MATH/AMSC 673 (Fall 2015)
PARTIAL DIFFERENTIAL EQUATIONS I
HOMEWORK # 5 (Pbs 1-2 due Nov 24, Pbs 3-4 due Dec 3, Pbs 5-7 due Dec 11)

1 (15 pts) *Semilinear Wave Equation.* Consider the following equation
\[
\begin{aligned}
&u_{tt} - u_{xx} = F(u(x,t)) \quad (-\infty < x < \infty, \ t \geq 0), \\
&u(x,0) = g(x), \quad u_t(x,0) = h(x) \quad (-\infty < x < \infty),
\end{aligned}
\]
where \( F \in C^1(\mathbb{R}) \) with \( \|F'\|_{L^\infty(\mathbb{R})} \leq M \), and \( g \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) and \( h \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \).

An important example is the sign Gordon equation for which \( F(u) = \sin u \). This problem shows how to prove existence of a unique solution \( u \in C^2(\mathbb{R} \times [0, \infty)) \) of (1) via the Contraction Mapping Theorem.

(a) Let \( u_0 \) be the d’Alembert solution of the initial value problem with \( F = 0 \). Consider the integral equation
\[
u(x,t) = u_0(x,t) + \frac{1}{2} \int_{C(x,t)} F(u(y,s))dyds,
\]
based on Duhamel’s formula. Show that if \( u \in C(\mathbb{R} \times [0, T]) \) satisfies (2), then \( u \in C^2(\mathbb{R} \times [0, T]) \) and satisfies (1). Hint: use HW #4 - Pb 6 to prove first that \( u \in C^1(\mathbb{R} \times [0, T]) \), and then (2) together with HW #4 - Pb 6 to conclude the assertion. This is an instance of the so-called bootstrap argument.

(b) Consider the normed space \( \mathcal{X} = \{ v \in L^\infty(\mathbb{R} \times [0, T]) : v \text{ continuous} \} \) and the operator \( \mathcal{L} : \mathcal{X} \to \mathcal{X} \)
\[
\mathcal{L}v(x,t) = u_0(x,t) + \frac{1}{2} \int_{C(x,t)} F(v(y,s))dyds.
\]
Show that \( \mathcal{L} \) is a contraction in \( \mathcal{X} \) provided \( MT^2 < 1 \) (see Evans p.498).

(c) Use the Contraction Mapping Theorem to conclude that there is a unique solution to the fixed point equation \( \mathcal{L}v = v \).

(d) Show that by repeated application of this argument on \([kT, (k+1)T]\) for \( k \geq 1 \) there is a unique solution \( u \) of (1) for all \( t > 0 \).

2 (15 pts) *Method of Characteristics* (Evans 3.5.3). Solve the following 1st order PDEs. Derive the full system of ODE including the correct initial conditions before you solve the system!

(a) \( x_1 u_{x_1} + x_2 u_{x_2} = 2u, \ u(x_1,1) = g(x_1) \);
(b) \( uu_{x_1} + u_{x_2} = 1, \ u(x_1,1) = \frac{1}{2} x_1 \).
(c) \( x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u, \ u(x_1, x_2, 0) = g(x_1, x_2) \).

3 (15 pts). *Hamilton-Jacobi Equation.* Weak solutions of the equation \( u_t + H(Du) = 0 \) in \( \mathbb{R}^n \) can be obtained by solving the parabolic regularization
\[
u_t + H(Du) - \varepsilon \Delta u = 0
\]
in \( \mathbb{R}^n \) and passing to the limit as \( \varepsilon \downarrow 0 \). The purpose of this problem is to prove a stability result for solutions of (3) that is uniform in \( \varepsilon \), and thus also valid for solutions of H-J
equations. For \( i = 1, 2 \), let \( u_i \) be a solution to (3) with initial condition \( u_i(x,0) = g_i(x) \) for \( x \in \mathbb{R}^n \) and \( \int_{\mathbb{R}^n} |u_i(x,t)|^2 + |\nabla u_i(x,t)|^2 dx < \infty \) for all \( t \geq 0 \). Let \( H \) be globally Lipschitz with constant \( L \). Show the \( L^\infty \)-contraction property

\[
\|(u_1 - u_2)(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|g_1 - g_2\|_{L^\infty(\mathbb{R}^n)}.
\]

To this end proceed as follows. Set \( G = \|g_1 - g_2\|_{L^\infty(\mathbb{R}^n)} \) and \( \phi = \max(u_1 - u_2 - G, 0) \).

