1 (10 pts) **Projection method.** Show that the residual $R_{n+1}$ of the momentum equation for Stokes flow satisfies the inequality

$$
\| R_{n+1} \|_{H^{-1}(\Omega)} \leq C(u, p) k
$$

in terms of the time step $k$. Specify the regularity of velocity $u$ and pressure $p$.

2 (15 pts). **Gradient Flows.** Let $H$ be a Hilbert space, $\phi : H \to \mathbb{R}$ be convex, and $E_n \geq 0$ be the energy dissipation estimator

$$
E_n := \langle F_n - \delta U_n, \delta U_n \rangle - \delta \phi(U_n).
$$

Let $F$ be the subdifferential of $\phi$.

(a) Prove the *a posteriori* error bound for any $F_n \in H$ and $U_0 \in D(F)$

$$
\tau_n^2 E_n \leq \tau_n \langle F(U_n) - F(U_{n-1}), U_n - U_{n-1} \rangle,
$$

where $D(F)$ stands for the domain of $F$, i.e. $\|F(U_0)\| < \infty$.

(b) Let $F(v) = -\Delta v$ and $H = L^2(\Omega)$. Use (1) to derive an a posteriori error estimator for the heat equation $\partial_t u - \Delta u = f$.

(c) Let $F(v) = -\Delta \beta(v)$ and $H = H^{-1}(\Omega)$, with $\beta : \mathbb{R} \to \mathbb{R}$ (non-strictly) monotone. Use (1) to derive an a posteriori error estimator for the *degenerate* parabolic PDE $\partial_t u - \Delta \beta(u) = f$.

(d) Let $u_0 = U_0 \in D(F)$ and $F_n = 0$ for all $n$. Use (1) to prove the *a priori* error estimate

$$
\max_{0 \leq t \leq T} \| u - U \| \leq \frac{\tau}{\sqrt{2}} \| F(u_0) \|.
$$

3 (15 pts). **Stability.** (a) Problem 11.13 of Larsson and Thomée.

(b) Consider $f = 0$ and apply to the wave equation. Draw conclusions.

(c) Apply to the Maxwell Equations in Problem 11.15 of Larsson and Thomée.

4 (20 pts). **MATLAB.** Problem 13.2 of Larsson and Thomée. Solve the discrete problem for $k = M^{-1} = 10^{-4} 2^{-j}$ for $j = 1 : 5$, compute the $L^2$ and $H^1$ errors at $t = 3/4$ and find the experimental rate of convergence. Compare with theory.

5 (15 pts). **Stability.** Problem 13.3 of Larsson and Thomée. To compare with the method (13.5) in Larsson and Thomée, consider the time-discrete scheme only (i.e. without space discretization).

6 (15 pts). **Error estimate.** Problem 13.4 of Larsson and Thomée.
7 (10 pts). *Unconditional instability.* Consider the explicit finite difference method for the wave equation:

\[
\frac{1}{k^2} \left( U_{j}^{n+1} - 2U_{j}^{n} + U_{j}^{n-1} \right) - \frac{c^2}{h^2} \left( U_{j-1}^{n-1} - 2U_{j}^{n-1} + U_{j+1}^{n-1} \right) = 0.
\]

Let \( \lambda = \frac{c^2 k^2}{h^2} \). Find the symbol \( S(h\xi) \) of the discrete operator and show that \( |S(h\xi)| > 1 \) for any value of \( \lambda \). Relate the discrete domain of dependence to the continuous one (see Larsson-Thomée p.190).