AMSC 612 Spring 2015

NUMERICAL METHODS FOR EVOLUTION PDE

HOMEWORK # 3

Pbs 1-2 due Th Apr 9, Pbs 3-7 due Tu Apr 21

1 (10 pts) *Projection method.* Show that the residual $\mathcal{R}^{n+1}$ of the momentum equation for Stokes flow satisfies the inequality

$$\|\mathcal{R}^{n+1}\|_{H^{-1}(\Omega)} \leq C(\mathbf{u}, p)k$$

in terms of the time step k. Specify the regularity of velocity $\mathbf{u}$ and pressure p.

2 (15 pts). *Gradient Flows.* Let H be a Hilbert space, $\phi : H \to \mathbb{R}$ be convex, and $\mathcal{E}_n \geq 0$ be the *energy dissipation* estimator

$$\mathcal{E}_n := \langle F_n - \delta U_n, \delta U_n \rangle - \delta\phi(U_n).$$

Let $\mathcal{F}$ be the subdifferential of ϕ.
(a) Prove the *a posteriori* error bound for any $F_n \in H$ and $U_0 \in D(\mathcal{F})$

$$\tau_n^2 \mathcal{E}_n \leq \tau_n \langle \mathcal{F}((U_n) - \mathcal{F}(U_{n-1}), U_n - U_{n-1} \rangle, \tag{1}$$

where $D(\mathcal{F})$ stands for the domain of $\mathcal{F}$, i.e. $\|\mathcal{F}(U_0)\| < \infty$.
(b) Let $\mathcal{F}(v) = -\Delta v$ and $H = L^2(\Omega)$. Use (1) to derive an a posteriori error estimator for the *heat equation* $\partial_t u - \Delta u = f$.
(c) Let $\mathcal{F}(v) = -\Delta\beta(v)$ and $H = H^{-1}(\Omega)$, with $\beta : \mathbb{R} \to \mathbb{R}$ (non-strictly) monotone. Use (1) to derive an a posteriori error estimator for the *degenerate* parabolic PDE $\partial_t u - \Delta\beta(u) = f$.
(d) Let $u_0 = U_0 \in D(\mathcal{F})$ and $F_n = 0$ for all n. Use (1) to prove the *a priori* error estimate

$$\max_{0 \leq t \leq T} \|u - U\| \leq \frac{\tau}{\sqrt{2}} \|\mathcal{F}(u_0)\|.$$

3 (15 pts). *Stability.* (a) Problem 11.13 of Larsson and Thomée.
(b) Consider $f = 0$ and apply to the wave equation. Draw conclusions.
(c) Apply to the Maxwell Equations in Problem 11.15 of Larsson and Thomée.

4 (20 pts). *MATLAB.* Problem 13.2 of Larsson and Thomée. Solve the discrete problem for $k = M^{-1} = 10^{-1}2^{-j}$ for $j = 1 : 5$, compute the L^2 and H^1 errors at $t = 3/4$ and find the experimental rate of convergence. Compare with theory.

5 (15 pts). *Stability.* Problem 13.3 of Larsson and Thomée. To compare with the method (13.5) in Larsson and Thomée, consider the time-discrete scheme only (i.e. without space discretization).

6 (15 pts). *Error estimate.* Problem 13.4 of Larsson and Thomée.

7 (10 pts). *Unconditional instability.* Consider the explicit finite difference method for the wave equation:

$$\frac{1}{k^2}\big(U_j^{n+1} - 2U_j^n + U_j^{n-1}\big) - \frac{c^2}{h^2}\big(U_{j-1}^{n-1} - 2U_j^{n-1} + U_{j+1}^{n-1}\big) = 0.$$

Let $\lambda = \frac{c^2k^2}{h^2}$. Find the symbol $S(h\xi)$ of the discrete operator and show that $|S(h\xi)| > 1$ for any value of λ. Relate the discrete domain of dependence to the continuous one (see Larsson-Thomée p.190).