1 (15 pts). 2-point boundary value problem. Consider the following boundary value problem in (0, 1):

\[-u'' + u = f, \quad u'(0) = 1, u(1) = 1.\]

(a) Write the variational formulation.

(b) Let \( x_1 = 0 < x_2 < \cdots < x_N < x_{N+1} = 1 \) be a partition of (0, 1) with \( h_i = x_{i+1} - x_i \). Let \( \{\phi_i\}_{i=1}^{N+1} \) be the corresponding basis of hat functions, and let \( f = \sum_{i=1}^{N+1} f_i \phi_i \) be piecewise linear. Let \( U = \sum_{i=1}^{N} U_i \phi_i \) be the piecewise linear finite element solution. Show that the matrix equation satisfied by \( U \) reads

\[(K + M) \cdot U = \hat{M} \hat{F},\]

for suitable entries \( \hat{f}_i \). Compute the entries of the \( N \times N \) stiffness matrix \( K \) and mass matrix \( M \), as well as the \( N \times (N+1) \) mass matrix \( \hat{M} \), by assembling the elementary contributions of each interval \((x_i, x_{i+1})\).

2 (15 pts). \( L^2 \)-interpolation estimate. Let \( u \in H^2(0, 1) \), that is \( u \) admits two weak derivatives in \( L^2(0, 1) \). Let \( u_I \) be the piecewise linear interpolant of \( u \) over a partition \( T = \{T\} \) of size \( h \), i.e. \( h = \max_{T \in T} h_T \). Prove that

\[\|u - u_I\|_{L^2(0, 1)} \leq C \left( \sum_{T \in T} h_T^4 \|u''\|^2_{L^2(T)} \right)^{\frac{1}{2}} \leq C h^2 \|u''\|_{L^2(0, 1)}.\]

Hint: Show the validity of Poincaré inequality for \( v \in W^1_2(0, 1) \) with \( v(0) = 0 \):

\[\|v\|_{L^2(0, 1)} \leq \|v'\|_{L^2(0, 1)}.\]

Then use a scaling argument as explained in class and take \( v = u - u_I \).

3 (20 pts). Write a MATLAB program 2-POINT that implements piecewise linear finite elements over a general mesh \( T = \{x_i\}_{i=0}^N \) for the 2-point boundary value problem of HW#1-Pb6.

(a) Organize the program in such a way that computations are done elementwise and then assemble to give rise to the stiffness and mass matrices \( K, M \) and right-hand side \( F \) of Pb#1. Compute the integrals using the midpoint rule.

(b) Check the rate of convergence in the \( H^1 \)-norm and \( L^\infty \)-norm over a uniform mesh with meshsize \( h = \frac{1}{2} 2^{-k} \) for \( 0 \leq k \leq 5 \) and \( b = 0, 100 \). Plot the \( H^1 \)-error and \( L^\infty \)-error vs \( h \) in a log-log plot. Explain the results.

(c) Plot the exact solution and the finite element solution for \( h = 1/20, 1/80 \) and \( b = 0, 100 \). How these plot compare with those in HW#1. Draw conclusions.

(d) Consider a graded mesh \( T \) of the form \( x_i = 1 - \left( \frac{N-i}{N} \right)^{\beta} \) for some \( \beta > 1 \). Experiment with different values of \( \beta \) and see the consequences for \( b = 100 \). Try to find a suitable value of \( \beta \) using the principle of error equilibration.

4 (15 pts). Robin Problem. Let \( \Omega := (0, 1) \) and \( u \in H^1(\Omega) \) be the solution of HW#1-Pb2.
(a) Write a finite element discretization using piecewise linear elements.

(b) Prove an error estimate for the error in $H^1(\Omega)$. Specify clearly the required regularity of $u$ and that of the coefficients and forcing function.

5 (15 pts). Graded meshes. Consider $u(x) = \sqrt{x}$ over $I = [0, 1]$, and $T = \{x_j\}_{j=0}^J$. The behavior $u \approx \sqrt{x}$ corresponds to a crack in a two dimensional problem (reentrant corner with internal angle $\omega = 2\pi$).

(a) Show that
$$\|u - I_T u\|_{L^\infty(x_i, x_{i+1})} = \left(\frac{\sqrt{x_{i+1}} - \sqrt{x_i}}{4(\sqrt{x_{i+1}} + \sqrt{x_i})}\right)^2.$$ 

Conclude that $\|u - I_T u\|_{L^\infty(I)} \geq \frac{1}{4\sqrt{N}}$ provided $T$ is uniform, i.e. $h = h_i$ for all $i$.

(b) Suppose $T$ is graded so that $x_i = \left(\frac{i}{N}\right)^4$. Show that
$$\|u - I_T u\|_{L^\infty(x_i, x_{i+1})} = \frac{1}{4N^2} \left(2 - \frac{1}{\sqrt{i+(i-1)^2}}\right).$$

Conclude that the global maximum error is $\leq \frac{1}{2N^2}$, which is the best approximation possible with $N + 1$ points. Compare with (a) and draw conclusions.

6 (10 pts). Quadrature. Let $T = \{x_i\}_{i=0}^N$ be a partition of $\Omega = (0, 1)$. Let $Q(w) = \sum_{i=1}^N Q_i(w)$ be the trapezoidal quadrature rule, where
$$Q_i(w) := \frac{h_i}{2} \left(w(x_{i-1}) + w(x_i)\right)$$
and $h_i = x_i - x_{i-1}$ is the local meshsize. Show that for all $w \in W^2_1(\Omega)$ the following error estimate holds
$$\left|Q(w) - \int_\Omega w\right| \leq C \sum_{i=1}^N h_i^2 \int_{x_{i-1}}^{x_i} |w''|.$$ 

Hint: Use the fact that $Q$ is exact for piecewise linear functions and apply the Bramble-Hilbert lemma.

7 (10 pts). Advection-diffusion equation. Consider the HW#1-Pb7, namely
$$-\epsilon u'' + u' = 0, \quad u(0) = u(1) = 0.$$ 

(a) Write the finite element discretization with piecewise linear elements over a uniform partition with meshsize $h$. Compare with the centered finite difference method of HW#1-Pb7.

(b) Consider the modified equation
$$-(\epsilon + \frac{h}{2}) u'' + u' = 0.$$ 

Write the corresponding finite element discretization. Compare with the upwind finite difference method of HW#1-Pb7.

(c) The streamline diffusion method is a Petrov-Galerkin method that utilizes test functions different from the trial ones. In 1D it reads
$$U \in \mathbb{V}_h : \quad B[U, V] = \int_0^1 \epsilon U'V' + U'\left(V + \frac{h}{2}V'\right) = 0 \quad \forall V \in \mathbb{V}_h.$$ 

Show that this is equivalent to approximating the modified equation with a Galerkin method.