AMSC 715 Spring 2019

NUMERICAL METHODS FOR EVOLUTION PDE

HOMEWORK # 1

Pbs 1-3 due Feb 12, Pbs 4-6 due Feb 21

1 (15 pts). Problem 8.5 of Larsson and Thomée.

2 (15 pts). Problem 8.7 of Larsson and Thomée.

3 (15 pts). Problem 8.13 of Larsson and Thomée.

4 (15 pts). *Fourth order finite differences*: Problem 9.4 of Larsson and Thomée. To derive the 5-point stencil proceed as follows. Suppose that $u \in C^6$ and show an error estimate for $u''(0) - D_h^2(0)$ where

$$D_h^2(0) = h^{-2}\big(u(-h) - 2u(0) + u(h)\big),$$

is the usual 3-point stencil; keep all terms in the error expression up to order 4. Combine now $D_h^2(0)$ with $D_{2h}^2(0)$ in order to eliminate the leading error term of order 2. This process is called *extrapolation*.

5 (15 pts). *Crank-Nicolson method*: Consider the semidiscrete method in $\Omega \subset \mathbb{R}^d$

$$\frac{U^{n+1} - U^n}{k} - \Delta \frac{U^{n+1} + U^n}{2} = f^{n+\frac{1}{2}} = \frac{f^{n+1} + f^n}{2}.$$

Prove the energy bounds:

$$\|U^N\|^2 + \frac{k}{4} \sum_{n=0}^{N-1} \|\nabla(U^{n+1} + U^n)\|^2 \leq \|U^0\|^2 + k \sum_{n=0}^{N-1} \|f^{n+\frac{1}{2}}\|^2_{H^{-1}(\Omega)}$$

$$\sum_{n=0}^{N-1} k\|k^{-1}(U^{n+1} - U^n)\|^2 + \|\nabla U^N\|^2 \leq \|\nabla U^0\|^2 + \sum_{n=0}^{N-1} k\|f^{n+\frac{1}{2}}\|^2.$$

Note that if $U^0 \notin H_0^1(\Omega)$ then there is no bound on $\|\nabla U^n\|$. Compare with the *regularizing effect* of backward Euler.

6 (25 pts). *Finite differences for the heat equation*: Consider $\partial_t u - \partial_x^2 u = 0$ in $\Omega = (-1, 1)$ with $T = 1$, initial condition $v = v(x)$, and Dirichlet boundary condition $g = g(x, t)$.
(a) Implement in MATLAB the forward Euler (FE), backward Euler (BE) and Crank-Nicolson methods (CN) with $v(x)$ and $g(x, t)$ given by

$$v(x) = \sin(\pi x) - \sin(3\pi x), \quad g(\pm 1, t) = 0,$$
$$v(x) = \text{sign(x)}, \quad g(x, t) = \text{erf}(\pm 1/\sqrt{4t});$$

erf is the solution of the heat equation in the whole line with initial condition $\text{sign}(x)$.
(b) Let $h = 1/10$. Compute the discrete solution with $k = 1/100, 1/300, 1/600$ for FE and BE and with $k = 1/10$ for CN.
(c) Find the exact solutions and compute the errors at $(0.5, 1)$. Draw conclusions.