AMSC 667 Section 0101 Spring 2011

NUMERICAL ANALYSIS II

HOMEWORK #4 (Pbs 1-2 due Apr 14, Pbs 3-5 due Apr 19)

1 (20 pts) *Robin boundary condition*: Consider the following strong form of a boundary value problem

$$-(p(x)u')' = f(x) \quad x \in (0,1), \qquad p(0)u'(0) - \alpha_1 u(0) = \gamma_1, \quad p(1)u'(1) + \alpha_2 u(1) = \gamma_2, \tag{1}$$

with constants $\alpha_1, \alpha_2 > 0$ and $\gamma_1, \gamma_2 \in \mathbb{R}$.

(a) Derive a variational formulation for (1), namely find a function space $\mathbb{V}$, a bilinear form $B : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ and a linear form $L : V \to \mathbb{R}$ such that

$$u \in \mathbb{V} : \quad B(u,v) = L(v) \quad \forall v \in \mathbb{V}. \tag{2}$$

(b) Show that B is coercive and continuous in $\mathbb{V}$ and $L \in \mathbb{V}^*$, namely L is linear and continuous.

(c) Show that (2) admits a unique solution $u \in \mathbb{V}$.

2 (20 pts) *Green's function*: Let δ_{x_0} be the Dirac delta at $x = x_0$ in the domain $\Omega = (0,1)$, namely

$$\langle \delta_{x_0}, v \rangle = v(x_0) \quad \forall v \in C_0^\infty(\Omega). \tag{3}$$

(a) Show that $\delta_{x_0} \in H^{-1}(\Omega)$.

(b) Consider the boundary value problem in Ω with constant $\alpha \geq 0$

$$Lg_{x_0} = -g''_{x_0} + \alpha g_{x_0} = \delta_{x_0} \qquad g_{x_0}(0) = g_{x_0}(1) = 0 \tag{4}$$

Find the solution $g_{x_0} \in H_0^1(\Omega)$ by solving the strong form (4) on the left and right of x_0 and using the condition at $x = x_0$. Show that g_{x_0} is piecewise linear provided $\alpha = 0$.

(c) Relate $g_{x_0} \in H_0^1(\Omega)$ to the Riesz representative of $\delta_{x_0} \in H^{-1}(\Omega)$.

(d) Let $u \in H_0^1(\Omega)$ be the weak solution of $Lu = f \in L^1(\Omega)$. Show the representation formula

$$u(x_0) = \int_0^1 f(x) g_{x_0}(x) dx \quad \forall x_0 \in \Omega. \tag{5}$$

3 (20 pts) *Interpolation error estimates*: Let $\mathcal{T}$ be the partition $0 = x_0 < x_1 < \cdots < x_{n+1} = 1$ of $\Omega = [0,1]$, and let $\omega_i = (x_i, x_{i+1}), h_i = x_{i+1} - x_i$ for $0 \leq i \leq n$, and $h = \max_i h_i$ be the meshsize of $\mathcal{T}$.

(a) Let $\xi_i \in (x_i, x_{i-1})$ be fixed, and let $J_\mathcal{T}$ be an interpolation operator onto the space $\mathbb{V}_0$ of piecewise constant functions over $\mathcal{T}_h$ defined by

$$J_\mathcal{T} u(\xi_i) = u(\xi_i) \qquad \forall\ u \in H^1(\Omega).$$

Show the *local* error estimate

$$\|u - J_\mathcal{T} u\|_{L^2(\omega_i)} \leq h_i \|u'\|_{L^2(\omega_i)}, \tag{6}$$

and deduce a *global* estimate in Ω involving h.

(b) Let $J_\mathcal{T}$ be now the L^2-projection onto the space $\mathbb{V}_0$. Show that $J_\mathcal{T} u = h_i^{-1} \int_{\omega_i} u$ and prove (6).