Multiply the difference of the PDEs by \( \phi \) to deduce

\[
\frac{1}{2} \int_{\mathbb{R}^n} \phi(x,t)^2 + \varepsilon \int_0^t \int_{\mathbb{R}^n} |\nabla \phi|^2 \leq L \int_0^t \int_{\mathbb{R}^n} |\phi||\nabla \phi|.
\]

Suitably manipulate the right-hand side and then use Gronwall’s lemma to conclude that \( \phi = 0 \), and thereby that \( u_1 - u_2 \leq G \).

4 (10 pts). Precise blow-up time. Consider the Cauchy problem for a scalar conservation law,

\[
\begin{align*}
  u_t + a(u)u_x &= 0, \quad -\infty < x < \infty, \ t > 0, \\
  u(x,0) &= u_0(x), \quad -\infty < x < \infty.
\end{align*}
\]

Even for compactly supported smooth initial data, the classical solution may become discontinuous at some \( T^* \). The precise breakdown time is signaled by the space derivative \( u_x \) becoming infinite at some point, that is \( u(x,t) \) is smooth for \( 0 < t < T^* \), but \( \sup_{x \in \mathbb{R}} u_x(x,t) \to \infty \) as \( t \uparrow T^* \) (think of an infinite compression of characteristics).

(a) Show that \( T^* \) satisfies

\[
1 + T^* \min_{y \in \mathbb{R}} \partial_y a(u_0(y)) = 0.
\]

(b) Determine the blow-up time for the Burgers’ equation \( u_t + uu_x = 0 \) with initial condition \((u_L > u_R)\)

\[
u_0(x) = \begin{cases} 
u_L & \text{for } x < 0 \\ 
u_L - \frac{u_L - u_R}{l} x & \text{for } 0 \leq x \leq l \\ 
u_R & \text{for } x > l. \end{cases}
\]

5 (15 pts). (a) Profile of rarefaction waves. Suppose that the conservation law

\[
\begin{align*}
  u_t + f(u)_x &= 0
\end{align*}
\]

has a solution of the form \( u(x,t) = v(x/t) \). Show that the profile of \( v \) is given by

\[
v(s) = (f')^{-1}(s).
\]

(b) Profile of traveling waves. Show that the viscous approximation to Burgers’s equation, that is

\[
\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon = \varepsilon \partial_{xx} u_\varepsilon
\]

has a solution of the form \( u_\varepsilon(x,t) = v(x - st) \) with

\[
v(y) = u_R + \frac{1}{2}(u_L - u_R)
\left(1 - \tanh \left( \frac{(u_L - u_R)y}{4\varepsilon} \right) \right),
\]

\]
and $s = (u_L + u_R)/2$ is the shock speed. Sketch this solution and compare with the limit function as $\varepsilon \downarrow 0$.

6 (15 pts) Convex flux. Consider the Cauchy problem for Burgers’ equation

$$u_t + uu_x = 0, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = \begin{cases} 1 & \text{for } |x| > 1 \\ |x| & \text{for } |x| < 1. \end{cases}$$

(a) Sketch the characteristics in the $(x,t)$ plane. Find a classical solution (continuous and piecewise $C^1$). Determine the time of breakdown (shock formation).

(b) Find a weak solution globally for $t > 0$, containing a shock curve. Note that the shock does not move with constant speed. Therefore, find first the solution away from the shock. Then, use the Rankine-Hugoniot condition to find a differential equation for the position of the shock given by $(x = s(t), t)$ in the $(x, t)$-plane.

7 (15 pts). Nonconvex flux. The Buckley-Leverett equations are a simple model for two-phase fluid flow in a porous medium with flux

$$f(u) = \frac{u^2}{u^2 + \frac{1}{3}(1-u)^2}.$$ 

In secondary oil recovery, water is pumped into some wells to displace the oil remaining in the underground rocks. Therefore $u$ represents the saturation of water, namely the percentage of water in the water-oil fluid, and varies between 0 and 1. Find the entropy solution to the Riemann problem with initial states

$$u(x, 0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0. \end{cases}$$

Hint: The line through the origin that is tangent to the graph of $f$ on the interval $[0, 1]$ has slope $1/(\sqrt{3} - 1)$ and touches the curve at $u = 1/\sqrt{3}$. 