tfrac{2\theta^2}{\nu^2}\|g - \bar G\|^2_{L^2(\Sigma)} + \eta^2 \sum_{n=1}^N \tau_n^2 \mathcal{D}_n\Big)^{1/2} \\ + \sqrt{\kappa} \int_0^T |i - \bar I|\, dt, \end{aligned}$$

with $\eta^2 := \max(1, (1 + b_0^2/\nu^4 p_N^2)/2)$. Moreover, if

$$U_0 := u_0, \quad \partial_t\, g \in L^2(0,T;L^2(\Omega)), \quad G_n(\cdot) := g(\cdot, t_n),$$

then the right-hand side of the previous estimate can be bounded uniformly by

$$\tau\Big(\eta \|\mathfrak{A} u_0\|_{L^2(\Omega)} + \tfrac{(1+\eta)\theta}{\nu} \int_0^T \|\partial_t g\|_{L^2(\Omega)}\, dt + (1 + \sqrt{2}\eta)\sqrt{\kappa}\, \mathrm{Var}\, i\Big).$$

Proof. Thanks to (5.8) and the Green formula, we can easily check that

$$[v]^2 = \mathfrak{a}(v,v) = \nu^2 \int_\Omega |\nabla v|^2\, dx \geq \nu^2 p_N^2 \|v - v_\Omega\|^2_{L^2(\Omega)} \quad \forall\, v \in \mathcal{V}.$$

Morever, since

$$\int_\Omega \boldsymbol{b} \cdot \nabla v\, dx = -\int_\Omega v \operatorname{div} \boldsymbol{b}\, dx + \int_{\partial\Omega} v \boldsymbol{b} \cdot \boldsymbol{n}\, dS = 0 \quad \forall\, v \in \mathcal{V},$$

we have

$$2\mathfrak{a}_a(v,w) = \int_\Omega \big(\boldsymbol{b}\cdot\nabla v(w-w_\Omega) - \boldsymbol{b}\cdot\nabla w(v-v_\Omega)\big)\,dx \le \frac{2b_0}{\nu^2 p_N}[v]\,[w],$$

i.e. $\mathfrak{a}$ satisfies (4.41c) with $\mu := b_0/\nu^2 p_N$. In particular, $\mathfrak{F}$ is angle-bounded and satisfies (4.18) with the asserted η.

Our next task is to split the source term $\langle f, w\rangle = \int_{\partial\Omega} gw$ into components in $\mathcal{V}^*$ and $\mathcal{H}$. To do so, we introduce the auxiliary function $\tilde{g}$, defined up to an additive constant by

$$-\Delta\tilde{g} = -\kappa\, i \quad \text{in } \Omega, \qquad \partial_{\boldsymbol{n}}\tilde{g} = g \quad \text{on } \partial\Omega;$$

the existence of $\tilde{g}$ is ensured by the compatibility condition between $\kappa\, i$ and g

$$\int_\Omega -\kappa i\,dx + \int_{\partial\Omega} g\,dS = -i + i = 0.$$

Since

$$\int_\Omega \nabla\tilde{g}\cdot\nabla v\,dx = \int_{\partial\Omega} gv\,dS - \kappa\, i\int_\Omega v\,dx = \int_{\partial\Omega} g(v-v_\Omega)\,dS \quad \forall\, v\in\mathcal{V},$$

we deduce that

$$\|\nabla\tilde{g}\|_{L^2(\Omega)} \le \theta\|g\|_{L^2(\partial\Omega)}.$$

Therefore f admits the natural splitting $f = f_1 + f_2$ where

$$\langle f_1(t), v\rangle := \int_\Omega \nabla\tilde{g}(x,t)\cdot\nabla v(x)\,dx, \quad \langle f_2(t), v)\rangle = \kappa\, i\int v(x)\,dx,$$

and

$$[f_1]_* = \nu^{-1}\|\nabla\tilde{g}\|_{L^2(\Omega)} \le \tfrac{\theta}{\nu}\|g\|_{L^2(\partial\Omega)}, \quad \|f_2\|_{L^2(\Omega)} = \kappa\|i\|_{L^2(\Omega)} = \sqrt{\kappa}\,|i|.$$

