ON A PROBLEM BY HANS FEICHTINGER

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Abstract. In this paper, we solve a spectral problem about positive semi-definite trace-class pseudodifferential operators on modulation spaces which was posed by H. Feichtinger. Later, C. Heil and D. Larson rephrased the problem in the broader setting of positive semi-definite trace-class operators on a separable Hilbert space. Our solution consists in constructing a counterexample that solves Hans Feichtinger's problem by first solving this second problem.

1. Introduction

In this paper we answer the following question posed by Feichtinger at an Oberwolfach mini-workshop on wavelets [4].

Problem 1.1. *Let T be a positive semi-definite trace class operator on $L^2(\mathbb{R})$ given by*

$$Tf(x) = \int_{\mathbb{R}} k(x,y) f(y) dy,$$

where $f \in L^2(\mathbb{R})$ and $k \in M^1(\mathbb{R}^2)$, the so-called Feichtinger algebra. Suppose that

$$T = \sum_{k=1}^{\infty} h_k \otimes \overline{h_k},$$

where $\{h_k\}_{k=1}^{\infty} \subset L^2(\mathbb{R})$ is a set of orthogonal eigenfunctions of T corresponding to the eigenvalues $\{\|h_k\|_2^2\}_{k=1}^{\infty}$, such that $\|h_k\|_{M^1(\mathbb{R})} < \infty$, and the bar denotes the complex conjugation. In particular, Trace$(T) = \sum_{k=1}^{\infty} \|h_k\|_2^2 < \infty$.

Must we have: $\sum_{k=1}^{\infty} \|h_k\|_{M^1(\mathbb{R})}^2 < \infty$?

Heil and Larson later put the problem in the broader setting of positive semi-definite trace-class operators on a separable Hilbert space $\mathbb{H}$ [9]. To state this generalization we first set some notations. Let $\mathbb{H}$ be a separable Hilbert space and choose an orthonormal basis

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$\{w_n\}_{n\geq 1}$ for $\mathbb{H}$. We define a subspace $\mathbb{H}^1$ of $\mathbb{H}$ by

$$\mathbb{H}^1 = \Big\{ f \in \mathbb{H} : |||f||| := \sum_{n=1}^{\infty} |\langle f, w_n \rangle| < \infty \Big\}. \tag{1.1}$$

It follows that $|||w_n||| = \|w_n\| = 1$ for every n, and that if $f \in \mathbb{H}^1$ then $f = \sum_{n=1}^{\infty} \langle f, w_n \rangle w_n$, with convergence of this series in *both* norms $\|\cdot\|$ and $|||\cdot|||$.

We define an operator $T : \mathbb{H} \to \mathbb{H}$ by

$$T = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} (w_m \otimes \overline{w_n}), \tag{1.2}$$

where the scalars c_{mn} are such that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |c_{mn}| < \infty$$

and the tensor product $w_m \otimes \overline{w_n}$ maps linearly $\mathbb{H}$ to $\mathbb{H}$ via

$$f \in \mathbb{H} \ \mapsto \ w_m \otimes \overline{w_n}(f) = \langle f, w_n \rangle w_m.$$

It is easy to see that $T \in \mathcal{I}_1$, the space of all trace-class operators, with

$$\|T\|_{\mathcal{I}_1} \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \|c_{mn}(w_m \otimes \overline{w_n})\|_{\mathcal{I}_1} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |c_{mn}| < \infty.$$

In addition, note that the series defining T converges not only in the strong operator topology and operator norm, but also in trace-class norm.

Now suppose that the operator T given by (1.2) is positive semi-definite. Let $\{h_n\}_{n\geq 1}$ be an orthonormal basis of eigenvectors of T and $\{\lambda_n\}_{n\geq 1} \subset [0, \infty)$ be the corresponding eigenvalues. It follows that

$$T = \sum_{n=1}^{\infty} \lambda_n (h_n \otimes \overline{h_n}) = \sum_{n=1}^{\infty} g_n \otimes \overline{g_n}, \tag{1.3}$$

where $g_n = \lambda_n^{1/2} h_n$. In addition,

$$\|T\|_{\mathcal{I}_1} = \sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty} \lambda_n \|h_n\|^2 < \infty.$$

Heil and Larson's generalization of Problem 1.1 is the following question [9].

