AMSC 614

NUMERICAL METHODS FOR STATIONARY PDE

HOMEWORK # 2 (Pbs 1-2 due Oct 7, Pbs 3-4 due Oct 9, Pbs 5-7 due Oct 14)

1 (15 pts). *2-point boundary value problem.* Consider the following boundary value problem in $(0,1)$:

$$-u'' + u = f, \qquad u'(0) = 1, u(1) = 1.$$

(a) Write the variational formulation.

(b) Let $x_1 = 0 < x_2 < \cdots < x_N < x_{N+1} = 1$ be a partition of $(0,1)$ with $h_i = x_{i+1} - x_i$. Let $\{\phi_i\}_{i=1}^{N+1}$ be the corresponding basis of hat functions, and let $f = \sum_{i=1}^{N+1} f_i\phi_i$ be piecewise linear. Let $U = \sum_{i=1}^{N} U_i\phi_i$ be the piecewise linear finite element solution. Show that the matrix equation satisfied by $\mathbf{U} = \{U_i\}_{i=1}^N$ and $\widehat{\mathbf{F}} = \{\hat{f}_i\}_{i=1}^{N+1}$ reads

$$(\mathbf{K} + \mathbf{M}) \cdot \mathbf{U} = \widehat{\mathbf{M}}\widehat{\mathbf{F}},$$

for suitable entries $\hat{f}_i$. Compute the entries of the $N \times N$ stiffness matrix $\mathbf{K}$ and mass matrix $\mathbf{M}$, as well as the $N \times (N+1)$ mass matrix $\widehat{\mathbf{M}}$, by assembling the elementary contributions of each interval (x_i, x_{i+1}).

2 (15 pts). *L^2-interpolation estimate.* Let $u \in H^2(0,1)$, that is u admits two week derivatives in $L^2(0,1)$. Let u_I be the piecewise linear interpolant of u over a partition $\mathcal{T} = \{T\}$ of $(0,1)$ of size h, i.e. $h = \max_{T\in\mathcal{T}} h_T$. Prove that

$$\|u - u_I\|_{L^2(0,1)} \le C\Big(\sum_{T\in\mathcal{T}} h_T^4 \|u''\|_{L^2(T)}^2\Big)^{\frac{1}{2}} \le Ch^2\|u''\|_{L^2(0,1)}.$$

Hint: Show the validity of Poincarè inequality for $v \in W_2^1(0,1)$ with $v(0) = 0$:

$$\|v\|_{L^2(0,1)} \le \|v'\|_{L^2(0,1)}.$$

Then use a scaling argument as explained in class and take $v = u - u_I$.

3 (20 pts). Write a MATLAB program `2-POINT` that implements piecewise linear finite elements over a general mesh $\mathcal{T} = \{x_i\}_{i=0}^N$ for the 2-point boundary value problem of HW#1-Pb6.

(a) Organize the program in such a way that computations are done elementwise and then assemble to give rise to the stiffness and mass matrices $\mathbf{K}, \mathbf{M}$ and right-hand side $\mathbf{F}$ of Pb#1. Compute the integrals using the midpoint rule.

(b) Check the rate of convergence in the H^1-norm and L^∞-norm over a uniform mesh with meshsize $h = \frac{1}{5}2^{-k}$ for $0 \le k \le 5$ and $b = 0, 100$. Plot the H^1-error and L^∞-error vs h in a log-log plot. Explain the results.

(c) Plot the exact solution and the finite element solution for $h = 1/20, 1/80$ and $b = 0, 100$. How these plot compare with those in HW#1. Draw conclusions.

(d) Consider a graded mesh $\mathcal{T}$ of the form $x_i = 1 - \big(\frac{N-i}{N}\big)^\beta$ for some $\beta > 1$. Experiment with different values of β and see the consequences for $b = 100$. Try to find a suitable value of β using the principle of error equilibration.

4 (15 pts). *Robin Problem.* Let $\Omega := (0,1)$ and $u \in H^1(\Omega)$ be the solution of HW#1-Pb2.

(a) Write a finite element discretization using piecewise linear elements.

(b) Prove an error estimate for the error in $H^1(\Omega)$. Specify clearly the required regularity of u and that of the coefficients and forcing function.

5 (15 pts). *Graded meshes.* Consider $u(x) = \sqrt{x}$ over $I = [0,1]$, and $\mathcal{T} = \{x_j\}_{j=0}^J$. The behavior $u \approx \sqrt{x}$ corresponds to a crack in a two dimensional problem (reentrant corner with internal angle $\omega = 2\pi$).

(a) Show that

$$\|u - I_{\mathcal{T}} u\|_{L^\infty(x_i, x_{i+1})} = \frac{\left(\sqrt{x_{i+1}} - \sqrt{x_i}\right)^2}{4\left(\sqrt{x_{i+1}} + \sqrt{x_i}\right)}.$$

Conclude that $\|u - I_{\mathcal{T}} u\|_{L^\infty(I)} \geq \frac{1}{4\sqrt{N}}$ provided $\mathcal{T}$ is uniform, i.e. $h = h_i$ for all i.

(b) Suppose $\mathcal{T}$ is graded so that $x_i = (\frac{i}{N})^4$. Show that

$$\|u - I_{\mathcal{T}} u\|_{L^\infty(x_i, x_{i+1})} = \frac{1}{4N^2}\left(2 - \frac{1}{i^2 + (i-1)^2}\right).$$

Conclude that the global maximum error is $\leq \frac{1}{2N^2}$, which is the best approximation possible with $N+1$ points. Compare with (a) and draw conclusions.

6 (10 pts). *Quadrature.* Let $\mathcal{T} = \{x_i\}_{i=0}^N$ be a partition of $\Omega = (0,1)$. Let $Q(w) = \sum_{i=1}^N Q_i(w)$ be the trapezoidal quadrature rule, where

$$Q_i(w) := \frac{h_i}{2}\Big(w(x_{i-1}) + w(x_i)\Big)$$

and $h_i = x_i - x_{i-1}$ is the local meshsize. Show that for all $w \in W_1^2(\Omega)$ the following error estimate holds

$$\Big|Q(w) - \int_\Omega w\Big| \leq C \sum_{i=1}^N h_i^2 \int_{x_{i-1}}^{x_i} |w''|.$$

Hint: Use the fact that Q is exact for piecewise linear functions and apply the Bramble-Hilbert lemma.

7 (10 pts). *Advection-diffusion equation.* Consider the HW#1-Pb7, namely

$$-\epsilon u'' + u' = 0, \qquad u(0) = u(1) = 0.$$

(a) Write the finite element discretization with piecewise linear elements over a uniform partition with meshsize h. Compare with the centered finite difference method of HW#1-Pb7.

(b) Consider the *modified* equation

$$-\Big(\epsilon + \frac{h}{2}\Big)u'' + u' = 0.$$

Write the corresponding finite element discretization. Compare with the upwind finite difference method of HW#1-Pb7.

(c) The *streamline diffusion method* is a Petrov-Galerkin method that utilizes test functions different from the trial ones. In 1D it reads

$$U \in \mathbb{V}_h: \quad B[U,V] = \int_0^1 \epsilon U'V' + U'(V + \frac{h}{2}V') = 0 \quad \forall V \in V_h.$$

Show that this is equivalent to approximating the modified equation with a Galerkin method.