Since the same arguments are valid for F_n, it only remains to apply Theorem 9 to conclude the proof. ■

Remark 5.4 For an advection-dominated diffusion operator $\mathfrak{A}$, the stability constants in the *a posteriori* and *a priori* error estimates of Theorem 11 are proportional to the advection/diffusion ratio b_0/ν^2, which is consistent with Remark 5.2. □

5.3 Quasi-Linear Operators

We consider two relevant examples of quasi-linear operators, namely the minimal surface operator and the p-Laplacian, both briefly discussed in Example 2.5.

Minimal Surface Operator. Let u be the *pseudo-solution* of the evolution minimal surface equation (2.34)

$$\partial_t u - \operatorname{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \quad \text{in } Q, \tag{5.11}$$

subject to an initial and lateral boundary condition $u = u_0$. This definition involves the area-type functional ϕ in $\mathcal{H} := L^2(\Omega)$ defined in (2.36) [42]

$$\phi(w) := \int_\Omega \sqrt{1+|\nabla w|^2}\, dx + \int_{\partial\Omega} |w-u_0|\, dS, \quad D(\phi) := L^2(\Omega)\cap W^{1,1}(\Omega).$$

We consider the following minimization problem

$$U_n := \arg\min\nolimits_{V\in D(\phi)} \Phi_n(V; U_{n-1}), \tag{5.12}$$

where Φ_n is the quadratic perturbation of the area functional ϕ

$$\Phi_n(V;U_{n-1}) := \int_\Omega \big(\tfrac{1}{2}\tau_n^{-1}|V-U_{n-1}|^2 + \sqrt{1+|\nabla V|^2}\big)\, dx + \int_{\partial\Omega} |V-u_0|\, dS.$$

Equivalently, we consider the following elliptic variational inequality

$$\int_\Omega \tau_n^{-1}(U_n - U_{n-1})(U_n - v) + \phi(U_n) - \phi(v) \le 0 \quad \forall\, v\in D(\phi). \tag{5.13}$$

Since ϕ is not l.s.c. in $L^2(\Omega)$, it is not obvious that either (5.12) or (5.13) admits a solution. We could, in principle, substitute ϕ with its l.s.c. envelope

$$\bar\phi(u) := \inf\big\{\liminf_{n\to+\infty}\phi(u_n):\ u_n\in\mathcal{H},\ \lim_{n\to+\infty} u_n = u\big\},$$

with proper domain $D(\bar\phi) = BV(\Omega)$. Since $\bar\phi(v)\le\phi(v)$, we could use $\bar\phi$ instead of ϕ to define the minimizing sequence U_n. However, in Theorem 12 we resort to a direct approach that avoids dealing with $BV(\Omega)$. To do so, we introduce the (modified) descent speed $\mathcal{E}_n$ of the functional Φ_n at time t_n

$$\mathcal{E}_n := -\tfrac{1}{2}|\delta U_n|^2 - \tau_n^{-1}\big(\Phi_n(U_n;U_{n-1}) - \Phi_n(U_{n-1};U_{n-1})\big) = -|\delta U_n|^2 - \delta\phi(U_n).$$

Theorem 12 *If $U_0\in W^{1,\infty}(\Omega)$, then Problem (5.12) is always uniquely solvable and it defines a family of uniformly bounded and locally Lipschitz functions $\{U_n\}_{n=1}^N$. We have the optimal a posteriori and a priori error estimates*

$$\begin{aligned}\|u-U\|^2_{L^\infty(0,T;L^2(\Omega))} &\le \|u_0-U_0\|^2_{L^2(\Omega)} + \sum_{n=1}^N \tau_n^2\mathcal{E}_n \\ &\le \|u_0-U_0\|^2_{L^2(\Omega)} + \tau\|u_0-U_0\|_{L^1(\partial\Omega)} + \tau\int_\Omega \sqrt{1+|\nabla U_0(x)|^2}\, dx.\end{aligned} \tag{5.14}$$

