

NUMERICAL METHODS FOR STATIONARY PDEs
 HOMEWORK # 5 (Pbs 1-2 due Th Nov 29, Pbs 3-4 due Th Dec 6)

1 (30 pts). *Implementation of FEM.* Modify the MATLAB finite element code `fem.m` so that it solves the general elliptic PDE with constant coefficients

$$-\operatorname{div}(A\nabla u) + \mathbf{b}\nabla u + cu = f,$$

where $A \in \mathbb{R}^{2 \times 2}$ is symmetric positive definite, $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$, and $c \in \mathbb{R}$. We impose Dirichlet boundary condition $u = g_D$ in the whole boundary $\partial\Omega$ of Ω .

(a) Replace the scalar diffusion coefficient `coef_a` in `fem.m` by a constant symmetric 2-by-2 matrix `coef_a` and implement the convection term $\mathbf{b}\nabla u$, following the tutorial. Note that the resulting matrix \mathbf{A} is nonsymmetric if $\mathbf{b} \neq 0$.

(b) Let Ω be the L -shape domain of HW#4-Pb4. Use the finite element code `gen_mesh_L_shape.m` to generate uniform meshes with meshsize $h = 2^{-k}$ with $k = 2, 3, 4, 5, 6$, and `fem.m` with Dirichlet condition $g_D = 0$, forcing function $f = 1$, and the following PDE coefficients:

(i) $A = 0.01I$, $\mathbf{b} = (0, 0)$, $c = 1$;

(ii) $A = \begin{bmatrix} 3 & -11 \\ -11 & 45 \end{bmatrix}$, $\mathbf{b} = (0, 0)$, $c = 1$;

(iii) $A = I$, $\mathbf{b} = (30, 60)$, $c = 0$;

(iv) $A = I$, $\mathbf{b} = (0, 0)$, $c = -180$.

(c) Show plots of u and $\|\nabla u\|_2$ for the finest mesh (without the mesh). Comment on the behavior of the discrete solutions on coarse meshes (relate to discussion in class on indefinite forms).

2 (20 pts). *Relation between norms and nodal values.* Let $\{\mathcal{T}\}$ be a family of shape-regular meshes with nodes $\{z_i\}_{i=1}^N$, and let $\mathbb{V}_{\mathcal{T}}$ be a finite element space over \mathcal{T} in \mathbb{R}^d . Let $\{\phi_i\}_{i=1}^N$ be the canonical Lagrange basis functions of $\mathbb{V}_{\mathcal{T}}$. Use *inverse inequalities* to prove the existence of a geometric constant such that for all $v = \sum_{i=1}^N v_i \phi_i \in \mathbb{V}_{\mathcal{T}}$ and all $T \in \mathcal{T}$

$$C^{-1} \|v\|_{L^2(T)}^2 \leq h_T^d \sum_{z_i \in T} v_i^2 \leq C \|v\|_{L^2(T)}^2; \quad (1)$$

$$\|v\|_{H^1(T)}^2 \leq Ch_T^{d-2} \|v\|_{L^\infty(T)}^2 \leq Ch_T^{d-2} \sum_{z_i \in T} v_i^2. \quad (2)$$

Examine whether or not a reverse inequality to (2) is valid. Study how (1) and (2) change if $v = \sum_{i=1}^N \hat{v}_i \psi_i$ and ψ_i is a scaled basis function, namely $\psi_i = C_i \phi_i$.

3 (25 pts). *Condition numbers.* Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be nonsingular. The condition number of \mathbf{A} in the 2-norm is $k(\mathbf{A}) = \|\mathbf{A}\|_2 \|\mathbf{A}^{-1}\|_2$. This number dictates the performance of both direct and iterative solvers. Show that

(a) if \mathbf{A} is symmetric and positive definite, then $k(\mathbf{A}) = \frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})}$, where $\lambda_{\max}(\mathbf{A})$ (resp. $\lambda_{\min}(\mathbf{A})$) is the largest (resp. smallest) eigenvalue of \mathbf{A} .

(b) the condition number of the mass matrix $\mathbf{M} = (\int_{\Omega} \phi_i \phi_j)_{i,j=1}^N$ over a *quasi-uniform* triangulation \mathcal{T} of size h satisfies $k(\mathbf{M}) = \mathcal{O}(1)$. Hint: use (1).

