MATH/AMSC 673 (Fall 2015)
PARTIAL DIFFERENTIAL EQUATIONS I
HOMEWORK # 2 (Pbs 1-3 due Sep 29; Pbs 4-6 due Oct 8)

1 (15 pts). *Mean value formula* (Evans 2.5.3).

2 (20 pts). C^2 *Regularity.* This problem shows that the solution of $\Delta u = f$ with f continuous may *not* be C^2. However if f is Hölder continuous, then so is $D^2 u$.
(a) Let $k \geq 0$ be an integer, and consider the following Poisson equation in $\mathbf{R}^2$ with polar coordinates (r, θ) (here $i = \sqrt{-1}$):

$$\Delta u = e^{ik\theta} g(r).$$

If $g : [0, \infty) \to \mathbf{R}$ is continuous, we seek a solution u of the form

$$u(x, y) = r^k e^{ik\theta} v(r) = (x + iy)^k v(r). \tag{1}$$

Find a solution formula for v in terms of g.
Now suppose that g is smooth on $(0, \infty)$ and continuous on $[0, \infty)$ with $g(0) = 0$, and that $k = 2$.
(b) Use (a) to show that $rv'(r)$ and $r^2 v''(r)$ are also continuous at $r = 0$ and vanish there.
(c) Compute u_x, u_y and u_{xx} directly from (1) in terms of v and its derivatives. Show that $u_{xx} - 2v(r)$ is necessarily continuous in $\mathbf{R}^2$.
(d) Prove that is it possible that u is C^1 in $\mathbf{R}^2$, but u_{xx} does not remain bounded as $r \to 0$ (examine $v(r)$).

3 (15 pts). (Extension of Evans 2.5.5) Let Ω be a bounded domain in $\mathbf{R}^n$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Show that there is a constant depending only on n and Ω such that

$$\max_{x \in U} |u(x)| \leq C\Big(\max_{x \in \bar{\Omega}} |f(x)| + \max_{x \in \partial\Omega} |g(x)| \Big).$$

Hint: Construct a supersolution of (2) (that is a function $u^+ \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying $-\Delta u^+ \geq f$ in Ω and $u^+ \geq g$ on $\partial\Omega$), and use the maximum principle.

4 (20 pts) *Elliptic PDE.* The following is a generalization of the Poisson's equation and corresponding methods.
(a) *Energy Method.* Let $\mathbf{A}(x) \in \mathbf{R}^{n \times n}$ be a smooth, bounded, uniformly positive definite matrix, and let $c(x) \geq 0$ and $f(x)$ be smooth functions. Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a minimizer of the functional

$$I[u] = \int_U \Big(\frac{1}{2} \nabla u \cdot \mathbf{A}(x) \nabla u + \frac{1}{2} c(x) u^2 - f(x) u \Big) dx,$$

subject to the Dirichlet condition $u = g$ on ∂U, with g continuous. Proceed as with the Laplace's equation to derive the *weak formulation*

$$\int_U \nabla u \cdot \mathbf{A}(x) \nabla v + c(x) u v = \int_U f v, \qquad \forall v \in \mathcal{V} := \{v \text{ Lipschitz} : v = 0 \text{ on } \partial U\},$$

and the PDE satisfied by u.
(b) *Weak Maximum Principle.* Suppose $f, g \geq 0$. Show that $u \geq 0$. Hint: Show that $v = \min(u, 0)$ is an admissible test function, and use it to conclude that $u \geq 0$. Use the nontrivial fact that $\nabla v = \chi_{\{u<0\}} \nabla u$ a.e.
(c) *Uniqueness.* Show that there can be only one minimizer of $I[u]$ or, equivalently, only one solution $u \in C^2(U) \cap C^0(\bar{U})$ of the corresponding PDE.

5 (15 pts) *Lack of Regularity.* This problem shows that a harmonic function may not always be smooth up to the boundary. Let $z = x + iy$ be a complex number. Consider the analytic function $f(z) = z^\alpha$ with $0 < \alpha = \pi/\omega < 1$. It is known that both the real and imaginary parts of f are harmonic functions.
(a) Determine the imaginary part u of f in polar coordinates (r, θ) and show that it solves the homogeneous Dirichlet problem in the sector $S = \{0 < \theta < \omega\}$, that is u vanishes on the boundary of S. Show that ∇u is not bounded as $r \to 0$ provided $\omega > \pi$. This shows the effect of reentrant corners in the regularity of u.
(b) Consider $\alpha = 1/2$. Show that u is harmonic in the semispace $\{y > 0\}$ and satisfies the mixed boundary conditions:

$$u = 0 \quad \text{on } \{x > 0, y = 0\}, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \{x < 0, y = 0\}.$$

Examine the regularity of u as $r \to 0$. This illustrates the effect of changing boundary conditions.

6 (20 pts) *Poisson's formula* (Evans 2.5.8). Let u be harmonic in $\Omega = \mathbf{R}^n_+$ and given by Poisson's formula. Assume the Dirichlet condition g is bounded and $g(x) = |x|$ for $x \in \mathbf{R}^{n-1}, |x| \leq 1$. Show that ∇u is *not* bounded near $x = 0$. This shows that the solution of Laplace's equation with Lipschitz Dirichlet datum may not be Lipschitz. Hint: Estimate $\frac{u(\lambda e_n) - u(0)}{\lambda}$ as $\lambda \downarrow 0$.