Problem 1.2. *With the above notations, must we have*

$$\sum_{n=1}^{\infty} \lambda_n |||h_n|||^2 < \infty? \tag{1.4}$$

In Section 3 we show that the solution to each of these problems is negative by providing counterexamples for each of them. But first, we provide some necessary background in Section 2

2. Preliminaries

In this section we recall the definition of the modulation spaces and some of their properties. In the second half of the section, we introduce two classes of trace-class operators that capture the behaviors of the operators in Problems 1.1 and 1.2.

2.1. **Modulation spaces.** Let $g \in \mathcal{S}(\mathbb{R})$ be a function in the Schwartz space of smooth and rapidly decaying functions, e.g., $g(x) = e^{-\pi x^2}$, and let $1 \leq p \leq \infty$. We say that a tempered distribution f is in the modulation space $M^p(\mathbb{R})$ if and only if

$$\|f\|_{M^p}^p := \iint_{\mathbb{R}^2} |V_g f(x,\omega)|^p dx d\omega < \infty,$$

with the usual modification for $p = \infty$, where

$$V_g f(x,\omega) = \int_{\mathbb{R}} f(t)\overline{g(t-x)} e^{-2\pi i \omega t} dt$$

is the *short-time Fourier transform* (STFT) of a function f with respect to g. A simple application of the Plancherel formula shows that if $f \in L^2(\mathbb{R})$ then

$$\|V_g f\|_{L^2(\mathbb{R}^2)}^2 = \iint_{\mathbb{R}^2} |V_g f(x,\omega)|^2 dx d\omega = \|g\|_2^2 \|f\|_2^2.$$

Consequently, V_g is a multiple of an isometry from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^2)$ and $M^2(\mathbb{R}) = L^2(\mathbb{R})$, [7]. The other modulation space that will be of interest in the sequel is $M^1(\mathbb{R})$, which is also known as the Feichtinger algebra [5, 7]. In particular, we note that

$$\mathcal{S}(\mathbb{R}) \subset M^1(\mathbb{R}) \subset M^2(\mathbb{R}) = L^2(\mathbb{R}) \subset M^\infty(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}).$$

We also need a discrete characterization of L^2 and M^1. Such a characterization exists for all the modulation spaces in terms of the so-called Wilson basis, see [2, 6, 12]. In particular, it is known that there exists an orthonormal basis $\mathcal{W} := \{w_n\}_{n\geq 1}$ for $L^2(\mathbb{R})$ where for each $n \geq 1$, $w_n \in M^1(\mathbb{R})$. In addition, for $1 \leq p \leq \infty$ and for all $f \in M^p$,

$$f = \sum_{n\geq 1} \langle f, w_n \rangle w_n,$$

where the series converges unconditionally in the norm of M^p if $1 \leq p < \infty$, and is weak* convergent if $p = \infty$. Moreover,

$$\|f\|_{M^p} = \left(\sum_{n \geq 1} |\langle f, w_n \rangle|^p \right)^{1/p}$$

is an equivalent norm for M^p; we refer to [7, Theorem 8.5.1] for details. In the sequel, we shall only be interested in $p = 1$, and $p = 2$. In the latter case, $\{w_n\}_{n \geq 1}$ is an orthonormal basis for $L^2(\mathbb{R})$.

It is trivial to extend these characterizations to modulation spaces defined on $\mathbb{R}^d$. In particular, one defines a Wilson orthonormal basis for $L^2(\mathbb{R}^2)$ by taking the tensor product of 1-dimensional Wilson ONBs. For example, $\{W_{n,m} : n, m \geq 1\} \subset L^2(\mathbb{R}^2)$ is given by

$$W_{n,m}(x, y) := w_n \otimes \overline{w_m}(x, y) = w_n(x)\overline{w_m(y)}, \quad n, m \geq 1,$$

and it acts by

$$W_{n,m}(f) = \langle f, w_m \rangle w_n = \left(\int_{\mathbb{R}} f(y)\overline{w_m(y)}dy \right) w_n.$$

In addition, $\{W_{n,m} : n, m \geq 1\}$ is an unconditional basis for $M^1(\mathbb{R}^2)$.

Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a compact integral operator associated with the kernel $k \in M^1(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and defined by

$$Tf(x) = \int_{\mathbb{R}} k(x, y)\, f(y)dy.$$

Then, T is a trace-class operator [9], and

$$k = \sum_{m,n \geq 1} \langle k, W_{m,n} \rangle W_{m,n}, \tag{2.1}$$

with convergence of the series in the M^1-norm. In addition,

$$\|k\|_{M^1} = \sum_{m,n \geq 1} |\langle k, W_{mn} \rangle| < \infty. \tag{2.2}$$

It now follows that for $f \in L^2(\mathbb{R})$,

$$Tf = \sum_{m,n \geq 1} \langle k, W_{mn} \rangle (w_m \otimes \overline{w_n})(f) = \sum_{m,n \geq 1} \langle k, W_{mn} \rangle (W_{m,n})(f).$$

The discrete version of the integral operator T is given by the matrix $K = (\langle k, W_{m,n}\rangle)_{m,n\geq 1}$, or equivalently

$$T = \sum_{m,n\geq 1} \langle k, W_{m,n}\rangle W_{m,n}. \tag{2.3}$$

Suppose in addition that T is positive semi-definite. Then, by the spectral theorem,

$$T = \sum_{k=1}^{\infty} \lambda_k t_k \otimes \overline{t_k} = \sum_{k=1}^{\infty} h_k \otimes \overline{h_k},$$

where $\{\lambda_k\}_{k=1}^{\infty} \subset (0,\infty)$ is the set of eigenvalues of T and $\{t_k\}_{k=1}^{\infty}$ is an orthonormal basis of corresponding eigenfunctions, and $h_k = \sqrt{\lambda_k} t_k$ for each $k \geq 1$. It was proved in [1, 9] that $h_k \in M^1(\mathbb{R})$.

2.2. **Type A and type B operators.** Let $\mathbb{H}$ denote an infinite-dimensional separable Hilbert space, with norm $\|\cdot\|$ and inner product $\langle \cdot,\cdot\rangle$. Let $\mathcal{I}_1 \subset \mathcal{B}(\mathbb{H})$ be the subspace of trace-class operators. A positive semi-definite operator T belongs to $\mathcal{I}_1$ if and only if

$$\|T\|_{\mathcal{I}_1} = \sum_{n=1}^{\infty} \lambda_n(T) < \infty,$$

where $\{\lambda_n(T)\}_{n\geq 1}$ is the set of eigenvalues of T arranged in a decreasing order and repeated according to multiplicity. For a detailed study on trace-class operators see [3, 10].

We fix now an orthonormal basis $\{w_n\}_{n\geq 1}$ for $\mathbb{H}$, once and for all. This basis induces the norm $|||\cdot|||$ on the dense subset $\mathbb{H}^1$ introduced in (1.1), and repeated here for the convenience of the reader:

$$|||f||| = \sum_{n=1}^{\infty} |\langle f, w_n\rangle|, \quad \mathbb{H}^1 = \Big\{ f \in \mathbb{H} : \sum_{n=1}^{\infty} |\langle f, w_n\rangle| < \infty \Big\}.$$

Definition 2.1. *An operator T given by* (1.2) *is of* Type A *with respect to the orthonormal basis $\{w_n\}_{n\geq 1}$ if, for an orthogonal set of eigenvectors $\{g_n\}_{n\geq 1}$ of T such that $T = \sum_{n=1}^{\infty} g_n \otimes \overline{g_n}$, with convergence in the strong operator topology, we have that*

$$\sum_{n=1}^{\infty} |||g_n|||^2 < \infty.$$

Definition 2.2. *An operator T given by* (1.2) *is of* Type B *with respect to the orthonormal basis $\{w_n\}_{n\geq 1}$ if there is some sequence of vectors $\{v_n\}_{n\geq 1}$ in $\mathbb{H}$ such that $T = \sum_{n=1}^{\infty} v_n \otimes \overline{v_n}$*

with convergence in the strong operator topology and we have that

$$\sum_{n=1}^{\infty} |||v_n|||^2 < \infty.$$

It is clear that if T is of Type A then it is of Type B. However, it was shown in [9, Example 2.2] that not every positive trace-class operator is of Type A or Type B, even when the operator is finite-rank.

Problem 1.2 can now be reformulated as follows.