Moreover, if the initial graph u_0 has a square-integrable curvature and $U_0 := u_0$, we obtain the following optimal a priori error estimate

$$\sum_{n=1}^N \tau_n^2\mathcal{E}_n \le \tfrac{1}{2}\tau^2\int_\Omega \Big|\operatorname{div}\frac{\nabla u_0}{\sqrt{1+|\nabla u_0|^2}}\Big|^2 dx. \tag{5.15}$$

Proof. In order to apply the abstract theory of §3 we make the following simple observation: lower-semicontinuity of ϕ just guarantees the *existence* of both a strong continuous solution u of (5.11) and $\{U_n\}$ of (5.12), whereas *convexity* of ϕ alone is used in the error analysis. The existence of a strong solution u of (2.4) follows from [42, Theorems 1.1, 1.2]. The well-posedness of the minimum problem (5.12) follows from the gradient interior estimates of Bombieri, De Giorgi, and Miranda for the (time-independent) minimal surface equation and their extension to a more general class of equations with a bounded source term [58, Theorem 5.1, Ex. 5.2]. Such a boundedness is a consequence of the maximum principle, which yields a uniform $L^\infty(\Omega)$ bound for U_n.

Since ϕ is convex, we are now ready to apply the abstract theory of §3. We easily verify that the a posteriori estimate in (5.14) results from Theorem 1, whereas the a priori estimate is a consequence of Theorem 3. The linear rate (5.15) is a by-product of Theorem 5. ■

Remark 5.5 As a by-product of Remark 2.9 and Theorem 1, we can also prove *interior* estimates for $\nabla(u-U)$ similar to those in (5.14) and (5.15). In fact, for every $\Omega' \Subset \Omega$ there exists a constant $C' > 0$ depending on the $W^{1,\infty}(Q')$-norms of u and U such that

$$\|\nabla u - \nabla U\|^2_{L^2(Q')} + \|\nabla u - \nabla \bar{U}\|^2_{L^2(Q')} \le C'\Big(\|u_0 - U_0\|^2_{L^2(\Omega)} + \sum_{n=1}^{N} \tau_n^2 \mathcal{E}_n\Big),$$

where $Q' := \Omega' \times (0,T)$. Moreover, if $\partial\Omega$ is a surface of strictly positive mean curvature (e.g. if Ω is strictly convex), this estimate holds up to the boundary, i.e. we can choose $\Omega' \equiv \Omega$. In particular, if $U_0 := u_0$ has square-integrable curvature, then we obtain an optimal *linear* a priori estimate. □

Remark 5.6 A remarkable property of the linear rate of convergence stated in (5.15) is its validity even if the solutions u and U detach from the Dirichlet datum u_0. This implies a regularity result of $\partial_t u$, and so of the curvature $\operatorname{div}(\nabla u/\sqrt{1+|\nabla u|^2})$ past singularities. □

Remark 5.7 If Ω is strictly convex, then the Dirichlet condition $U_n = u_0$ can be imposed strongly. In such a case, the a posteriori estimator becomes

$$\mathcal{E}_n \le \mathcal{D}_n = \frac{1}{\tau_n}\int_\Omega \Big(\frac{\nabla U_n}{\sqrt{1+|\nabla U_n|^2}} - \frac{\nabla U_{n-1}}{\sqrt{1+|\nabla U_{n-1}|^2}}\Big)\nabla(U_n - U_{n-1}). \quad \square$$

