MATH/AMSC 673 (Fall 2015)

PARTIAL DIFFERENTIAL EQUATIONS I

HOMEWORK # 5 (Pbs 1-2 due Nov 24, Pbs 3-4 due Dec 3, Pbs 5-7 due Dec 11)

1 (15 pts) *Semilinear Wave Equation.* Consider the following equation

$$\begin{cases} u_{tt} - u_{xx} = F(u(x,t)) \quad (-\infty < x < \infty,\ t > 0), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) \quad (-\infty < x < \infty), \end{cases} \tag{1}$$

where $F \in C^1(\mathbf{R})$ with $\|F'\|_{L^\infty(\mathbf{R})} \leq M$, and $g \in C^2(\mathbf{R}) \cap L^\infty(\mathbf{R})$ and $h \in C^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$. An important example is the sign Gordon equation for which $F(u) = \sin u$. This problem shows how to prove existence of a unique solution $u \in C^2(\mathbf{R} \times [0,\infty))$ of (1) via the Contraction Mapping Theorem.

(a) Let u_0 be the d'Alembert solution of the initial value problem with $F = 0$. Consider the integral equation

$$u(x,t) = u_0(x,t) + \frac{1}{2} \int_{C(x,t)} F(u(y,s))dyds, \tag{2}$$

based on Duhamel's formula. Show that if $u \in C(\mathbf{R} \times [0,T])$ satisfies (2), then $u \in C^2(\mathbf{R} \times [0,T])$ and satisfies (1). Hint: use HW #4 - Pb 6 to prove first that $u \in C^1(\mathbf{R} \times [0,T])$, and then (2) together with HW #4 - Pb 6 to conclude the assertion. This is an instance of the so-called *bootstrap* argument.

(b) Consider the normed space $\mathcal{X} = \{v \in L^\infty(\mathbf{R} \times [0,T]) : v \text{ continuous}\}$ and the operator $\mathcal{L} : \mathcal{X} \to \mathcal{X}$

$$\mathcal{L}v(x,t) = u_0(x,t) + \frac{1}{2} \int_{C(x,t)} F(v(y,s))dyds.$$

Show that $\mathcal{L}$ is a contraction in $\mathcal{X}$ provided $MT^2 < 1$ (see Evans p.498).

(c) Use the Contraction Mapping Theorem to conclude that there is a unique solution to the fixed point equation $\mathcal{L}v = v$.

(d) Show that by repeated application of this argument on $[kT, (k+1)T]$ for $k \geq 1$ there is a unique solution u of (1) for all $t > 0$.

2 (15 pts) *Method of Characteristics* (Evans 3.5.3). Solve the following 1st order PDEs. Derive the full system of ODE including the correct initial conditions before you solve the system!

(a) $x_1 u_{x_1} + x_2 u_{x_2} = 2u, \quad u(x_1, 1) = g(x_1)$;

(b) $uu_{x_1} + u_{x_2} = 1, \quad u(x_1, x_1) = \frac{1}{2}x_1$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u, \quad u(x_1, x_2, 0) = g(x_1, x_2)$.

3 (15 pts). *Hamilton-Jacobi Equation.* Weak solutions of the equation $u_t + H(Du) = 0$ in $\mathbf{R}^n$ can be obtained by solving the parabolic regularization

$$u_t + H(Du) - \varepsilon \Delta u = 0 \tag{3}$$

in $\mathbf{R}^n$ and passing to the limit as $\varepsilon \downarrow 0$. The purpose of this problem is to prove a stability result for solutions of (3) that is uniform in ε, and thus also valid for solutions of H-J equations. For $i = 1, 2$, let u_i be a solution to (3) with initial condition $u_i(x, 0) = g_i(x)$ for $x \in \mathbf{R}^n$ and $\int_{\mathbf{R}^n} |u_i(x,t)|^2 + |\nabla u_i(x,t)|^2 dx < \infty$ for all $t \geq 0$. Let H be globally Lipschitz with constant L. Show the *L^∞-contraction property*

$$\|(u_1 - u_2)(\cdot, t)\|_{L^\infty(\mathbf{R}^n)} \leq \|g_1 - g_2\|_{L^\infty(\mathbf{R}^n)}.$$

