AMSC 667 Section 0101 Spring 2011

NUMERICAL ANALYSIS II

HOMEWORK #1 (Pbs 1-3 due Feb 10, Pbs 4-6 due Feb 17)

1 (20 pts). *Convexity and Dissipativity*: Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a C^2 *convex* function, namely ϕ satisfies for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $0 \le s \le 1$

$$\phi\big((1-s)\mathbf{x} + s\mathbf{y}\big) \le (1-s)\phi(\mathbf{x}) + s\phi(\mathbf{y}) \tag{1}$$

(a) Show that ϕ satisfies the inequality

$$\phi(\mathbf{y}) - \phi(\mathbf{x}) - \mathbf{f}(\mathbf{x})(\mathbf{y}-\mathbf{x}) \ge \frac{\gamma}{2}\|\mathbf{y}-\mathbf{x}\|^2 \quad \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \tag{2}$$

with $\gamma = 0$ and $\mathbf{f}(\mathbf{x}) = D\phi(\mathbf{x})$ the gradient of ϕ. If ϕ satisfies (2) with $\gamma > 0$ we say that ϕ is *strictly convex*. Give a geometric interpretation to (2).

(b) Show that $\mathbf{f}$ is *monotone* (*uniformly monotone* if $\gamma > 0$), namely $\mathbf{f}$ satisfies

$$\langle \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \ge \gamma \|\mathbf{y}-\mathbf{x}\|^2 \qquad \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \tag{3}$$

(Hint: consider (2) with $\mathbf{x}$ and $\mathbf{y}$ interchanged);

(c) Show that $D^2\phi(\mathbf{x})$, the Hessian of ϕ, is uniformly positive definite, namely

$$\mathbf{z}^T D^2\phi(\mathbf{x})\mathbf{z} \ge \gamma\|\mathbf{z}\|^2 \quad \forall\, \mathbf{z} \in \mathbb{R}^n.$$

(Hint: write $\mathbf{y} = \mathbf{x} + t\mathbf{z}$ with $\mathbf{z}$ fixed, use the integral representation of (2), and take $t \to 0$).

(d) Consider the autonomous system of *ODE*, so-called *gradient flow*,

$$\mathbf{y}' = -D\phi(\mathbf{y}). \tag{4}$$

Show that any solution of (4) decreases the energy, namely

$$\frac{d}{dt}\phi(\mathbf{y}(t)) = -|D\phi(\mathbf{y}(t))|^2 \le 0.$$

2 (15 pts). *Discrete Gronwall inequality*: Let ϕ_n, η_n, χ_n, with $\chi_n \ge 0$, satisfy the recurrence relation:

$$\phi_n \le \eta_n + \sum_{k=1}^{n} \chi_{k-1}\phi_{k-1} \qquad (n \ge 1). \tag{5}$$

(a) Set $R_n = \sum_{k=1}^n \chi_{k-1}\phi_{k-1}$, $R_0 = 0$, and write (5) as $R_{n+1} - R_n - \chi_n R_n \le \chi_n \eta_n$ for $n \ge 0$. Multiply by the integrating factor $\prod_{k=0}^n (1+\chi_k)^{-1}$, with $\prod_{k=n}^{n-1}(1+\chi_k)^{-1} = 1$, to derive the inequality:

$$\phi_n \le \eta_n + \sum_{k=0}^{n-1}\Big(\eta_k \chi_k \prod_{i=k+1}^{n-1}(1+\chi_i)\Big).$$

(b) Assume that η_n is non-decreasing. Prove

$$\phi_n \le \eta_n \prod_{i=0}^{n-1}(1+\chi_i) \le \eta_n e^{\sum_{i=0}^{n-1}\chi_i}. \tag{6}$$

3 (15 pts). *Trapezoidal Method*: Consider the following method with variable stepsize h_{n+1}:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h_{n+1}}{2}\Big(\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})\Big). \tag{7}$$

(a) Find a condition on h_{n+1} such that the nonlinear problem (7) admits a unique solution $\mathbf{y}_{n+1}$.
(b) Find an expression for the truncation error and prove a global error estimate (Hint: use Taylor's formula and the discrete Gronwall's inequality).
(c) Remove the exponential constant in (b) provided $-\mathbf{f}$ is monotone, namely it satisfies (3) with $\gamma = 0$.

4 (15 pts). Consider the following Predictor-Corrector Method as a way to approximate (7):

$$\mathbf{y}_{n+1}^{(0)} = \mathbf{y}_n + h_{n+1}\mathbf{f}(\mathbf{x}_n, \mathbf{y}_n)$$
$$\mathbf{y}_{n+1}^{(j)} = \mathbf{y}_n + \frac{h_{n+1}}{2}\Big(\mathbf{f}(\mathbf{x}_n, \mathbf{y}_n) + \mathbf{f}(\mathbf{x}_{n+1}, \mathbf{y}_{n+1}^{(j-1)})\Big), \qquad j \geq 1.$$

(a) Determine under which condition(s) this fixed point iteration converges, that is $\mathbf{y}_{n+1}^{(j)} \to \mathbf{y}_{n+1}$ as $j \to \infty$ where $\mathbf{y}_{n+1}$ is the solution of (7).
(b) Show that $\|\mathbf{y}_{n+1} - \mathbf{y}_{n+1}^{(j)}\| = O(h_{n+1}^{2+j})$. Conclude that one iteration is sufficient to preserve the truncation error of (7). The resulting scheme for $j = 1$ is an explicit *2nd order Runge-Kutta Method* with variable stepsize.

5 (15 pts). Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix. For $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ being given, consider the linear system of n ODEs

$$\mathbf{y}' + \mathbf{A}\mathbf{y} = \mathbf{f}. \tag{8}$$

(a) Determine the magnification factor for the Forward Euler (FE), Backward Euler (BE), and Trapezoidal Methods (TM). Such a factor is expressed in terms of the spectral radius of a suitable matrix that is to be found. Hint: use the spectral decomposition of $\mathbf{A}$.
(b) Discuss the resulting stability constraints and relate to absolute stability.
(c) The system (8) arises, for instance, from a space discretization of the Heat Equation $u_t - \Delta u = f$. The resulting matrix $\mathbf{A}$ exhibits very disparate eigenvalues (*stiff* system). Explain why BE and TM are always preferred for time stepping (8).
(d) Write the Crank-Nicolson (CN) method for (8) and compare with TM. What order do you expect FE, BE, TM, and CN to have? What type of error estimate do you expect?

6 (20 pts). Consider the following second order initial value problem modeling a spring-dashpot system:

$$y'' + 101y' + 100y = \sin x, \qquad y(0) = 2, y'(0) = 0. \tag{9}$$

(a) Solve this problem by hand. To this end find first the solution to the homogeneous equation (natural modes), and next a particular solution using the method of undetermined coefficients. Explain whether the problem is *stiff* or not.
(b) Convert (9) into a first order system, and write the forward Euler (FE), backward Euler (BE) and Trapezoidal methods (TM).
(c) Write MATLAB programs that implement FE, BE, and TM with step-sizes $h = 10^{-k}$ for $k = 1, 2, 3$ on the interval $(0, 10)$.
(d) Find the error between computed solutions of (c) and exact solution of (a) at $t_n = nh$ and plot the results. Explain the results in terms of absolute stability, and draw conclusions.