1 (20 pts). *Approximation with smooth functions:* Let \( u \in W_p^1(\Omega) \) with \( 1 \leq p \leq \infty \) and dimension \( d \geq 1 \). Let \( \rho \in C_0^\infty(\mathbb{R}^d) \) be a mollifier with properties:

\[
supp \rho = B_1, \quad \rho \geq 0, \quad \int_{B_1} \rho = 1,
\]

where \( B_1 \) is the unit ball centered at the origin. Let \( \rho_\varepsilon(x) = \varepsilon^{-d} \rho(\varepsilon^{-1} x) \) and \( u_\varepsilon \) be defined as

\[
u_\varepsilon(x) = \int_{\Omega} u(x-y) \rho_\varepsilon(y) dy \quad \forall x \in K,
\]

where \( K \) is a compact set of \( \Omega \) and \( \varepsilon \) is sufficiently small. Show the error estimate

\[
\|u - u_\varepsilon\|_{L_p(K)} \leq C\varepsilon|u|_{W_p^1(\Omega)}.
\]

2 (20 pts). *Equivalent norms (Deni-Lions):* Consider the Sobolev space \( W^{k+1}_p(\Omega) \) with \( k \geq 0, 1 \leq p \leq \infty \) and a Lipschitz domain \( \Omega \) in \( \mathbb{R}^d \). Let \( \{f_i\}_{i=1}^N \) be linear continuous functionals in \( W^{k+1}_p(\Omega) \) such that for any polynomial \( v \in \mathbb{P}_k \) of degree \( \leq k \):

\[
f_i(v) = 0 \quad \forall 1 \leq i \leq N \quad \iff \quad v = 0.
\]

(a) Show that \( \|v\|_{W^{k+1}_p(\Omega)} \) is equivalent to the seminorm

\[
|v|_{W^{k+1}_p(\Omega)} + \sum_{i=1}^N |f_i(v)|.
\]

Hint: Proceed by contradiction assuming that there is a sequence \( \{v_n\} \subset W^{k+1}_p(\Omega) \) such that \( \|v_n\|_{W^{k+1}_p(\Omega)} = 1 \) but the latter seminorm tends to \( 0 \). Use that \( W^{k+1}_p(\Omega) \) is compactly imbedded in \( W^k_p(\Omega) \) (Rellich Theorem), namely that each bounded sequence in \( W^{k+1}_p(\Omega) \) admits a convergence subsequence in \( W^k_p(\Omega) \).

(b) Show the polynomial interpolation bound

\[
\inf_{g \in \mathbb{P}_k} \|v - g\|_{W^{k+1}_p(\Omega)} \leq C(\Omega)|v|_{W^{k+1}_p(\Omega)} \quad \forall v \in W^{k+1}_p(\Omega).
\]

3 (20 pts). *Nonhomogeneous Dirichlet Problem:* Given a bounded Lipschitz domain \( \Omega \) in \( \mathbb{R}^n \), and \( g \in H^1(\Omega), f \in H^{-1}(\Omega) \), set

\[
L(v) = \langle f, v \rangle - \int_{\Omega} \nabla g \nabla v \quad \forall v \in H^1(\Omega).
\]

(a) Prove that there exists a unique solution to the variational problem

\[
z \in H^1_0(\Omega) : \quad \int_{\Omega} \nabla z \nabla v = L(v) \quad \forall v \in H^1_0(\Omega).
\]

(b) Show that such a problem is equivalent to the minimization of \( J(v) = \int_{\Omega} \frac{1}{2} |\nabla v|^2 - \langle f, v \rangle \) over the subspace \( V = \{ v \in H^1(\Omega) : v - g \in H^1_0(\Omega) \} \).

(c) Prove that \( u = z + g \) formally solves \(-\Delta u = f \) in \( \Omega \) with boundary condition \( u = g \) on \( \partial \Omega \).

4 (20 pts). *Third boundary value problem:* Given \( f \in L^2(\Omega), g \in H^2(\Omega) \) and \( 0 < P_1 \leq p \leq P_2 \) on \( \partial \Omega \), consider the Robin problem

\[
-\Delta u = f \quad \text{in} \quad \Omega, \quad \partial_\nu u + p(u - g) = 0 \quad \text{on} \quad \partial \Omega,
\]

\[
\|u\|_{P} \leq C\|f\|_{L_p(\Omega)}.
\]
(a) Find a variational formulation which amounts to solving this problem.

(b) Show that Lax-Milgram theorem applies and conclude that there exists a unique solution \( u \in H^1(\Omega) \). To this end, show that the bilinear form is coercive in \( H^1(\Omega) \).

(c) Suppose that \( p = \epsilon^{-1} \to \infty \) and denote the corresponding solution by \( u_\epsilon \). Determine the boundary value problem satisfied by \( u_0 = \lim_{\epsilon \to 0} u_\epsilon \).

(d) Derive an error estimate for \( \|u_\epsilon - u_0\|_{H^1(\Omega)} \).

5 (20 pts). Darcy’s flow. Let \( u \) be the pressure and \( \sigma = -K\nabla u \) be the flux of the model problem for flow in porous media, which can be written as

\[
K^{-1}\sigma + \nabla u = 0, \quad \text{div } \sigma = f.
\]

(a) Let \( V = H_0(\text{div};\Omega) := \{\tau \in [L^2(\Omega)]^d : \text{div } \tau \in L^2(\Omega), \tau \cdot \nu = 0 \text{ on } \partial \Omega\} \) and \( Q = L_0^2(\Omega) \). Write a variational formulation for this problem, and show that the inf-sup condition is satisfied.

(b) Deduce existence, uniqueness, and stability of the solution pair \( (u, \sigma) \).