***p*-Laplacian.** Given data $f \in L^1(0,T;L^2(\Omega))$ and $u_0 \in L^2(\Omega)$, we consider the evolution problem associated with the *p-Laplacian* ($p > 2$)

$$\begin{gathered}\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \quad \text{in } Q,\\ u(x,0) = u_0(x) \quad \text{in } \Omega, \quad u(x,t) = 0 \quad \text{on } \Sigma.\end{gathered}$$

This problem is equivalent to an evolution variational inequality (2.4) associated with the p-energy functional

$$\phi(v) = \int_\Omega \tfrac{1}{p}|\nabla v|^2 \quad \forall\, v \in W_0^{1,p}(\Omega);$$

thus $D(\phi) = W_0^{1,p}(\Omega)$. Given discrete data $U_0, \{F_n\}_{n=0}^N$, and the quadratic perturbation Φ_n of the p-energy functional ϕ

$$\Phi_n(V;U_{n-1}) := \int_\Omega \big(\tfrac{1}{2}\tau_n^{-1}|V(x) - U_{n-1}(x)|^2 + \tfrac{1}{p}|\nabla V(x)|^p - F_n(x)V(x)\big)\,dx,$$

let U_n be the solution of the recursive minimization problem

$$U_n := \arg\min_{V\in D(\phi)} \Phi_n(V;U_{n-1}).$$

To state the error estimates, we note that the discrete estimator $\mathcal{E}_n$ defined in (3.8) is the (modified) descent speed of the functional Φ_n at time t_n

$$\mathcal{E}_n = -\tfrac{1}{2}|\delta U_n|^2 - \tau_n^{-1}\big(\Phi_n(U_n;U_{n-1}) - \Phi_n(U_{n-1};U_{n-1})\big),$$

and indicate by E the total discretization error with $c_p > 0$ defined in (2.53)

$$\begin{aligned} E := \max\Big(&\|u-U\|_{L^\infty(0,T;L^2(\Omega))},\\ &\tfrac{\sqrt{2}c_p}{p}\big(\|\nabla u - \nabla U\|_{L^p(Q)} + \|\nabla u - \nabla \bar{U}\|_{L^p(Q)}\big)^{p/2}\Big). \end{aligned}$$

Theorem 13 *If $p \geq 2$, then the following optimal a posteriori and a priori error estimates are valid*

$$\begin{aligned} E &\leq \Big(\|u_0 - U_0\|^2_{L^2(\Omega)} + \sum_{n=1}^N \tau_n^2 \mathcal{E}_n\Big)^{1/2} + \|f-\bar{F}\|_{L^1(0,T;L^2(\Omega))}\\ &\leq \Big(\|u_0 - U_0\|^2_{L^2(\Omega)} + \tfrac{\tau}{p}\int_\Omega |\nabla U_0(x)|^p\,dx + \tfrac{\tau}{4}\|\bar{F}\|^2_{L^2(Q)}\Big)^{1/2}\\ &\quad + \|f-\bar{F}\|_{L^1(0,T;L^2(\Omega))}. \end{aligned} \tag{5.16}$$

Moreover, if $f\in L^2(Q)$ and we choose $U_0 := u_0$ and $F_n(x) := \tau_n^{-1}\int_{t_{n-1}}^{t_n} f(x,t)\,dt$, we obtain the uniform a priori error estimate

$$E \leq \sqrt{3\tau}\Big(\tfrac{1}{p}\int_\Omega |\nabla U_0(x)|^p\,dx + \|f\|^2_{L^2(Q)}\Big)^{1/2}. \tag{5.17}$$

