AMSC 667 Section 0101 Spring 2011

NUMERICAL ANALYSIS II

HOMEWORK #2 (Pbs 1-3 due Feb 24, Pbs 4-7 due Mar 10)

1 (20 pts). *Discontinuous Galerkin (dG) Method*: Let $\lambda > 0$ and $y(t)$ be the solution of the following linear ODE with initial condition $y(0) = y_0$:

$$y' + \lambda y = f(t).$$

Let $\mathbb{V}_\mathcal{T}$ be the space of discontinuous piecewise linear functions over a partition $\mathcal{T} = \{t_n\}_{n=0}^N$ of the interval $[0, T]$, with $0 = t_0 < t_1 < \cdots < t_N = 1$. The dG method of order 2 is a variational method to approximate $y(t)$ and reads as follows: find $Y \in \mathbb{V}_\mathcal{T}$ such that $Y_0 = y_0$ and

$$\int_{t_n}^{t_{n+1}} (Y'V + \lambda YV)dt + (Y_n^+ - Y_n)V_n^+ = \int_{t_n}^{t_{n+1}} fV\,dt \qquad \forall V \in \mathbb{V}_\mathcal{T},$$

where Y_n^+ is the value of $Y(t)$ from the right and $Y_n = Y(t_n)$ is the value from the left. To study this method one uses energy techniques similar to those of 2-point boundary value problems.

(a) Prove the identity $(b-a)b = \frac{1}{2}(b-a)^2 + \frac{1}{2}b^2 - \frac{1}{2}a^2$ for all $a, b \in \mathbb{R}$, and use it to show the stability bound

$$\frac{1}{2}|Y_{n+1}|^2 - \frac{1}{2}|Y_n|^2 + \frac{1}{2}|Y_n^+ - Y_n|^2 + \int_{t_n}^{t_{n+1}} \lambda |Y|^2 dt = \int_{t_n}^{t_{n+1}} fY\,dt.$$

(b) Use (a) to show that there exists a unique solution Y of the dG method.

(c) Let $I_\mathcal{T} y$ be the dG interpolant of y with values in $\mathbb{V}_\mathcal{T}$ defined by

$$I_\mathcal{T} y(t_{n+1}) = y(t_{n+1}), \qquad \int_{t_n}^{t_{n+1}} \big(I_\mathcal{T} y(t) - y(t)\big)dt = 0.$$

Show the error estimate ($h_{n+1} = t_{n+1} - t_n$)

$$\int_{t_n}^{t_{n+1}} |I_\mathcal{T} y(t) - y(t)|^2 dt \lesssim h_{n+1}^4 \int_{t_n}^{t_{n+1}} |y''(t)|^2 dt.$$

(d) Use integration by parts to show that $I_\mathcal{T} y$ satisfies

$$\int_{t_n}^{t_{n+1}} \big((I_\mathcal{T} y)'V + \lambda I_\mathcal{T} yV\big)dt + (I_\mathcal{T} y_n^+ - I_\mathcal{T} y_n)V_n^+ = \int_{t_n}^{t_{n+1}} \big(fV + \lambda(I_\mathcal{T} y - y)V\big)dt \qquad \forall V \in \mathbb{V}_\mathcal{T}.$$

(e) Write the equation satisfied by the error $E = I_\mathcal{T} y - Y$, and combine (a) and (c) to derive the following error estimate:

$$|E_n|^2 + \sum_{i=1}^{n-1} |E_i^+ - E_i|^2 + \int_0^{t_n} \lambda |E(t)|^2 dt \lesssim \lambda \sum_{i=0}^{n-1} h_{i+1}^4 \int_{t_i}^{t_{i+1}} |y''(t)|^2 dt.$$

2 (15 pts). *Adams-Moulton Formula*: (a) Let the stepsize h be uniform. Let $p(t)$ be the polynomial of degree 2 that interpolates $f(t_i, y_i)$ $(i = n-1, n, n+1)$. Integrate $p(t)$ on $[t_n, t_{n+1}]$ to derive

$$y_{n+1} = y_n + \frac{h}{12}\Big(5f(t_{n+1}, y_{n+1}) + 8f(t_n, y_n) - f(t_{n-1}, y_{n-1})\Big).$$

(b) Prove that the truncation error is $O(h^4)$ and verify the consistency conditions directly.
(c) Derive an error estimate.

