AMSC 715 Spring 2020

**NUMERICAL METHODS FOR EVOLUTION PDE**

**HOMEWORK # 2**

Pbs 1-2 due Th Mar 5, Pbs 3-4 due Th Mar 12, Pbs 5-6 due Th Mar 26

1 (15 pts). *Neumann condition and discrete maximum principle.* Given $\alpha, \beta \in \mathbb{R}$ and $f \in C^1[0,1]$, consider the heat equation with mixed boundary conditions:

$$\partial_t u - \partial_x^2 u = f(x) \quad x \in (0,1), \qquad u(0,t) = \alpha, \quad \partial_x u(1,t) = \beta.$$

(a) *Finite difference method (FDM).* Write an implicit FDM on a uniform partition $\mathcal{T} = \{x_j\}_{j=0}^M$ with $0 = x_0 < x_1 < \cdots x_M = 1$ and meshsize $h$ and time step $k$ with approximate Neumann condition

$$\frac{U_M^{n+1} - U_{M-1}^{n+1}}{h} = \beta.$$

Show that the truncation error $\mathbf{T}^n = (\tau_j^n)_{j=1}^M$ satisfies $\|\mathbf{T}^n\|_\infty \le C_1(u)h + C_2(u)k$. Explain the regularity of $u$ involved in the constants $C_1(u)$ and $C_2(u)$.
(b) *Maximum principle.* Write the discrete problem for the vector $\mathbf{U}^{n+1} = (U^{n+1})_{j=1}^M$ as

$$\mathbf{K}\mathbf{U}^{n+1} = \widehat{\mathbf{U}}^n + k\mathbf{F},$$

where $\widehat{U}_j^n = U_j^n$ if $j < M$ and $\widehat{U}_M^n = 0$. Both the matrix $\mathbf{K}$ and right-hand side $\mathbf{F}$ are obtained by multiplying all the equations by $k$. Deduce the $\ell^\infty$-stability bounds $\max_{1\le j\le M} U_j^{n+1} \le \|\mathbf{U}^n\|_\infty$ provided $\mathbf{F} \le 0$ and $\min_{1\le j\le M} U_j^{n+1} \ge -\|\mathbf{U}^n\|_\infty$ provided $\mathbf{F} \ge 0$. Conclude that the matrix $\mathbf{K}$ is nonsingular.
(c) *Discrete barrier.* Let $w = w(x)$ be the solution of the 2-point boundary value problem:

$$w'' = 1 \quad x \in (0,1), \qquad w(0) = 0, \quad w'(1) = -1.$$

Show that $\mathbf{W} = (w(x_j))_{j=1}^M$ satisfies $(\mathbf{K}\mathbf{W})_j \le W_j - \frac{k}{2}$ for $1 \le j < M$ and $(\mathbf{K}\mathbf{W})_M \le -\frac{k}{2}$ provided $h, k$ are sufficiently small.
(d) *Error estimate.* Let $\mathbf{E}^n = (u(x_j, t^n) - U_j^n)_{j=1}^M$ be the finite difference error. Show that the auxiliary vector $\mathbf{V}^{n+1} = \mathbf{E}^{n+1} + \gamma\mathbf{W}$ satisfies componentwise

$$\mathbf{K}\mathbf{V}^{n+1} \le \widehat{\mathbf{V}}^n + k\mathbf{T}^n - \frac{\gamma k}{2} \le \widehat{\mathbf{V}}^n$$

provided $\gamma > 0$ is suitably chosen: how does it relate to $k$ and $h$? Deduce the $\ell^\infty$-upper estimate $\max_{1\le j\le M} E_j^n \le \widehat{C}_1(u)h + \widehat{C}_2(u)k$ where the constants $\widehat{C}_1(u), \widehat{C}_2(u)$ are proportional to $C_1(u), C_2(u)$. Modify the definition of $\mathbf{V}^{n+1}$ to derive a lower bound for $\min_{1\le j\le M} E_j^n$ and finally the estimate $\|\mathbf{E}^n\|_\infty \le \widehat{C}_1(u)h + \widehat{C}_2(u)k$.

