

AMSC 612 Spring 2015
NUMERICAL METHODS FOR EVOLUTION PDE
HOMEWORK # 2

Pbs 1-3 due Fr Mar 13, Pbs 4-6 due Fr Mar 27

1 (15 pts). *Neumann condition and discrete maximum principle.* Given $\alpha, \beta \in \mathbb{R}$ and $f \in C^1[0, 1]$, consider the heat equation with mixed boundary conditions:

$$\partial_t u - \partial_x^2 u = f(x) \quad x \in (0, 1), \quad u(0) = \alpha, \quad u'(1) = \beta.$$

(a) *Finite difference method (FDM).* Write an implicit FDM on a uniform partition $\mathcal{T} = \{x_j\}_{j=0}^M$ with $0 = x_0 < x_1 < \dots < x_M = 1$ and meshsize h and time step k with approximate Neumann condition

$$\frac{U_M - U_{M-1}}{h} = \beta.$$

Show that the truncation error $\mathbf{T}^n = (\tau_j^n)_{j=1}^M$ satisfies $\|\mathbf{T}^n\|_\infty \leq C_1(u)h + C_2(u)k$. Explain the regularity of u involved in the constants $C_1(u)$ and $C_2(u)$.

(b) *Maximum principle.* Write the discrete problem for the vector $\mathbf{U}^{n+1} = (U_j^{n+1})_{j=1}^M$ as

$$\mathbf{K}\mathbf{U}^{n+1} = \hat{\mathbf{U}}^n + k\mathbf{F},$$

where $\hat{U}_j^n = U_j^n$ if $j < M$ and $\hat{U}_M^n = 0$. Both the matrix \mathbf{K} and right-hand side \mathbf{F} are obtained by multiplying all the equations by k . Deduce the ℓ^∞ -stability bounds $\max_{1 \leq j \leq M} U_j^{n+1} \leq \|\mathbf{U}^n\|_\infty$ provided $\mathbf{F} \leq 0$ and $\min_{1 \leq j \leq M} U_j^{n+1} \geq -\|\mathbf{U}^n\|_\infty$ provided $\mathbf{F} \geq 0$. Conclude that the matrix \mathbf{K} is nonsingular.

(c) *Discrete barrier.* Let $w = w(x)$ be the solution of the 2-point boundary value problem:

$$w'' = 1 \quad x \in (0, 1), \quad w(0) = 0, \quad w'(1) = -1.$$

Show that $\mathbf{W} = (w(x_j))_{j=1}^M$ satisfies $(\mathbf{KW})_j \leq W_j - \frac{k}{2}$ for $1 \leq j < M$ and $(\mathbf{KW})_M \leq -\frac{k}{2}$ provided h, k are sufficiently small.

(d) *Error estimate.* Let $\mathbf{E}^n = (u(x_j, t^n) - U_j^n)_{j=1}^M$ be the finite difference error. Show that the auxiliary vector $\mathbf{V}^{n+1} = \mathbf{E}^{n+1} + \gamma\mathbf{W}$ satisfies componentwise

$$\mathbf{KV}^{n+1} \leq \hat{\mathbf{V}}^n + k\mathbf{T}^n - \frac{\gamma k}{2} \leq \hat{\mathbf{V}}^n$$

provided $\gamma > 0$ is suitably chosen: how does it relate to k and h ?. Deduce the ℓ^∞ -upper estimate $\max_{1 \leq j \leq M} E_j^n \leq \hat{C}_1(u)h + \hat{C}_2(u)k$ where the constants $\hat{C}_1(u), \hat{C}_2(u)$ are proportional to $C_1(u), C_2(u)$. Modify the definition of \mathbf{V}^{n+1} to derive a lower bound for $\min_{1 \leq j \leq M} E_j^n$ and finally the estimate $\|\mathbf{E}^n\|_\infty \leq \hat{C}_1(u)h + \hat{C}_2(u)k$.

2 (15 pts). *Advection-diffusion PDE and upwinding.* Consider the PDE

$$\partial_t u - a\partial_x^2 u + b\partial_x u = f(x) \quad x \in \mathbb{R},$$

with constants $a, b > 0$. Consider a uniform lattice $\{x_j\}$ of size h and uniform time-step k . Write an implicit FDM with *upwinding*.