(c) Let $I_\mathcal{T}$ be the (Lagrange) interpolation operator over the space $\mathbb{V}_1$ of continuous piecewise linear functions, that is

$$I_\mathcal{T} u(x_i) = u(x_i) \qquad \forall\ u \in H^1(\Omega).$$

Show the *local* error estimates

$$\|u' - (I_\mathcal{T} u)'\|_{L^2(\omega_i)} \leq h_i \|u''\|_{L^2(\omega_i)} \quad \|u - I_\mathcal{T} u\|_{L^2(\omega_i)} \leq C h_i^2 \|u''\|_{L^2(\omega_i)}, \qquad \forall\ u \in H^2(\Omega), \tag{7}$$

and derive *global* estimates in Ω involving h. Hint: In both cases argue with a smooth u, and use Rolle's theorem. Next extend the estimates to $H^1(\Omega)$ and $H^2(\Omega)$ by a density argument.

4 (20 pts). *Graded meshes*: Let $u(x) = x^\alpha$ with $\frac{1}{2} < \alpha < 1$ and $x \in \Omega = (0,1)$. Consider the 2-point boundary value problem

$$-u'' = f(x), \qquad u(0) = 0 \quad u(1) = 1. \tag{8}$$

(a) Show that $u \in H^1(\Omega)$, $f \in H^{-1}(\Omega)$, and $f \notin L^1(\Omega)$. This illustrates that $H^{-1}(\Omega)$ is strictly larger than $L^1(\Omega)$.

(b) Let $\mathcal{T} = \{x_i\}_{i=0}^{N+1}$ be a partition of Ω with $x_0 = 0$ and $x_N = 1$, let $\mathbb{V}(\mathcal{T})$ be the finite element space of continuous piecewise linear functions over $\mathcal{T}$, and let $U \in \mathbb{V}(\mathcal{T})$ be the Galerkin solution. Write the equation satisfied by U, including boundary conditions for both U and the test function, and show that

$$\|u' - U'\|_{L^2(\Omega)} = \inf_{V \in \mathbb{V}_\mathcal{T}} \|u' - V'\|_{L^2(\Omega)}.$$

(c) Let $\mathcal{T}$ be a *quasi-uniform* mesh. Explain whether or not the following estimate is valid

$$\|u' - U'\|_{L^2(\Omega)} \lesssim N^{-1}.$$

(d) Let $\mathcal{T}$ be a *graded* mesh so that $x_i = (\frac{i}{N})^\beta$ with $\beta > \frac{2}{2\alpha - 1}$. Show that

$$\|u' - U'\|_{L^2(\Omega)} \lesssim N^{-1}.$$

This shows that optimal estimates can be achieved by suitable mesh grading. Hint: Use the local interpolation estimates (7) on $\omega_i = (x_i, x_{i+1})$ for $i > 0$ and add over i. Consider the interval $(0, x_1)$ separately.

5 (20 pts) *Duality argument*: Let u be the weak solution to the Dirichlet problem on $\Omega = (0,1)$

$$Lu = -(p(x)u')' + q(x)u = f(x), \qquad u(0) = u(1) = 0,$$

with $0 < P_1 \le p(x) \le P_2$ and $0 \le q(x) \le Q$ smooth. Let $U \in \mathbb{V}(\mathcal{T})$ be the finite element solution on the space of C^0 piecewise linear functions $\mathbb{V}(\mathcal{T})$ over $\mathcal{T}$. Prove the error estimate

$$\|u - U\|_{L^2(\Omega)} \le Ch^2 \|f\|_{L^2(\Omega)}. \tag{9}$$

In view of (7), this is the best approximation result. Note also that the quadratic rate (9) is achieved without the restrictions of mesh uniformity and regularity $u \in C^4(\Omega)$ (compare with finite differences). Hint: Let $\phi \in H_0^1(\Omega)$ be the solution of $L\phi = u - U$. Integrate by parts to express $\|u - U\|_{L^2(\Omega)}^2$ in terms of ϕ, and next use the error equation for $u - U$ (Galerkin orthogonality). Finally employ the regularity estimate $\|v''\|_{L^2(\Omega)} \le C\|Lv\|_{L^2(\Omega)}$ for both $v = u$ and $v = \phi$.