2 (15 pts). *Advection-diffusion PDE and upwinding.* Consider the PDE

$$\partial_t u - a\partial_x^2 u + b\partial_x u = f(x) \quad x \in \mathbb{R},$$

with constants $a, b > 0$. Consider a uniform lattice $\{x_j\}$ of size $h$ and uniform time-step $k$. Write an implicit FDM with *upwinding.*

(a) Find the symbol $S(h\xi)$ of the implicit discrete operator and show that $|S(h\xi)| \leq 1$ for all $h, k > 0$.
(b) Derive the $\ell^2_h$-stability bound $\|\mathbf{U}^{n+1}\|_2 \leq \|\mathbf{U}^n\|_2 + k\|\mathbf{F}^{n+1}\|_2$ using von Neumann analysis; here $\|\mathbb{V}\|_2^2 = h\sum_{j\in\mathbb{Z}} V_j^2$.
(c) Examine the truncation error $\mathbf{T}^n = (\tau_j^n)_{j\in\mathbb{Z}}$ and show that $\|\mathbf{T}^n\|_{2,h} \leq C_1(u)h + C_2(u)k$. Make the regularity of $u$ entering in the constants $C_1(u), C_2(u)$ explicit.
(d) Derive an error estimate in $\ell^2_h$.

3 (15 pts). *Second order backward difference.* The so-called BDF(2) is a popular 2-step method for stiff equations. Given $U^0, U^1$, its semidiscrete version for the heat equation reads:

$$\frac{3U^{n+1} - 4U^n + U^{n-1}}{2k} - \Delta U^{n+1} = f^{n+1}.$$

(a) Derive the method upon combining two BE methods with steps $k$ and $2k$ in such a way that the truncation error is of second order.
(b) Determine the truncation error directly using Taylor expansion.
(c) Use (a) to derive the following identity of $L^2(\Omega)$-norms upon multiplying the discrete PDE by $U^{n+1}$, integrating by parts, and assuming that $U^{n+1}$ has vanishing trace:

$$\|U^{n+1}-U^n\|^2 - \frac{1}{4}\|U^{n+1}-U^{n-1}\|^2 + \frac{3}{4}\|U^{n+1}\|^2 - \|U^n\|^2 + \frac{1}{4}\|U^{n-1}\|^2 + k\|\nabla U^{n+1}\|^2 = k\langle f^{n+1}, U^{n+1}\rangle.$$

(d) Use (c) to prove the following $L^2$-stability bound

$$\|U^{N+1}\|^2 + 2\sum_{n=1}^{N} k\|\nabla U^{n+1}\|^2 \leq 5\|U^0\|^2 + 6\|U^1\|^2 + 2\sum_{n=1}^{N} k\|f^{n+1}\|^2_{H^{-1}(\Omega)}.$$

Hint: use the triangle inequality in the form $\|\phi_1 + \phi_2\|^2 \leq (1+\delta)\|\phi_1\|^2 + (1+\delta^{-1})\|\phi_2\|^2$ for any $\delta > 0$, add over $n$, and look for cancellations.

4 (15 pts). *Semidiscrete finite element method.* Problem 10.4 in Larsson and Thomée.

5 (25 pts). *MATLAB: FEM for the Heat Equation.* (a) Modify the MATLAB code `fem` in the website `http://www.math.umd.edu/~rhn/teaching.html` by adding a loop $1 \leq n \leq N$ to account for a *backward Euler* discretization of the time variable.
(b) Let $\Omega = (0,1)^2$ and $T = 1$. Let a Neumann condition $g_N$ be imposed on the side $x = 1$, and a Dirichlet condition $g_D$ on the rest of the boundary $\partial\Omega$. Let

$$u(x,y,t) = \sin(3\pi x)e^{-y-2t}.$$

be the exact solution. Find $g_D, g_N$ and the forcing $f = \partial_t u - \Delta u$.
(c) Compute the discrete solution $U$ at $T = 1$, along with the $L^2$, $H^1$, and $L^\infty$ errors. To this end, use the relation $k = h^2$ between time-step and meshsize, and find the discrete solution for meshsizes $h = 2^{-j}$ with $j = 3, 4, 5, 6$. Plot the discrete solution at $T = 1$ for $h = 2^{-4}$ and plot the three errors in terms of $h$ in a log-log scale. Verify the decay $\|u(T) - U(T)\| \approx h^s$ and relate $s$ to theory.
(d) Repeat (a) and (c) for the *Crank-Nicolson* method, this time with a relation $k = h$. Compare with (c) and draw conclusions.

6 (15 pts). *Crank-Nicolson-Galerkin method.* Problem 10.7 of Larsson and Thomée.