Finally, if the continuous data u_0, f satisfy

$$\operatorname{div}(|\nabla u_0|^{p-2}\nabla u_0) \in L^2(\Omega), \quad \partial_t f \in L^1(0,T;L^2(\Omega))$$

and we set $U_0 := u_0$, $F_n(x) = f(x,t_n)$, then we get an a priori bound for $\mathcal{E}_n$

$$\begin{aligned} \Big(\sum_{n=1}^N \tau_n^2\mathcal{E}_n\Big)^{1/2} &\leq \frac{\tau}{\sqrt{2}}\|\operatorname{div}(|\nabla u_0|^{p-2}\nabla u_0) + f(0)\|_{L^2(\Omega)}\\ &\quad + \tau\|\partial_t f\|_{L^1(0,T;L^2(\Omega))}, \end{aligned} \tag{5.18}$$

and we end up with the optimal a priori linear rate of convergence

$$E \leq \tau\big(\tfrac{1}{\sqrt{2}}\|\operatorname{div}(|\nabla u_0|^{p-2}\nabla u_0) + f(0)\|_{L^2(\Omega)} + 2\|\partial_t f\|_{L^1(0,T;L^2(\Omega))}\big). \tag{5.19}$$

Proof. Owing to the p-growth of $\Phi_n(V;U_{n-1})$ with respect to ∇V, we conclude that $\Phi_n(\cdot;U_{n-1})$ is l.s.c. in $\mathcal{H} = L^2(\Omega)$ and so the theory of §3 applies. In particular, (2.53) combined with Lemma 2.8 yields

$$\sigma(w;v) \geq \tfrac{c_p}{p}\int_\Omega |\nabla(w-v)|^p\,dx, \quad \forall\ w,v\in W_0^{1,p}(\Omega).$$

Consequently, the estimates in (5.16) follow from Theorems 1 and 3, whereas Theorem 4 yields (5.17). In light of (3.7), we finally deduce (5.18) from Theorem 5 and (5.19) from Corollary 6. ■

Remark 5.8 Taking into account Corollary 2, in the first inequality of (5.16) as well as in (5.18) we can substitute $\mathcal{E}_n$ with

$$\mathcal{D}_n = \tau_n^{-1} \int_\Omega \left(|\nabla U_n|^{p-2} \nabla U_n - |\nabla U_{n-1}|^{p-2} \nabla U_{n-1} \right) \cdot \nabla (U_n - U_{n-1}) \, dx. \quad \square$$

Remark 5.9 If $1 < p < 2$, then we still obtain the above estimates for the error in $L^\infty(0,T;L^2(\Omega))$. We can also prove that there exists a constant C depending only on the norms of $f, \bar{F}$ in $L^2(Q)$ and of $\nabla u_0, \nabla U_0$ in $L^p(\Omega)$ such that

$$\|\nabla u - \nabla U\|_{L^2(Q)} + \|\nabla u - \nabla \bar{U}\|_{L^2(Q)} \le C \Big(\|u_0 - U_0\|^2_{L^2(\Omega)} + \sum_{n=1}^N \tau_n^2 \mathcal{E}_n \Big)^{1/2}.$$

This follows easily from two simple observations. First, we point out that the coercivity term ϱ associated with the p-Laplacian

$$\varrho(v,w) = \int_\Omega \left(|\nabla v|^{p-2} \nabla v - |\nabla w|^{p-2} \nabla w \right) \cdot \nabla (v - w) \, dx$$

satisfies the following lower bound (cf. e.g. [56, §2, Ex. E3(ii)])

$$\exists\, c_p > 0 : \quad \varrho(v,w) \ge c_p \left(\|\nabla v\|_{L^p(\Omega)} + \|\nabla w\|_{L^p(\Omega)} \right)^{p-2} \|\nabla (v-w)\|^2_{L^p(\Omega)}.$$

Making use of Lemma 2.8 and the subsequent Remark 2.9, there exists a constant C depending *only* on the $L^p(\Omega)$-norms of $\nabla v, \nabla w$ such that the smaller coercivity term σ for the p-Laplacian satisfies

$$\sigma(v;w) \ge C \|\nabla (v-w)\|^2_{L^p(\Omega)}.$$

Finally, we observe that the $L^p(\Omega)$ norms of ∇u and ∇U are uniformly bounded in $(0,T)$, once we know an analogous bound for the initial data u_0, U_0 and for the $L^2(Q)$-norms of $f, \bar{F}$. This is in turn an immediate consequence of the energy identity (2.6) and its discrete counterpart (3.12). $\square$