To this end proceed as follows. Set $G = \|g_1 - g_2\|_{L^\infty(\mathbf{R}^n)}$ and $\phi = \max(u_1 - u_2 - G, 0)$. Multiply the difference of the PDEs by ϕ to deduce

$$\frac{1}{2}\int_{\mathbf{R}^n} \phi(\cdot, t)^2 + \varepsilon \int_0^t \int_{\mathbf{R}^n} |\nabla \phi|^2 \leq L \int_0^t \int_{\mathbf{R}^n} |\phi||\nabla \phi|.$$

Suitably manipulate the right-hand side and then use Gronwall's lemma to conclude that $\phi = 0$, and thereby that $u_1 - u_2 \leq G$.

4 (10 pts). *Precise blow-up time.* Consider the Cauchy problem for a scalar conservation law,

$$\begin{aligned} u_t + a(u)u_x = 0, &\quad -\infty < x < \infty,\ t > 0, \\ u(x, 0) = u_0(x), &\quad -\infty < x < \infty. \end{aligned}$$

Even for compactly supported smooth initial data, the classical solution may become discontinuous at some T^*. The precise breakdown time is signaled by the space derivative u_x becoming infinite at some point, that is $u(x,t)$ is smooth for $0 < t < T^*$, but $\sup_{x\in\mathbf{R}} u_x(x,t) \to \infty$ as $t \uparrow T^*$ (think of an infinite compression of characteristics).
(a) Show that T^* satisfies

$$1 + T^* \min_{y\in\mathbf{R}} \partial_y a(u_0(y)) = 0.$$

(b) Determine the blow-up time for the Burgers' equation $u_t + uu_x = 0$ with initial condition $(u_L > u_R)$

$$u_0(x) = \begin{cases} u_L & \text{for } x < 0 \\ u_L - \frac{u_L - u_R}{l} x & \text{for } 0 \leq x \leq l \\ u_R & \text{for } x > l. \end{cases}$$

5 (15 pts). (a) *Profile of rarefaction waves.* Suppose that the conservation law

$$u_t + f(u)_x = 0$$

has a solution of the form $u(x,t) = v(x/t)$. Show that the profile of v is given by

$$v(s) = (f')^{-1}(s).$$

(b) *Profile of traveling waves.* Show that the viscous approximation to Burgers's equation, that is

$$\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon = \varepsilon \partial_{xx} u_\varepsilon$$

has a solution of the form $u_\varepsilon(x,t) = v(x - st)$ with

$$v(y) = u_R + \frac{1}{2}(u_L - u_R)\Big(1 - \tanh\frac{(u_L - u_R)y}{4\varepsilon}\Big),$$

and $s = (u_L + u_R)/2$ is the shock speed. Sketch this solution and compare with the limit function as $\varepsilon \downarrow 0$.

6 (15 pts) *Convex flux.* Consider the Cauchy problem for Burgers' equation

$$u_t + uu_x = 0, \quad -\infty < x < \infty,\ t > 0$$
$$u(x,0) = \begin{cases} 1 & \text{for } |x| > 1 \\ |x| & \text{for } |x| < 1. \end{cases}$$

(a) Sketch the characteristics in the (x,t) plane. Find a classical solution (continuous and piecewise C^1). Determine the time of breakdown (shock formation).
(b) Find a weak solution globally for $t > 0$, containing a shock curve. Note that the shock does not move with constant speed. Therefore, find first the solution away from the shock. Then, use the Rankine-Hugoniot condition to find a differential equation for the position of the shock given by $(x = s(t), t)$ in the (x,t)-plane.

7 (15 pts). *Nonconvex flux.* The Buckley-Leverett equations are a simple model for two-phase fluid flow in a porous medium with flux

$$f(u) = \frac{u^2}{u^2 + \frac{1}{2}(1-u)^2}.$$

In secondary oil recovery, water is pumped into some wells to displace the oil remaining in the underground rocks. Therefore u represents the saturation of water, namely the percentage of water in the water-oil fluid, and varies between 0 and 1. Find the entropy solution to the Riemann problem with initial states

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0. \end{cases}.$$

Hint: The line through the origin that is tangent to the graph of f on the interval $[0,1]$ has slope $1/(\sqrt{3}-1)$ and touches the curve at $u = 1/\sqrt{3}$.