3 (15 pts). *Gear's Formula*: (a) Let the stepsize h be uniform. Interpolate (t_i, y_i) $(i = n-1, n, n+1)$ with a polynomial $p(t)$ of degree 2 and next set $p'(t_{n+1}) = f(t_{n+1}, y_{n+1})$ to deduce the following *backward differentiation formula* BDF(2):

$$y_{n+1} = \frac{4}{3}y_n - \frac{1}{3}y_{n-1} + \frac{2}{3}hf(t_{n+1}, y_{n+1}).$$

(b) Using Taylor's formula prove that the truncation error is $O(h^3)$. Even though this implies consistency, verify the consistency conditions directly.

4 (15 pts). *Root condition*: Consider the Adams-Moulton multistep method of Pb2.

(a) Study stability and convergence using the root condition. Find the order of this method.
(b) Show that the interval of absolute stability is given by $-6 \le \lambda h \le 0$. Explain why this method is not well suited for stiff problems.

Consider the Gear multistep method of Pb3.

(c) Study stability and convergence using the root condition. Find the order of this method.
(d) Check the strong root condition and examine relative stability.
(e) Verify absolute stability for all $\lambda \le 0$. Explain why this method is well-suited for *stiff* problems.

5 (10 pts). Show that the two-step method

$$y_{n+1} = 2y_{n-1} - y_n + h\Big(\frac{5}{2}f(t_n, y_n) + \frac{1}{2}f(t_{n-1}, y_{n-1})\Big)$$

is of order 2 but does not satisfy the root condition.
(b) Construct an example to show that this method need not converge when solving $y' = f(t, y)$.

6 (15 pts). Solve HW1-Pb6 with the following variable-order MATLAB built-in solvers:

- `ode15s`: this is similar to the Gear's multistep method or BDF;
- `ode113`: this is the multistep method which combines Adams-Bashforth and Adams-Moulton formulas;
- `ode23`: this is the single step method of Bogacki and Shampine of order 2-3.
- `ode45`: this is a single step method of Dormand and Prince similar to Runge-Kutta-Fehlberg (4-5).

The basic syntax is

```
[T,Y]=solver('function',[t0 tf],y0,options),
```

where `solver` is any of the above solvers, `function` is an m file with the system of ODEs, `t0` is the initial condition, `tf` is the final condition, `y0` is the vector of initial values, `T` is a column vector of time points and `Y` is a solution array (see handout for details). The argument `options` sets, among other things, the relative and absolute accuracy, tells the solver if the Jacobian is constant in case of linear systems, and provides statistics of the computation useful to determine the computational cost.

Note that multistep methods are selfstarting, namely they do not need starting values; this is achieved by combining different methods of increasing order and adaptively selecting the mesh-size. Run the programs with relative accuracy `RelTol`$=10^{-2}$ and 10^{-4}, `JConstant=on` and `Stats=on`. The syntax could be

```
options = odeset('RelTol',1e-4,'JConstant',on,'Stats',on)
```

Compare the results and computational costs. Plot pictures of the computed and true solutions, and draw conclusions.

7 (10 pts). *Error Control*: This problem illustrates how to achieve global error control, a highly desirable goal, via local error control. Consider the case of practical interest consisting of a system of n linear ODEs

$$\mathbf{y}' + \mathbf{A}\mathbf{y} = \mathbf{f}, \tag{1}$$

where $\mathbf{A}$ is a symmetric and positive definite $n \times n$ matrix. Let $0 < \varepsilon \ll 1$ be a given tolerance, and let a single-step method produce the stepsize h_{n+1} and discrete solution $\mathbf{Y}_{n+1}$ so that the following *local* error bound holds

$$\|\mathbf{u}_n(t_{n+1}) - \mathbf{Y}_{n+1}\|_2 \leq h_{n+1}\varepsilon, \tag{2}$$

where $\mathbf{u}_n$ is the true solution to (7) passing through $\mathbf{y}_n$ at t_n. Prove the following *global* error estimate:

$$\|\mathbf{y}(t_n) - \mathbf{Y}_n\|_2 \leq \varepsilon(t_n - t_0). \tag{3}$$

This not only shows the global effect of local error control, but also the fact that error estimates depending exponentially of $t_n - t_0$ can be overly pessimistic.
Hint: Convert (1) into a diagonal system in which the equations decouple. Show $|y(t_{n+1}) - u_n(t_{n+1})| \leq |y(t_n) - Y_n|$ for scalar equations such as (1), and next combine this with the scalar version of (2) to derive the crucial inequality

$$|y(t_{n+1}) - Y_{n+1}| \leq |y(t_n) - Y_n| + h_{n+1}\varepsilon.$$

Show that this leads to (3) in the scalar case and, finally, extend the argument to the vector-valued case.