Problem 2.3. *If T is of Type B with respect to an orthonormal basis $\{w_n\}_{n\geq 1}$, must it be of Type A with respect to the same ONB $\{w_n\}_{n\geq 1}$?*

3. Main results

We answer negatively Problems 1.2 and 2.3 by constructing a counterexample for the complex Hilbert space $\mathbb{H}$, in Proposition 3.1. This example is then modified to generate an example when the Hilbert space $\mathbb{H}$ is over the real field, in Proposition 3.3. From there, we answer the Feichtinger original problem in Theorem 3.4.

Proposition 3.1. *Let $\mathbb{H} = \ell^2(\{1,2,...\})$, and choose $p > 1$. Let $\{w_\ell\}_{\ell=1}^{\infty}$ denote the standard orthonormal basis of $\mathbb{H}$, i.e., $w_\ell = \delta_\ell$. Then $\mathbb{H}^1 = \ell^1(\{1,2,...\})$. For each $n \geq 1$, let $\{e_{n,k}\}_{k=0}^{n-1}$ be the Fourier ONB of $\mathbb{C}^n$ defined by*

$$e_{n,k} = \frac{1}{\sqrt{n}}\left(e^{-\frac{2\pi i k\ell}{n}}\right)_{\ell=0}^{n-1} = \frac{1}{\sqrt{n}}\left(1, e^{-\frac{2\pi ik}{n}}, e^{-\frac{4\pi ik}{n}}, ..., e^{-\frac{2\pi ik(n-1)}{n}}\right)^T,$$

and consider the $n \times n$ matrix T_n given by

$$T_n = \sum_{k=0}^{n-1} \lambda_{n,k}(e_{n,k} \otimes \overline{e_{n,k}}) = \frac{1}{n^3}\sum_{k=0}^{n-1}\left(1 + \frac{k}{n^p}\right)(e_{n,k} \otimes \overline{e_{n,k}}) \in \mathbb{C}^{n\times n},$$

where $\lambda_{n,k} = \frac{1}{n^3}\left(1 + \frac{k}{n^p}\right)$. We define an infinite block-diagonal matrix T by

$$T = T_1 \oplus T_2 \oplus ... \oplus T_n \oplus ...$$

Then, T is a positive semi-definite trace-class operator of Type B but not of Type A with respect to the orthonormal basis $\{w_\ell\}$.

Proof. By construction, the blocks T_n that make up T are pairwise orthogonal. Furthermore, for each $n \geq 1$, the spectrum of T_n consists of simple eigenvalues $\lambda_{n,k}$ with corresponding eigenvectors $e_{n,k}$ for $k = 0, \ldots, n-1$. Consequently, for each $n \geq 1$, and each $k \in \{0, \ldots, n-1\}$, $e_{n,k}$ generates a one-dimensional eigenspace of T corresponding to the eigenvalue $\lambda_{n,k}$. It is clear that T is positive semi-definite. Since $\|e_{n,k}\|_2 = 1$ and $T = \bigoplus_{n=1}^{\infty} \sum_{k=0}^{n-1} \lambda_{n,k}(e_{n,k} \otimes \overline{e_{n,k}})$, we see that

$$\begin{aligned}
\|T\|_{\mathrm{op}} &\leq \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n^3} \left(1 + \frac{k}{n^p}\right) \|e_{n,k} \otimes \overline{e_{n,k}}\|_{\mathrm{op}} \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n^3} \left(1 + \frac{k}{n^p}\right) \|e_{n,k}\| \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n^3} \left(1 + \frac{k}{n^p}\right) < \infty.
\end{aligned}$$

Furthermore, since $p > 1$, we see that

$$\begin{aligned}
\|T\|_{\mathcal{I}_1} = \mathrm{trace}(T) \quad &= \quad \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n^3} \left(1 + \frac{k}{n^p}\right) \\
&= \quad \sum_{n=1}^{\infty} \frac{1}{n^3} \left(n + \frac{n(n-1)}{2n^p}\right) \\
&< \quad \infty.
\end{aligned}$$

Hence T is a well-defined trace-class operator on $\mathbb{H}$.

