MATH/AMSC 673 (Fall 2015)
PARTIAL DIFFERENTIAL EQUATIONS I
HOMEWORK # 1 (Pbs 1-3 due Tue Sep 15, Pbs 4-6 due Tue Sep 22)

1 (15 pts). *Transport PDE.* Let $\mathbf{b} \in \mathbf{R}^n$ and $c \in \mathbf{R}$ be constants. Derive an explicit expression for the solution u of

$$\begin{cases} u_t + \mathbf{b} \cdot Du + cu = 0 & \text{in } \mathbf{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbf{R}^n \times \{t = 0\}. \end{cases}$$

Hint: Use the method of characteristics. Alternatively, convert this PDE into the transport equation upon multiplying by a suitable exponential.

2 (15 pts). *Diffusion PDE.* Let $\Omega \subset \mathbf{R}^n$ be an open and bounded domain occupied by a pollutant of concentration $\rho(x, t)$. Assuming that the pollutant diffuses from higher to smaller concentrations according to *Fick's law* (that is the flux is proportional to the negative gradient of ρ with proportionality constant or *diffusivity* k), derive the PDE satisfied by ρ in the following two cases:

(a) There are distributed sources and sinks $f(x, t)$ of contaminant in Ω.

(b) The pollutant is transported by a fluid that moves with velocity $\mathbf{v}(x, t)$ in Ω.

Hint: do a mass balance calculation similar to the energy balance for the heat equation.

3 (20 pts). *Heat equation in* 1*d.* Solve the following Cauchy problem in $\mathbf{R}$

$$u_t - u_{xx} = 0, \qquad u(x, 0) = \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases}$$

Hint: Seek a solution in the form $u(x, t) = \phi(x/\sqrt{t})$ and derive an ODE for ϕ.

4 (20 pts). *Energy approach.* Let Ω be a smooth and bounded domain in $\mathbf{R}^n$ and let $f : \Omega \to \mathbf{R}$ be a smooth function. Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a minimizer of the functional

$$I[u] := \int_\Omega \Big(\sqrt{1 + |Du|^2} - fu \Big) dx$$

subject to the Dirichlet boundary condition $u = g$ on $\partial\Omega$. Proceed as with the Laplace's equation to s how that u satisfies the *prescribed mean curvature equation*

$$-\mathrm{div}\Big(\frac{Du}{\sqrt{1 + |Du|^2}} \Big) = f.$$

In case $f = 0$ the PDE is called the *minimal surface equation.* Could you elaborate on these names?

5 (15 pts). *Existence and uniqueness.* The following minimization problems on the interval $[0, 1]$ are simple examples exhibiting features typical of phase transitions.

(a) *Nonuniqueness*: Construct an infinite family of minimizers of

$$\mathcal{E}[u] = \int_0^1 \left(1 - (u'(t))^2\right)^2 dt$$

over the set of all continuous piecewise C^1 functions satisfying $u(0) = u(1) = 0$. Hint: try piecewise linear functions with slope ± 1.

(b) *Nonexistence*: Show that there is no piecewise C^1 minimizer of

$$\mathcal{E}[u] = \int_0^1 \left(u(t)^2 + \left(1 - (u'(t))^2\right)^2\right) dt$$

over the same set as in (a). Hint: Use a sequence of solutions of (a) to show that a potential minimizer $\bar{u}$ would necessarily satisfy $\mathcal{E}[\bar{u}] = 0$. Reach then a contradiction.

6 (15 pts). *Wave equation.* We say that $u(x,t)$ is a *weak* solution to the wave equation in one space variable if

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x,t)\ (\phi_{tt}(x,t) - \phi_{xx}(x,t))\ dxdt = 0$$

for every $\phi \in C_0^2(\mathbf{R}^2)$. Here $C_0^2(\mathbf{R}^2)$ is the set of functions in $C^2(\mathbf{R}^2)$ that have compact support, i.e. that are identically zero outside a bounded set.

(a) Show that any *strong* (classical C^2) solution of the wave equation is also a weak solution;

(b) Show that discontinuous functions of the form

$$u(x,t) = H(x - t) \qquad u(x,t) = H(x + t)$$

are weak solutions of the wave equation. Here H is the Heaviside function

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases}$$