MATH/AMSC 673 (Fall 2015)

PARTIAL DIFFERENTIAL EQUATIONS I

HOMEWORK # 4 (Pbs 1-3 due Nov 5, Pbs 4-6 due Nov 12)

1 (15 pts) *Regularizing effect.* Let u be the solution of the homogeneous heat equation in the cylinder $\Omega \times (0, \infty)$ with vanishing Dirichlet data and initial condition g. Prove that

$$\int_\Omega |\nabla u(x,t)|^2 dx \le \frac{1}{t} \int_\Omega |g(x)|^2 dx.$$

This shows that an initial datum merely in $L^2(\Omega)$ leads to a solution with gradient in $L^2(\Omega)$ for all $t > 0$. Hint: To this end multiply the PDE by tu_t and use the energy method.

2 (10 pts). *Backward heat equation.* Consider the equation

$$\begin{cases} u_t + u_{xx} = 0, & 0 < x < 1, \quad t > 0 \\ u(0,t) = u(1,t) = 0. \end{cases} \tag{1}$$

(a) Show that solutions of the form $u(x,t) = X(x)T(t)$ exist provided $X(x)$ is proportional to $\sin(n\pi x)$ and $T(t)$ to $e^{\pi^2 n^2 t}$ where n is any positive integer (separation of variables).

(b) Show that an initial value problem for (1) is *ill-posed*: given $\epsilon > 0$ and $M > 0$ there exists an initial condition $g = u(\cdot, 0)$ such that

$$\|g\|_{C(\mathbf{R})} \le 1 \qquad \|u(\cdot,t)\|_{C(\mathbf{R})} \ge M \quad \forall\, t \ge \epsilon.$$

3 (15 pts) *Method of Reflection.* (a) Find a representation formula for the solution of

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad (|x| < 1,\ t > 0), \\ u(-1,t) = u(1,t) = 0 \quad (t > 0), \qquad u(x,0) = g(x), \quad u_t(x,0) = h(x) \quad (|x| < 1). \end{cases}$$

using d'Alembert's formula and the method of reflection (Hint: consider intervals of the form $(2k-1, 2k+1)$ and $(2k+1, 2k+3)$ with k integer and find suitable extensions of g, h to them).

(b) Consider the initial condition

$$u(x,0) = \max(0, 1 - 2|x|), \qquad u_t(x,0) = 0.$$

Solve and plot the solution for $t = 0, 1/2, 3/4, 1, 5/4, 3/2, 7/4, 2$. This shows the reflection effect of the end points $x = \pm 1$ in the resulting wave.

4 (20 pts) *Equipartition of Energy* (Evans 2.5.17). Let $u \in C^2$ solve the 1D wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad (-\infty < x < \infty,\ t > 0), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) \quad (-\infty < x < \infty), \end{cases}$$

where g and h are smooth with compact support. The *kinetic energy* is $k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx$, and the *potential energy* is $p(t) = \frac{c^2}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx$. Prove

(a) $k(t) + p(t)$ is constant in time t;
(b) $k(t) = p(t)$ if t is sufficiently large. Can you quantify how big t should be? Hint: By d'Alembert's formula, for $F' = \frac{1}{2}(cg' + h)$ and $G' = \frac{1}{2}(cg' - h)$,

$$u_t(x,t) = F'(x+ct) - G'(x-ct), \quad u_x(x,t) = c^{-1}F'(x+ct) + c^{-1}G'(x-ct).$$

5 (15 pts) *Long time behavior* (Evans 2.5.18). Let u solve the wave equation

$$\begin{cases} u_{tt} = \Delta u & \text{in } \mathbf{R}^3 \times (0,\infty) \\ u = g, \quad u_t = h & \text{on } \mathbf{R}^3 \times \{t=0\}, \end{cases}$$

where g and h are smooth with compact support. Show that there exists a constant C such that

$$\max_{x\in\mathbf{R}^3} |u(x,t)| \leq \frac{C}{\max(1,t)} \quad \forall\, t > 0.$$

Can you make the constant C explicit in terms of g and h?

6 (25 pts). *Non-homogeneous wave equation.* Let $f \in C(\mathbf{R} \times [0,\infty))$ and $u(x,t) = \frac{1}{2}\int_{C(x,t)} f(y,s)dyds$ be the solution to

$$\begin{cases} u_{tt} - u_{xx} = f(x,t) & (-\infty < x < \infty,\ t > 0), \\ u(x,0) = 0, \quad u_t(x,0) = 0 & (-\infty < x < \infty). \end{cases}$$

(a) Show that $u \in C^1(\mathbf{R} \times [0,\infty))$ and

$$u_t(x,t) = \frac{1}{2}\int_0^t \Big(f(x+t-s,s) + f(x-t+s,s)\Big)ds$$
$$u_x(x,t) = \frac{1}{2}\int_0^t \Big(f(x+t-s,s) - f(x-t+s,s)\Big)ds$$

Hint: Write an integral representation for $h^{-1}(u(x+h,t) - u(x,t))$, decompose the domain $C(x+h,t) - C(x,t)$ in three appropriate regions, and then estimate the integral $\int_{C(x+h,t)-C(x,t)} f(y,s)dyds$.
(b) Suppose now that $f \in C^1(\mathbf{R} \times [0,\infty))$. Show that $u \in C^2(\mathbf{R} \times [0,\infty))$ and u satisfies the PDE.
(c) Conclude that $f \in C^k(\mathbf{R} \times [0,\infty))$ implies $u \in C^{k+1}(\mathbf{R} \times [0,\infty))$