(c) the condition number of the stiffness matrix $\mathbf{K} = (\int_{\Omega} \nabla \phi_i \nabla \phi_j)_{i,j=1}^N$ for vanishing Dirichlet boundary condition over a *quasi-uniform* triangulation \mathcal{T} of size h satisfies $k(\mathbf{K}) = \mathcal{O}(h^{-2}) = \mathcal{O}(N^{\frac{2}{d}})$. Hint: use (2).

- (d) the condition number of the mass matrix $\widehat{\mathbf{M}} = (\int_{\Omega} \psi_i \psi_j)_{i,j=1}^N$ over a *graded* shape-regular triangulation \mathcal{T} still satisfies $k(\widehat{\mathbf{M}}) = \mathcal{O}(1)$ provided that the basis functions are rescaled $\psi_i = C_i \phi_i$ in such a way that $\int_{\Omega} \psi_i^2 = 1$. What is the value of C_i ?
- (e) the condition number of the stiffness matrix $\widehat{\mathbf{K}} = (\int_{\Omega} \nabla \psi_i \nabla \psi_j)_{i,j=1}^N$ for vanishing Dirichlet boundary condition over a *graded* shape-regular triangulation \mathcal{T} still satisfies $k(\widehat{\mathbf{K}}) = \mathcal{O}(N^{\frac{2}{d}})$, provided that the basis functions are rescaled $\psi_i = C_i \phi_i$ in such a way that $\int_{\Omega} |\nabla \psi_i|^2 = 1$. What is the value of C_i ? Hint: If $v = \sum_{i=1}^N v_i \phi_i = \sum_{i=1}^N \hat{v}_i \psi_i$, use (2) to prove $\|v\|_{H^1(\Omega)}^2 \leq C_1 \sum_{i=1}^N \hat{v}_i^2$ and that this implies

$$\lambda_{max}(\widehat{\mathbf{K}}) \leq C_1.$$

Next combine the inverse estimate $\|v\|_{L^\infty(T)} \leq Ch_T^{-\frac{d-2}{2}} \|v\|_{L^{\frac{2d}{d-2}}(T)}$ with the definition of \hat{v}_i to deduce

$$\sum_{i=1}^N \hat{v}_i^2 \leq C \sum_{T \in \mathcal{T}} \|v\|_{L^{\frac{2d}{d-2}}(T)}^2,$$

Finally apply the Hölder inequality with suitable exponents and the Sobolev imbedding estimate $\|v\|_{L^{\frac{2d}{d-2}}(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)}$ to deduce

$$\lambda_{min}(\widehat{\mathbf{K}}) \geq C_2 N^{-2/d}.$$

4 (25 pts). *L²-projection*. Let \mathcal{T} be a shape-regular triangulation of a polygonal domain $\Omega \subset \mathbb{R}^n$. Let $\mathbf{V}_{\mathcal{T}}$ be a space of (possibly discontinuous) finite elements containing at least $P_k(T)$ as interpolating polynomials for all $T \in \mathcal{T}$. Given $u \in L^2(\Omega)$ let $u_{\mathcal{T}} \in \mathbf{V}_{\mathcal{T}}$ be the L^2 -projection of u onto $\mathbf{V}_{\mathcal{T}}$, i.e.

$$u_{\mathcal{T}} \in \mathbf{V}_{\mathcal{T}} : \quad \int_{\Omega} (u - u_{\mathcal{T}})v = 0 \quad \forall v \in \mathbf{V}_{\mathcal{T}}.$$

Prove the following error estimates for $0 \leq m \leq k+1$ ($H^{-m}(\Omega) = \text{dual of } H_0^m(\Omega)$):

$$\|u - u_{\mathcal{T}}\|_{L^2(\Omega)} \leq \inf_{v \in \mathbf{V}_{\mathcal{T}}} \|u - v\|_{L^2(\Omega)} \leq Ch^{k+1} |u|_{H^{k+1}(\Omega)}; \quad (\text{a})$$

$$\|u - u_{\mathcal{T}}\|_{H^{-m}(\Omega)} \leq Ch^{k+1+m} |u|_{H^{k+1}(\Omega)}; \quad (\text{b})$$

$$\|u - u_{\mathcal{T}}\|_{H^m(\Omega)} \leq Ch^{k+1-m} |u|_{H^{k+1}(\Omega)}, \quad (\text{c})$$

the latter provided \mathcal{T} is quasi-uniform. The inequality (b) is referred to as a *superconvergence* estimate. Hint: to prove (c) add and subtract $I_{\mathcal{T}}u$ and use an inverse inequality.