- (a) Find the symbol $S(h\xi)$ of the implicit discrete operator and show that $|S(h\xi)| \leq 1$ for all $h, k > 0$.
- (b) Derive an ℓ_h^2 -stability bound using von Neumann analysis.
- (c) Examine the truncation error $\mathbf{T}^n = (\tau_j^n)_{j \in \mathbb{Z}}$ and show that $\|\mathbf{T}^n\|_{2,h} \leq C_1(u)h + C_2(u)k$. Make the regularity of u entering in the constants $C_1(u), C_2(u)$ explicit.
- (d) Derive an error estimate in ℓ_h^2 .

3 (15 pts). *Second order backward difference.* The so-called BDF(2) is a popular 2-step method for stiff equations. Given U^0, U^1 , its semidiscrete version for the heat equation reads:

$$\frac{3U^{n+1} - 4U^n + U^{n-1}}{2k} - \Delta U^{n+1} = f^{n+1}.$$

- (a) Derive the method upon combining two BE methods with steps k and $2k$ in such a way that the truncation error is of second order.
- (b) Determine the truncation error directly using Taylor expansion.
- (c) Use (a) to derive the following identity of $L^2(\Omega)$ -norms upon multiplying the discrete PDE by U^{n+1} , integrating by parts, and assuming that U^{n+1} has vanishing trace:

$$\|U^{n+1} - U^n\|^2 - \frac{1}{4}\|U^{n+1} - U^{n-1}\|^2 + \frac{3}{4}\|U^{n+1}\|^2 - \|U^n\|^2 + \frac{1}{4}\|U^{n-1}\|^2 + k\|\nabla U^{n+1}\|^2 = k\langle f^{n+1}, U^{n+1} \rangle.$$

- (d) Use (c) to prove the following L^2 -stability bound

$$\|U^{N+1}\|^2 + 2 \sum_{n=1}^N k \|\nabla U^{n+1}\|^2 \leq 5\|U^0\|^2 + 6\|U^1\|^2 + 2 \sum_{n=1}^N k \|f^{n+1}\|_{H^{-1}(\Omega)}^2.$$

Hint: use the triangle inequality in the form $\|\phi_1 + \phi_2\|^2 \leq (1 + \delta)\|\phi_1\|^2 + (1 + \delta^{-1})\|\phi_2\|^2$ for any $\delta > 0$, add over n , and look for cancellations.

4 (15 pts). *Semidiscrete finite element method.* Problem 10.4 in Larsson and Thomée.

5 (25 pts). *MATLAB: FEM for the Heat Equation.* (a) Modify the MATLAB code `fem` in the website <http://www2.math.umd.edu/~rhn/teaching.html> by adding a loop $1 \leq n \leq N$ to account for a *backward Euler* discretization of the time variable.

(b) Let $\Omega = (0, 1)^2$ and $T = 1$. Let a Neumann condition g_N be imposed on the side $x = 1$, and a Dirichlet condition g_D on the rest of the boundary $\partial\Omega$. Let

$$u(x, y, t) = \sin(3\pi x)e^{-y-2t}.$$

be the exact solution. Find g_D, g_N and the forcing $f = \partial_t - \Delta u$.

(c) Compute the discrete solution U at $T = 1$, along with the L^2 , H^1 , and L^∞ errors. To this end, use the relation $k = h^2$ between time-step and meshsize, and find the discrete solution for meshsizes $h = 2^{-j}$ with $j = 3, 4, 5, 6$. Plot the discrete solution at $T = 1$ for $h = 2^{-4}$ and plot the three errors in terms of h in a log-log scale. Verify the decay $\|u(T) - U(T)\| \approx h^s$ and relate s to theory.

(d) Repeat (a) and (c) for the *Crank-Nicolson* method, this time with a relation $k = h$. Compare with (c) and draw conclusions.

5 (15 pts). *Crank-Nicolson-Galerkin method.* Problem 10.7 of Larsson and Thomée.