5.4 Degenerate Operators

We conclude this section with the classical Stefan problem (2.40) and (2.41)

$$\partial_t u - \Delta \beta(u) = f, \quad \beta(r) := (r-1)^+ - r^-,$$

which was formulated as a variational evolution inequality in $H^{-1}(\Omega)$ in Example 2.6.

We choose $U_0, u_0 \in L^2(\Omega)$, $f \in L^2(0,T;H^{-1}(\Omega))$, and a sequence of approximating $F_n \in H^{-1}(\Omega)$; at each step, we solve the semi-discrete problem

$$\delta U_n - \Delta \beta(U_n) = F_n, \quad \text{in } H^{-1}(\Omega)\ , \quad \beta(U_n) \in H^1_0(\Omega).$$

To carry out a *sharp* error analysis, we note that the residual estimator $\mathcal{D}_n$ defined in (3.20) reads

$$\mathcal{D}_n := \tau_n^{-1} \int_\Omega \left(\beta(U_n) - \beta(U_{n-1}) \right) (U_n - U_{n-1}) \, dx,$$

together with the total error E involving enthalpy u and temperature $\beta(u)$

$$E := \max \left(\|u - U\|_{L^\infty(0,T;H^{-1}(\Omega))}, \|\beta(u) - \beta(U)\|_{L^2(Q)} + \|\beta(u) - \beta(\bar{U})\|_{L^2(Q)} \right),$$

where U is obtained by piecewise linear interpolation of the discrete values U_n.

Theorem 14 *If u and U are the continuous and discrete solution of the Stefan problem with respect to the previous choice of data, then they satisfy the a posteriori error estimate*

$$E \le \Big(\|u_0 - U_0\|^2_{H^{-1}(\Omega)} + \sum_{n=1}^{N} \tau_n^2 \mathcal{D}_n \Big)^{1/2} + \|f - F\|_{L^1(0,T;H^{-1}(\Omega))}. \qquad (5.20)$$

Moreover, if $U_0 := u_0$ and we choose F_n as the mean value of f in the interval (t_{n-1}, t_n), we get the optimal a priori bound

$$E \le \sqrt{3}\tau \big(\tfrac{1}{2} \|\beta(u_0)\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q)} \big)^{1/2}. \qquad (5.21)$$

Finally, if the continuous data u_0, f satisfy

$$\beta(u_0) \in H^1_0(\Omega), \quad \partial_t f \in L^1(0,T; H^{-1}(\Omega))$$

and we choose $F_n(x) := f(x, t_n)$, we can bound the a posteriori estimator by

$$\Big(\sum_{n=1}^{N} \tau_n^2 \mathcal{D}_n \Big)^{1/2} \le \tfrac{\tau}{\sqrt{2}} \|\nabla \beta(u_0)\|_{L^2(\Omega)} + \tau \|\partial_t f\|_{L^1(0,T;H^{-1}(\Omega))}, \qquad (5.22)$$

and we end up with the optimal a priori bound

$$E \le \tau \big(\tfrac{1}{\sqrt{2}} \|\nabla\beta(u_0)\|_{L^2(\Omega)} + \tfrac{1}{\sqrt{2}} \|f(0)\|_{-1(\Omega)} + 2 \|\partial_t f\|_{L^1(0,T;H^{-1}(\Omega))} \big).$$

Proof. In view of Lemma 2.11 we are in a position to apply to abstract results of §3. We readily see that Theorem 1 yields (5.20), Theorem 4 implies (5.21), Theorem 5 leads to (5.22), and finally Corollary 6) gives rise to (5.22). ∎

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