MATH 401 – SECTION 0301 – SPRING 2016

Applications of Linear Algebra

REVIEW for EXAM #1 (March 31)

1. Find the equation $z = ax + by + c$ for the plane passing through the three points $\mathbf{p}_1 = (0, 2, -1), \mathbf{p}_2 = (-2, 4, 3), \mathbf{p}_3 = (2, -1, -3)$.

2. Find the LU decomposition of A with pivoting and solve the linear system $A\mathbf{x} = \mathbf{b}$:

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 6 & 2 & -1 \\ 1 & 1 & -7 & 2 \\ 1 & -1 & 2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}.$$

3. A block matrix D is called *block diagonal* if D can written as

$$D = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

with A and B squares matrices not necessarily of the same size, while the 0's are zero matrices of the appropriate sizes. Prove that D has an inverse if and only if A and B do, and

$$D^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix}.$$

4. True and False: (a) If A is symmetric then A^2 is symmetric.
(b) If A is nonsingular symmetric matrix then A^{-1} is symmetric.
(c) If A and B are symmetric $n \times n$ matrices, so is AB.
(d) If A is symmetric and D is diagonal then AD is symmetric.

5. (a) Let A be an $n \times n$ matrix. Which is faster to compute A^2 or A^{-1}.
(b) What about A^3 versus A^{-1}?
(c) How many operations (flops) are needed to compute A^k? Hint: When $k > 3$, you can get away with less than $k - 1$ matrix multiplications.
(d) Which is faster, back substitution of multiplying A by a vector?

5. Determine the rank(A) and the decomposition $PA = LU$ for

$$A = \begin{bmatrix} 0 & 0 & 0 & 3 & 1 \\ 1 & 2 & -3 & 1 & -2 \\ 2 & 4 & -2 & 1 & -2 \end{bmatrix}.$$

6. (a) Let V the space of integrable functions in $[0, 1]$. Show that the set of functions with integral zero form a subspace of V.
(b) Show that the set of solutions of the ordinary differential equation $y'' + 2y' - 3y = 0$ form a subspace of the functions with two continuous derivatives. Find a basis and the dimension.
(c) How about the nonhomogeneous ordinary differential equation $y'' + 2y' - 3y = 1$?

(d) The trace of a square matrix $a \in \mathbb{R}^{n\times n}$ is the sum of its diagonal entries. Prove that the set of trace zero matrices is a subspace of $\mathbb{R}^{n\times n}$.
(e) A planar vector field $\mathbf{p}(x, y) = (u(x, y), v(x, y))^T$ is called *irrotational* if it has zero divergence $\operatorname{div} \mathbf{p} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$. Show that the set of irrotational vector fields forms a subspace of the space of vector fields.

7. Determine whether the polynomials x^2+1, x^2-1, x^2+x+1 span the space of quadratic polynomials P_2.

8. Show that the functions $f(x) = x$ and $g(x) = |x|$ are linearly independent when considered as functions on all of $\mathbb{R}$, but are linearly dependent when considered as functions defined only on $\mathbb{R}^+ = \{x > 0\}$.

9. Given the following vectors, answer the following questions and provide a justification:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}.$$

(a) Do $(\mathbf{v}_i)_{i=1}^4$ span $\mathbb{R}^3$?
(b) Are $(\mathbf{v}_i)_{i=1}^4$ linearly independent?
(c) Do $(\mathbf{v}_i)_{i=1}^4$ form a basis for $\mathbb{R}^3$? If not, is it possible to choose some subset which is a basis?
(d) What is the dimension of the span of $(\mathbf{v}_i)_{i=1}^4$?

10. Find basis of the four spaces rng A, ker A, corng A and coker A of the following matrix A along with their dimensions:

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 3 \\ 2 & 3 & 7 & 0 \end{bmatrix}.$$

11. Show that $\mathbf{v}_1 = (1, 2, 0, -1)^T, \mathbf{v}_2 = (-3, 1, 1, -1)^T, \mathbf{v}_3 = (2, 0, -4, 3)^T$ and $\mathbf{w}_1 = (3, 2, -4, 2)^T, \mathbf{w}_2 = (2, 3, -7, 4)^T, \mathbf{w}_3 = (0, 3, -3, 1)^T$ are two bases for the same three dimensional subspace V of $\mathbb{R}^4$.