We now show that T is of Type B. To this end we observe that for each $n \geq 1$, $\sum_{k=0}^{n-1} e_{n,k} \otimes \overline{e_{n,k}} = I_n$, where I_n denotes the identity of order n. Then

$$\begin{aligned}
T_n \quad &= \quad \frac{1}{n^3} \sum_{k=0}^{n-1} \left(1 + \frac{k}{n^p}\right) (e_{n,k} \otimes \overline{e_{n,k}}) \\
&= \quad \frac{1}{n^3} \sum_{k=0}^{n-1} (e_{n,k} \otimes \overline{e_{n,k}}) + \frac{1}{n^{3+p}} \sum_{k=0}^{n-1} k(e_{n,k} \otimes \overline{e_{n,k}}) \\
&= \quad \frac{1}{n^3} I_n + \frac{1}{n^{3+p}} \sum_{k=0}^{n-1} k(e_{n,k} \otimes \overline{e_{n,k}}).
\end{aligned}$$

Thus T can be written as

$$\begin{aligned} T &= \bigoplus_{n\geq 1} T_n = \bigoplus_{n\geq 1}\left(\frac{1}{n^3}I_n + \frac{1}{n^{3+p}}\sum_{k=0}^{n-1} k(e_{n,k}\otimes\overline{e_{n,k}})\right) \\ &= \bigoplus_{n\geq 1}\left(\frac{1}{n^3}I_n\right) + \bigoplus_{n\geq 1}\frac{1}{n^{3+p}}\sum_{k=0}^{n-1} k(e_{n,k}\otimes\overline{e_{n,k}}) \\ &= \bigoplus_{n\geq 1}\frac{1}{n^3}\sum_{k=1}^{n}(w_{\frac{n(n-1)}{2}+k}\otimes\overline{w_{\frac{n(n-1)}{2}+k}}) + \bigoplus_{n\geq 1}\frac{1}{n^{3+p}}\sum_{k=0}^{n-1} k(e_{n,k}\otimes\overline{e_{n,k}}). \end{aligned}$$

Then we have

$$\left|\left|\left|w_{\frac{n(n-1)}{2}+k}\right|\right|\right| = 1, \quad |||e_{n,k}||| = \sqrt{n},$$

and

$$\begin{aligned} &\sum_{n\geq 1}\frac{1}{n^3}\cdot\sum_{k=1}^{n} 1^2 + \sum_{n\geq 1}\frac{1}{n^{3+p}}\sum_{k=0}^{n-1} k\cdot(\sqrt{n})^2 \\ = &\sum_{n\geq 1}\left(\frac{1}{n^2} + \frac{n-1}{2n^{1+p}}\right) < \infty, \quad \text{for any } p > 1. \end{aligned}$$

Hence, T is of Type B with respect to $\{w_\ell\}_{\ell\geq 1}$.

We now show that T is not of Type A with respect to $\{w_\ell\}_\ell$. The key point is that T has only one-dimensional eigenspaces, so

$$\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\lambda_{n,k}(e_{n,k}\otimes\overline{e_{n,k}}) = \sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=0}^{n-1}\left(1+\frac{k}{n^p}\right)(e_{n,k}\otimes\overline{e_{n,k}})$$

is the unique decomposition of T as a sum of rank one projections generated by orthogonal eigenfunctions of T. Note again that $|||e_{n,k}||| = \sqrt{n}$, and

$$\lambda_{n,k}|||e_{n,k}||| = \frac{1}{n^3}\left(1+\frac{k}{n^p}\right)\cdot\sqrt{n} < \infty.$$

However,

$$\begin{aligned} \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\lambda_{n,k}|||e_{n,k}|||^2 &= \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=0}^{n-1}\left(1+\frac{k}{n^p}\right) \\ &= \sum_{n=1}^{\infty}\frac{1}{n^2}\left(n+\frac{n(n-1)}{2n^p}\right) \\ &\geq \sum_{n=1}^{\infty}\frac{1}{n} = \infty. \end{aligned}$$

□

We can modify the counterexample in Proposition 3.1 to deal with the case of a real Hilbert space $\mathbb{H}$. This amounts to using a real-valued ONB for $\mathbb{R}^n$ instead of the Fourier ONB $\{e_{n,k}\}_{k=0}^{n-1}$. For this let $\{h_{n,k}\}_{k=0}^{n-1}$ denote the Hartley ONB basis for $\mathbb{R}^n$ (see [11]), where

$$h_{n,k} = \frac{1}{\sqrt{n}}\left(\cos\left(\frac{2\pi kl}{n}\right) + \sin\left(\frac{2\pi kl}{n}\right)\right)_{l=0}^{n-1} = \sqrt{\frac{2}{n}}\left(\cos\left(\frac{2\pi kl}{n} - \frac{\pi}{4}\right)\right)_{l=0}^{n-1}.$$

