1 (15 pts). Lax-Wendroff Scheme: (a) Problem 12.2 of Larsson and Thomée. 
(b) Show that the scheme in p.191 of Larsson and Thomée can be written as
\[ \frac{U_j^{n+1} - U_j^n}{k} - \frac{U_{j+1}^n - U_{j-1}^n}{2h} = \frac{ka^2 U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} = 0. \]
(c) Prove that the method is of second order by direct use of Taylor expansion.

2 (20 pts). MATLAB. (a) Consider the linear 1st order hyperbolic PDE
\[ u_t + u_x = 0 \]
in the space interval \( \Omega = (-1, 1) \) and time interval \( (0, 0.5) \). Let the initial conditions be either
\[ v(x) = \max(1 - 4|x + 0.25|, 0), \quad v(x) = \begin{cases} 1 & x \leq -0.25 \\ 0 & x > -0.25 \end{cases} \]
and boundary conditions \( U_0 = v(-1) \) at \( x_0 = -1 \) (inflow) and \( U^n_J = U_{j-1}^n \) at \( x_j = 1 \) (outflow).
Implement the following schemes for arbitrary \( k, h \): (i) Upwind, (ii) Lax-Friedrichs, (iii) Lax-Wendroff.
(b) Run the programs with \( \frac{k}{h} = 0.8 \) and \( h = 0.01, 0.005, 0.0025 \), and plot both the computed and true solutions at \( t = 0.5 \).
(c) Compare the methods in terms of the smearing effect around corners and jumps and the presence of oscillations.

3 (15 pts) Convex flux. Consider the Cauchy problem for Burgers’ equation \( u_t + uu_x = 0 \) with initial condition
\[ u_0(x) = \begin{cases} 1 & \text{for } |x| > 1; \\ |x| & \text{for } |x| < 1. \end{cases} \]
(a) Sketch the characteristics in the (x,t) plane. Find a classical solution (continuous and piecewise \( C^1 \)). Determine the time of breakdown (shock formation).
(b) Find a weak solution globally for \( t > 0 \), containing a shock curve. Note that the shock does not move with constant speed. Therefore, find first the solution away from the shock. Then, use the Rankine-Hugoniot condition to find a differential equation for the position of the shock given by \( (x = s(t), t) \) in the \((x, t)\)-plane.

4 (15 pts) Nonconvex flux. The Buckley-Leverett equation is a simple model for two-phase fluid flow in a porous medium with flux
\[ f(u) = \frac{u^2}{u^2 + \frac{1}{2}(1 - u)^2}. \]
In secondary oil recovery, water is pumped into some wells to displace the oil remaining in the underground rocks. Therefore \( u \) represents the saturation of water, namely the percentage of water in the water-oil fluid, and varies between 0 and 1. Find the entropy solution to the Riemann problem with initial states
\[ u_0(x) = 1 \quad \text{for } x < 0; \quad u_0(x) = 0 \quad \text{for } x > 0. \]
Hint: The line through the origin that is tangent to the graph of \( f \) on the interval \([0, 1]\) has slope \( 1/(\sqrt{3} - 1) \) and touches the curve at \( u = 1/\sqrt{3} \).

5 (10 pts) Engquist-Osher Scheme. Let \( f(0) = 0 \) and consider the numerical flux
\[ F(u_L, u_R) = \int_0^{u_L} \max(f'(s), 0) ds + \int_0^{u_R} \min(f'(s), 0) ds \]
(a) Show that the resulting method is consistent and monotone.
(b) Show that $F(u_L, u_R)$ can be equivalently written as
$$F(u_L, u_R) = \frac{1}{2} (f(u_L) + f(u_R) - \int_{u_L}^{u_R} |f'(s)| ds).$$
(c) If $f(u) = u^2/2$ show that $F(u_L, u_R)$ can be equivalently written as
$$F(u_L, u_R) = \frac{1}{2} \left( \max(u_L, 0)^2 + \min(u_R, 0)^2 \right).$$

6 (10 pts). **Lax-Friedrichs Scheme.** This method reads
$$U_{j+1}^k - U_{j+1}^{k-1} + \frac{f(U_{j+1}^k) - f(U_{j-1}^k)}{2h} = 0.$$
(a) Show that the corresponding numerical flux reads
$$F(u_L, u_R) = \frac{1}{2} (f(u_L) + f(u_R)) + \frac{h}{2\Delta t} (u_L - u_R).$$
(b) Show that this method is consistent and monotone.

7 (15 pts). **Godunov Scheme.** This method comes from solving exactly 1d Riemann problems with piecewise constant data (see Lucier’s notes).
(a) Assume that the flux $f$ is convex. Show that the corresponding numerical flux can be written as
$$F(u_L, u_R) = \begin{cases} 
\min_{u_L \leq u \leq u_R} f(u) & u_L \leq u_R \\
\max_{u_R \leq u \leq u_L} f(u) & u_R < u_L.
\end{cases}$$
This formula is still valid for any Lipschitz flux regardless of convexity.
(b) Show that this method is consistent and monotone.