Deriving the Formula for the Sum of a Double Geometric Series

In Chapter 13, in the section entitled "The analysis", I promise to supply the formula for the sum of a double geometric series and the mathematical derivation of it. Here it is.

Consider a sum of terms each of which contains a product of two quantities, one involving a number or variable raised to successively higher powers, and the other involving a different number or variable raised to successively lower powers:

$$y^n + xy^{n-1} + x^2 y^{n-2} + ... + x^{n-2} y^2 + x^{n-1} y + x^n \ .$$

The trick in adding up this double geometric series is to factor out an expression involving x in order to make it look like an ordinary geometric series. In fact, if you factor out x^n from every term, the double series becomes:

$$\begin{aligned} & x^n (\frac{y}{x^n}^n + \frac{y^{n-1}}{x^{n-1}} + \frac{y}{x} + 1) = \\ & x^n (1 + (\frac{y}{x}) + ... + (\frac{y}{x})^n). \end{aligned} \ .$$

Lo and behold, the expression inside the parentheses is an ordinary geometric series in the quantity y/x. Using Formula 1 on the page "Deriving the Formula for the Sum of a Geometric Series", we now obtain the expression

$$\begin{aligned} & x^n (\frac{1-(y/x)^{n+1}}{1-(y/x)}) = \frac{x^{n+1}}{x} (\frac{1-(y/x)^{n+1}}{1-(y/x)}) \\ & = \frac{x^{n+1} - y^{n+1}}{x-y}. \end{aligned}$$

Now the precise expression that we needed to add up in Chapter 2 was of the form

$$xy^{n-1} + x^2 y^{n-2} + ... + x^{n-2} y^2 + x^{n-1} y + x^n \ ,$$

that is, the leading term y^n was omitted. Therefore, to add that series up, we only need to factor out an x, and use the formula in the case that the largest power is n-1, namely:

$$x(y^{n-1} + xy^{n-2} + ... + x^{n-2} y + y^{n-1}) = x \frac{x^n - y^n}{x-y} \ .$$

This formula is then applied in Chapter 13 with $x = (1 + r)$ and $y = (1 + s)$, where r is an interest rate and s is an escalation rate.