Thus

$$\sum_{k=0}^{n-1} h_{n,k} \otimes \overline{h_{n,k}} = \sum_{k=0}^{n-1} h_{n,k} \otimes h_{n,k} = I_n,$$

where I_n denotes the identity of order n in $\mathbb{R}^n$.

Lemma 3.2. *For a fixed $n \geq 1$ and each $0 \leq k \leq n-1$ we have*

$$\sqrt{\frac{n}{2}} \leq |||h_{n,k}||| = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \left|\cos\left(\frac{2\pi kl}{n}\right) + \sin\left(\frac{2\pi kl}{n}\right)\right| \leq \sqrt{n}. \tag{3.1}$$

Proof. Denote by S_n the set

$$S_n := \left\{\frac{2\pi k}{n} : 0 \leq k \leq n-1\right\}.$$

It is easy to see that for each $0 \leq l \leq n-1$ we have

$$S_n = \left\{\frac{2\pi kl}{n} \pmod{2\pi} : 0 \leq k \leq n-1\right\} = \left\{-\frac{2\pi k}{n} \pmod{2\pi} : 0 \leq k \leq n-1\right\}.$$

Let $E := \sum_{x \in S_n} |\cos x + \sin x|$. Then

$$\begin{aligned} 2E &= \sum_{x\in S_n} |\cos x + \sin x| + \sum_{-x \in S_n} |\cos x + \sin x| \\ &= \sqrt{2}\sum_{k=0}^{n-1}\left|\cos\left(\frac{2\pi k}{n} - \frac{\pi}{4}\right)\right| + \sqrt{2}\sum_{k=0}^{n-1}\left|\cos\left(\frac{2\pi k}{n} + \frac{\pi}{4}\right)\right| \\ &= \sqrt{2}\sum_{k=0}^{n-1}\left[\left|\cos\left(\frac{2\pi k}{n} - \frac{\pi}{4}\right)\right| + \left|\sin\left(\frac{2\pi k}{n} - \frac{\pi}{4}\right)\right|\right]. \end{aligned} \tag{3.2}$$

Now for each $x \in \mathbb{R}$,

$$(|\sin x| + |\cos x|)^2 = |\sin x|^2 + |\cos x|^2 + 2|\sin x \cos x| = 1 + |\sin 2x| \geq 1,$$

$$\Rightarrow \sqrt{2} \geq |\sin x| + |\cos x| \geq 1.$$

It follows from (3.2) that $n \geq E \geq \frac{n}{\sqrt{2}}$ and therefore (3.1). $\square$

Proposition 3.3. *Let $\mathbb{H} = \ell^2(\{1, 2, ...\})$, and choose $p > 1$. Let $\{w_\ell\}_{\ell=1}^{\infty}$ denote the standard orthonormal basis of $\mathbb{H}$, i.e., $w_\ell = \delta_\ell$. For each $n \geq 1$ let T_n denote the $n \times n$ matrix given by*

$$T_n = \frac{1}{n^3} \sum_{k=0}^{n-1} \left(1 + \frac{k}{n^p}\right) (h_{n,k} \otimes h_{n,k}) \in \mathbb{R}^{n \times n}.$$

We define an infinite block-diagonal matrix T by

$$T = T_1 \oplus T_2 \oplus ... \oplus T_n \oplus ...$$

Then, T is a positive semi-definite trace-class operator of Type B but not of Type A with respect to the orthonormal basis $\{w_\ell\}_{\ell \geq 1}$.

Proof. The proof is almost identical to that of Proposition 3.1 where the Fourier ONB vectors $e_{n,k}$ are replaced by the Hartley ONB vectors $h_{n,k}$ and the estimate $|||e_{n,k}||| = \sqrt{n}$ is replaced by $\sqrt{\frac{n}{2}} \leq |||h_{n,k}||| \leq \sqrt{n}$, cf. Lemma 3.2. $\square$

We can now give an answer to Feichtinger's question, i.e., Problem 1.2.

