AMSC 715 Spring 2020

NUMERICAL METHODS FOR EVOLUTION PDE

HOMEWORK # 1

Pbs 1-4 due Feb 13, Pbs 5-7 due Feb 20

1 (10 pts). *Uniform regularity.* Consider the heat equation $\partial_t u - k\Delta u = 0$ in $\mathbb{R}^d \times (0,\infty)$ with initial condition $u(\cdot,0) = v$. Let v be uniformly *Hölder* continuous in $\mathbb{R}^d$ with exponent α, namely there exists a constant $M > 0$ such that $\|v\|_{L^\infty(\mathbb{R}^d)} \le M$ and

$$|v(x_1) - v(x_2)| \le M\omega(|x_1 - x_2|) \quad \forall\, x_1, x_2 \in \mathbb{R}^d,$$

where $\omega(s) = s^\alpha$ is the Hölder modulus of continuity and $0 < \alpha \le 1$; if $\alpha = 1$ then v is said to be *Lipschitz* continuous. Proceed as with the proof of continuity of $u(\cdot,t)$ as $t \to 0$ given in class to show

(a) *Uniform Hölder continuity in space*: for all $x_1, x_2 \in \mathbb{R}^d$ and $t \ge 0$

$$|u(x_1,t) - u(x_2,t)| \le N\,\omega(|x_1 - x_2|),$$

(b) *Uniform Hölder continuity in time*: for all $x \in \mathbb{R}^d$ and $t_1, t_2 \ge 0$

$$|u(x,t_1) - u(x,t_2)| \le N\sqrt{\omega(|t_1 - t_2|)},$$

and determine the dependence of the constant N on M and k. In particular this provides a rate of convergence for the max-norm error $\|u(\cdot,t) - v\|_{L^\infty(\mathbb{R}^d)}$.

2 (15 pts). *Weighted energy estimates.* Problem 8.5 of Larsson and Thomée.

3 (15 pts). *Max-norm stability.* Problem 8.7 of Larsson and Thomée.

4 (15 pts). *Neumann boundary condition.* Problem 8.13 of Larsson and Thomée.

5 (10 pts). *Fourth order finite differences*: Problem 9.4 of Larsson and Thomée. To derive the 5-point stencil proceed as follows. Suppose that $u \in C^6$ and show an error estimate for $u''(0) - D_h^2(0)$ where

$$D_h^2(0) = h^{-2}\big(u(-h) - 2u(0) + u(h)\big),$$

is the usual 3-point stencil; keep all terms in the error expression up to order 4. Combine now $D_h^2(0)$ with $D_{2h}^2(0)$ in order to eliminate the leading error term of order 2. This process is called *extrapolation.*

6 (15 pts). *Crank-Nicolson method*: Consider the semidiscrete method in $\Omega \subset \mathbb{R}^d$

$$\frac{U^{n+1} - U^n}{k} - \Delta\frac{U^{n+1} + U^n}{2} = f^{n+\frac{1}{2}} = \frac{f^{n+1} + f^n}{2}.$$

Prove the energy bounds:

$$\|U^N\|^2 + \frac{k}{4}\sum_{n=0}^{N-1}\|\nabla(U^{n+1}+U^n)\|^2 \le \|U^0\|^2 + k\sum_{n=0}^{N-1}\|f^{n+\frac{1}{2}}\|^2_{H^{-1}(\Omega)}$$

$$\sum_{n=0}^{N-1} k\|k^{-1}(U^{n+1}-U^n)\|^2 + \|\nabla U^N\|^2 \le \|\nabla U^0\|^2 + \sum_{n=0}^{N-1} k\|f^{n+\frac{1}{2}}\|^2.$$

Note that if $U^0 \notin H_0^1(\Omega)$ then there is no bound on $\|\nabla U^n\|$. Compare with the *regularizing effect* of backward Euler.

7 (20 pts). *Finite differences for the heat equation*: Consider $\partial_t u - \partial_x^2 u = 0$ in $\Omega = (-1,1)$ with $T = 1$, initial condition $v = v(x)$, and Dirichlet boundary condition $g = g(x,t)$.

(a) Implement in MATLAB the forward Euler (FE), backward Euler (BE) and Crank-Nicolson methods (CN) with $v(x)$ and $g(x,t)$ given by

$$v(x) = \sin(\pi x) - \sin(3\pi x), \quad g(\pm 1, t) = 0,$$
$$v(x) = \mathrm{sign(x)}, \quad g(\pm 1, t) = \mathrm{erf}(\pm 1/\sqrt{4t});$$

erf is the solution of the heat equation in the whole line with initial condition sign(x).

(b) Find the exact solutions and plot them along with the numerical solutions at $t = 0.1$ for $h = 1/10$ and $k = 1/400$.

(c) Given the integers $0 \le n \le 4$ let the meshsize be $h = 0.1 \times 2^{-n}$. Let the time step be $k = h^2, h^2/4, h^2/8$ for FE and BE and $k = h$ for CN. Compute the discrete solutions for the various combinations of h and k and plot the L^∞-error at $T = 1$ vs h in a log-log graph. Determine the computational orders of convergence.

(d) Draw conclusions in terms of rates of convergence vs total space-time degrees of freedom for the three time discretizations FE, BE, and CN. Compare with theory.