AMSC 715 Spring 2019

NUMERICAL METHODS FOR EVOLUTION PDE

HOMEWORK # 4

Pbs 1-2 due Th May 2, Pbs 3-4 due Th May 9, Pbs 5-7 due Tu May 21

1 (15 pts). *Lax-Wendroff Scheme*: (a) Problem 12.2 of Larsson and Thomée.
(b) Show that the scheme in p.191 of Larsson and Thomée can be written as

$$\frac{U_j^{n+1}-U_j^n}{k} - a\frac{U_{j+1}^n - U_{j-1}^n}{2h} - \frac{ka^2}{2}\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} = 0.$$

(c) Prove that the method is of second order by direct use of Taylor expansion.

2 (20 pts). *MATLAB*. (a) Consider the linear 1st order hyperbolic PDE $u_t + u_x = 0$. in the space interval $\Omega = (-1, 1)$ and time interval $(0, 0.5)$. Let the initial conditions be either

$$v(x) = \max(1 - 4|x + 0.25|, 0), \qquad v(x) = \begin{cases} 1 & x \le -0.25 \\ 0 & x > -0.25 \end{cases}$$

and boundary conditions $U_0 = v(-1)$ at $x_0 = -1$ (inflow) and $U_J^n = U_{J-1}^n$ at $x_J = 1$ (outflow). Implement the following schemes for arbitrary k, h: (i) Upwind, (ii) Lax-Friedrichs, (iii) Lax-Wendroff.
(b) Run the programs with $\frac{k}{h} = 0.8$ and $h = 0.01, 0.005, 0.0025$, and plot both the computed and true solutions at $t = 0.5$.
(c) Compare the methods in terms of the smearing effect around corners and jumps and the presence of oscillations.

3 (15 pts) *Convex flux.* Consider the Cauchy problem for Burgers' equation $u_t + uu_x = 0$ with initial condition

$$u_0(x) = 1 \quad \text{for } |x| > 1; \qquad u_0(x) = |x| \quad \text{for } |x| < 1.$$

(a) Sketch the characteristics in the (x,t) plane. Find a classical solution (continuous and piecewise C^1). Determine the time of breakdown (shock formation).
(b) Find a weak solution globally for $t > 0$, containing a shock curve. Note that the shock does not move with constant speed. Therefore, find first the solution away from the shock. Then, use the Rankine-Hugoniot condition to find a differential equation for the position of the shock given by $(x = s(t), t)$ in the (x, t)-plane.

4 (15 pts). *Nonconvex flux.* The Buckley-Leverett equation is a simple model for two-phase fluid flow in a porous medium with flux

$$f(u) = \frac{u^2}{u^2 + \frac{1}{2}(1-u)^2}.$$

In secondary oil recovery, water is pumped into some wells to displace the oil remaining in the underground rocks. Therefore u represents the saturation of water, namely the percentage of water in the water-oil fluid, and varies between 0 and 1. Find the entropy solution to the Riemann problem with initial states

$$u_0(x) = 1 \quad \text{for } x < 0; \qquad u_0(x) = 0 \quad \text{for } x > 0.$$

Hint: The line through the origin that is tangent to the graph of f on the interval $[0, 1]$ has slope $1/(\sqrt{3} - 1)$ and touches the curve at $u = 1/\sqrt{3}$.

5 (10 pts). *Engquist-Osher Scheme.* Let $f(0) = 0$ and consider the numerical flux

$$F(u_L, u_R) = \int_0^{u_L} \max(f'(s), 0)ds + \int_0^{u_R} \min(f'(s), 0)ds$$

(a) Show that the resulting method is consistent and monotone.
(b) Show that $F(u_L, u_R)$ can be equivalently written as

$$F(u_L, u_R) = \frac{1}{2}\big(f(u_L) + f(u_R) - \int_{u_L}^{u_R} |f'(s)|ds\big).$$

(c) If $f(u) = u^2/2$ show that $F(u_L, u_R)$ can be equivalently written as

$$F(u_L, u_R) = \frac{1}{2}\Big(\max(u_L, 0)^2 + \min(u_R, 0)^2\Big).$$

6 (10 pts). *Lax-Friedrichs Scheme.* This method reads

$$\frac{U_j^{k+1} - \frac{U_{j+1}^k + U_{j-1}^k}{2}}{\Delta t} + \frac{f(U_{j+1}^k) - f(U_{j-1}^k)}{2h} = 0.$$

(a) Show that the corresponding numerical flux reads

$$F(u_L, u_R) = \frac{1}{2}\Big(f(u_L) + f(u_R)\Big) + \frac{h}{2\Delta t}(u_L - u_R).$$

(b) Show that this method is consistent and monotone.

7 (15 pts). *Godunov Scheme.* This method comes from solving exactly 1d Riemann problems with piecewise constant data (see Lucier's notes).
(a) Assume that the flux f is convex. Show that the corresponding numerical flux can be written as

$$F(u_L, u_R) = \begin{cases} \min_{u_L \le u \le u_R} f(u) & u_L \le u_R \\ \max_{u_R \le u \le u_L} f(u) & u_R < u_L. \end{cases}$$

This formula is still valid for any Lipschitz flux regardless of convexity.
(b) Show that this method is consistent and monotone.