Theorem 3.4. *Suppose that $\{w_n\}_{n \geq 1}$ is a Wilson orthonormal basis for $L^2(\mathbb{R})$ with $g \in M^1(\mathbb{R})$. Let $p > 1$, and for each $n \geq 1$ set $\lambda_{n,k} = \frac{1}{n^3}(1 + \frac{k}{n^p})$.*

For fixed $n \geq 1$ and each $0 \leq k \leq n-1$, let $h_{n,k} \in L^2(\mathbb{R})$ where

$$h_{n,k} = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \left(\cos\left(\frac{2\pi k l}{n}\right) + \sin\left(\frac{2\pi k l}{n}\right)\right) w_{\frac{n(n-1)}{2}+l+1}.$$

Let T be the operator defined by

$$T = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \lambda_{n,k} h_{n,k} \otimes h_{n,k}.$$

The following statements hold:

(i) *$\{h_{n,k} : 0 \leq k \leq n-1, n \geq 1\}$ is an orthonormal basis for $L^2(\mathbb{R})$.*

(ii) *T is a positive semi-definite trace-class operator on $L^2(\mathbb{R})$ that provides a counter-example to Problem 1.2.*

Proof. (i) It is easy to see that for each $n \geq 1$, $\{h_{n,k}\}_{k=0}^{n-1}$ is an orthogonal set in $L^2(\mathbb{R})$. Indeed, $\langle h_{n,k}, h_{n',k'} \rangle = 0$, for $n \neq n'$. Furthermore, since $\langle w_n, w_m \rangle = \delta_{n,m}$ we have that $\|h_{n,k}\| = 1$ for all $n \geq 1$, and $k \in \{0, 1, \ldots, n-1\}$.

(ii) It is also easy to see that T is a well-defined operator on $L^2(\mathbb{R})$. In fact, the series defining T converges in the operator norm. Furthermore, since $\|h_{n,k} \otimes h_{n,k}\|_{\mathcal{I}_1} = 1$, it follows that

$$\begin{aligned}\|T\|_{\mathcal{I}_1} &= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\lambda_{n,k} \\ &= \sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=0}^{n-1}(1+\frac{k}{n^p}) \\ &= \sum_{n=1}^{\infty}\frac{1}{n^3}\left(n+\frac{n(n-1)}{2n^p}\right) < \infty.\end{aligned}$$

Consequently, T is a trace-class operator.

By Lemma 3.2,

$$\begin{aligned}\|h_{n,k}\|_{M^1} &= \sum_{m=1}^{\infty}|\langle h_{n,k}, w_m\rangle| \\ &= \frac{1}{\sqrt{n}}\sum_{m=1}^{\infty}\left|\left\langle \sum_{l=0}^{n-1}\left(\cos\left(\frac{2\pi kl}{n}\right)+\sin\left(\frac{2\pi kl}{n}\right)\right)w_{\frac{n(n-1)}{2}+l}, w_m\right\rangle\right| \\ &= \frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\left|\cos\left(\frac{2\pi kl}{n}\right)+\sin\left(\frac{2\pi kl}{n}\right)\right| \\ &\geq \sqrt{\frac{n}{2}}.\end{aligned}$$

Also each term

$$\begin{aligned}\lambda_{n,k}\|h_{n,k}\|_{M^1} &= \frac{1}{n^3}(1+\frac{k}{n^p})\cdot\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\left|\cos\left(\frac{2\pi kl}{n}\right)+\sin\left(\frac{2\pi kl}{n}\right)\right| \\ &\leq \frac{1}{n^3}(1+\frac{k}{n^p})\cdot\sqrt{n} < \infty.\end{aligned}$$

However,

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\lambda_{n,k}\|h_{n,k}\|_{M^1}^2 &\geq \sum_{n=1}^{\infty}\frac{1}{2n^2}\sum_{k=0}^{n-1}(1+\frac{k}{n^p}) \\ &= \sum_{n=1}^{\infty}\frac{1}{2n^2}\left(n+\frac{n(n-1)}{2n^p}\right) \\ &\geq \sum_{n=1}^{\infty}\frac{1}{2n} = \infty.\end{aligned}$$

□

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