MATH 401 – Section 0301 – Spring 2016

Applications of Linear Algebra

REVIEW for EXAM #2 (May 3)

Review of §3: 3.1.2(g), 12, 21(b); 3.2.7, 23, 26, 28, 40; 3.3.4, 16, 27, 34; 3.4.4, 12, 15; 3.5.11, 16, 19(a) .

Review of §4: 4.3.5, 15(d), 17; 4.4.2, 35, 45.

Review of §5: 5.1.6, 18, 22, 26; 5.2.2, 4, 17(b); 5.3.1(d), 3, 16, 27(b), 28(ii), 30, 32; 5.4.2, 12; 5.5.2, 6, 12, 13, 22, 28; 5.6.1(c), 4(b), 17(c), 19.

Additional problems (taken from P. Olver and C. Shakihan):

1. Let V be an inner product space.
(a) Prove that $\langle \mathbf{x}, \mathbf{v}\rangle = 0$ for all $\mathbf{v} \in V$ if and only if $\mathbf{x} = 0$.
(b) Prove that $\langle \mathbf{x}, \mathbf{v}\rangle = \langle \mathbf{y}, \mathbf{v}\rangle$ for all $\mathbf{v} \in V$ if and only if $\mathbf{x} = \mathbf{y}$.
(c) Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be a basis of V. Prove that $\langle \mathbf{x}, \mathbf{v}_i\rangle = \langle \mathbf{y}, \mathbf{v}_i\rangle$ for $1 \le i \le n$ if and only if $\mathbf{x} = \mathbf{y}$.

2. (a) Prove the identity

$$\langle \mathbf{u}, \mathbf{v}\rangle = \frac{1}{4}\Big(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2\Big),$$

which allows one to reconstruct an inner product from its norm. (b) Use this formula to find the inner product on $\mathbb{R}^2$ corresponding to the norm $\|\mathbf{v}\| = \sqrt{v_1^2 - 3v_1v_2 + 5v_2^2}$.

3. Which of the following formulas for $\langle f, g\rangle$ define inner products on the space of continuous functions in $[-1, 1]$?

$$\text{(a)} \int_{-1}^{1} f(x)g(x)x\,dx; \quad \text{(b)} \int_{-1}^{1} f(x)g(x)x^2dx; \quad \text{(c)} f(-1)g(-1) + f(0)g(0) + f(1)g(1).$$

4. A subset $S \subset \mathbb{R}^n$ is called *convex* if for any $\mathbf{x}, \mathbf{y} \in S$ then the line segment joining $\mathbf{x}$ to $\mathbf{y}$ is also in S, i.e. $t\mathbf{x} + (1-t)\mathbf{y} \in S$ for all $0 \le t \le 1$. Prove that the unit ball is a convex subset of a normed vector space.

5. Find the closest point or points to $\mathbf{w} = (-1, 2)^T$ that lie on (a) the x-axis; (b) the y-axis; (c) the line $y = x$; (d) the line $x + y = 0$; (e) the line $2x + y = 0$.

6. Find the closest point on the plane spanned by $(1, 2, -1)^T, (0, -1, 3)^T$ to the point $(1, 1, 1)^T$, using (a) the Euclidean inner product; (b) the weighted inner product $\langle \mathbf{v}, \mathbf{w}\rangle = 2v_1w_1 + 4v_2w_2 + 3v_3w_3$; (c) the inner product $\langle \mathbf{v}, \mathbf{w}\rangle = \mathbf{v}^T C\mathbf{w}$ with the matrix

$$C = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

7. Find the least squares solution to the overdetermined system $x_1 + x_2 = 2, x_2 + x_4 = 1, x_1 + x_3 = 0, x_3 - x_4 = 1, x_1 - x_4 = 2$.

8. The amount of waste (in millions of tons a day) generated in Lower Slobbovia from 1960 to 1995 was

1960	1965	1970	1975	1980	1985	1990	1995
86	99.8	115.8	125	132.6	143.1	156.3	169.5

(a) Find the equation of the least squares line that best fits these data
(b) Use the result to estimate the amount of waste in the year 2000.

9. Given the data values $t_i = 0, 0.5, 1$ and $y_i = 1, 0.5, 0.25$ construct the trigonometric function of the form $g(t) = a\cos \pi t + b \sin \pi t$ that best approximates the data in the least squares sense.

10. Approximate the function $f(t) = t^{1/3}$ using the least squares method based on the inner product $\int_0^1 f(x)g(x)dx$ on the interval $[0, 1]$ by (a) a straight line; (b) a parabola; (c) a cubic polynomial.

11. (a) Prove that the polynomials $p_0(t) = 1, p_1(t) = t - \frac{2}{3}, p_2(t) = t^2 - \frac{6}{5}t + \frac{3}{10}$ form an orthonormal basis for the space of polynomials P_2 of degree ≤ 2 with respect to the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)tdt$. (b) Find the corresponding orthonormal basis. (c) Write t^2 as a linear combination of p_0, p_1, p_2.

12. (a) Find the QR factorization of the coefficient matrix, and (b) use this factorization to solve the system:

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$