Horseshoes and Nonintegrability in the Restricted Case of a Rigid Body in a Central Gravitational Field

RADU BALAN

Princeton University ,
Program in Applied and Computational Mathematics,
Princeton, NJ 08544
e-mail address: rvbalan@math.princeton.edu

June 14, 1995

Abstract

The purpose of this paper is to study the motion of a spinless axially symmentric rigid body in a Newtonian field whose center of mass we suppose to be on a Keplerian orbit. In this case the system can be reduced to a Hamiltonian system with configuration space a two-dimensional sphere. We prove that the restricted planar motion has no analytic second integral and we find horseshoes due to the eccentricity of the orbit. In the case $I_3/I_1 > 4/3$, we prove that the system on the sphere is also analytically nonintegrable.

Key Words: transversal intersection, horseshoes, analytic nonintegrability

1 Introduction

The purpose of this paper is to study the integrability of a Hamiltonian dynamical system modelling the motion of a rigid body in a central gravitational field. We prove that the spinless axially symmetric rigid body, which is completely integrable in an uniform field (the Lagrange case), is analytically nonintegrable in a central gravitational field in the sense that chaotic motion of the internal rotation occurs.

In the restricted three-body problem there have been published many papers, from Poincaré 1899 [Poin899] to recent years [Xia92, Xia93]. On the other hand, the chaotic motion of a rigid body (namely the existence of horseshoes and Arnold diffusion) has been studied for some mechanical systems as in [HolMar83] or [Gray92].

The rigid body problem in celestial mechanics appeared with the paper by Duboshin in 1958 ([Dubo58]). Meantime many papers have appeared in two main areas: in one, the complete interaction between the motion of the centers of mass (CM) and the attitude motion has been considered and the studies have focussed, primary, on existence and stability analysis of special solutions (see for instance [Erem83], [CidEl85] or [WaMaKr92]); in the other, the motion of the CM has been decoupled from the attitude motion and usually just the first correction in the attitude motion has been kept (see for instance the study of [Belets66] or the papers of [TeoGra92] or [CelFal92]).

Our study is of the second type. The CM is supposed to move on an unperturbed Keplerian orbit. We also suppose to have an axially symmetric rigid body without spin. This sufficiently simplifies the equations of motion so that we are able to prove chaotic behaviour of the solution.

2 The Hamiltonians and Statement of the Problem

To describe the rigid body we use two coordinate systems whose origins are at the center of attraction: one fixed called the fixed system (ξ, η, ζ) and another corresponding to the principal axes of the body (i.e. in which the moment of inertia tensor diagonalizes), called the body system with coordinates (x, y, z) - see fig.1. The transition from one coordinate system to the other is given by a 3×3 matrix from $SO(3)$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A(t) \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \quad , \quad A(t) \in SO(3) \tag{1}$$

so that $r^2 = x^2 + y^2 + z^2 = \xi^2 + \eta^2 + \zeta^2$.

Suppose we have a rigid body of mass m in a Newtonian field whose center of attraction is denoted by O. In O suppose we have two coordinate systems as above. We denote by $\vec{r}$ the position vector of the center of mass of the rigid body (CM) and by (ξ, η, ζ) and, respectively, (x, y, z) the coordinates of CM in the two systems. For an element dm in the rigid body we denote by $\vec{r}_1$ its position vector with respect to O, $\vec{R}$ its position vector with respect to CM and by α the angle between $\vec{R}$ and $\vec{r}$. Then:

$$\vec{r}_1 = \vec{R} + \vec{r} \quad \text{and} \quad r_1^2 = r^2 + R^2 - 2rR\,cos\alpha.$$

The potential energy of dm is then given by (see formula [5-84] from [Gold80]):

$$dE_p = -\frac{GMdm}{r_1} = -\frac{GMdm}{r}\sum_{n=0}^{\infty} P_n(cos\alpha), \tag{2}$$

where $P_n(x)$ are the Legendre polynomials ($P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{3x^2-1}{2}$ ...) and G is the constant of attraction. Let us denote by (X, Y, Z) the coordinates of $\vec{R}$ in the body system. Then:

$$cos\alpha = -\frac{\vec{r}\cdot\vec{R}}{rR} = -\frac{xX + yY + zZ}{rR}. \tag{3}$$

On the other hand, integrating (2) we get:

$$E_p = -\frac{GM}{r}\sum_{n=0}^{\infty}\frac{1}{r^n}\int_m R^n P_n(cos\alpha)dm = -\frac{GM}{r}\sum_{n=0}^{\infty}\frac{1}{r^n}Q_n(\hat{\vec{r}}), \tag{4}$$

where $Q_n(\hat{\vec{r}})$ is the 2^n-polar inertial momentum given by $Q_n(\hat{\vec{r}}) = \int_m R^n P_n(cos\alpha)dm$. Using (3) and the fact that the body system is the principal axes system, we obtain:

$$Q_0(\hat{\vec{r}}) = \int_m dm = m,$$

$$Q_1(\hat{\vec{r}}) = \int_m R\,cos\alpha dm = -\frac{x}{r}\int_m X\,dm - \frac{y}{r}\int_m Y\,dm - \frac{z}{r}\int_m Z\,dm = 0,$$

$$Q_2(\hat{\vec{r}}) = \int_m R^2\frac{3cos^2\alpha - 1}{2}dm = -\frac{3}{2}(\frac{x^2}{r^2}I_1 + \frac{y^2}{r^2}I_2 + \frac{z^2}{r^2}I_3) + \frac{1}{2}(I_1 + I_2 + I_3),$$

where $I_1 = \int_m(Y^2 + Z^2)dm$ and I_2, I_3, obtained by circular permutations, are the principal moments of inertia.

Finally we get:

$$E_p = -\frac{GMm}{r} + \frac{GM}{2r^3}[3(\frac{x^2}{r^2}I_1 + \frac{y^2}{r^2}I_2 + \frac{z^2}{r^2}I_3) - (I_1 + I_2 + I_3)] + \mathcal{O}(\frac{1}{r^4}). \tag{5}$$

This is called MacCallagh's formula (see [Gold80]).

For the kinetic energy we use Köenig's theorem to obtain:

$$E_c = \frac{p_\xi^2 + p_\eta^2 + p_\zeta^2}{2m} + \frac{l_1^2}{2I_1} + \frac{l_2^2}{2I_2} + \frac{l_3^2}{2I_3} \tag{6}$$

where (p_ξ, p_η, p_ζ) are the components of the CM linear momentum in the fixed system $(p_\xi = m\dot{\xi},\ p_\eta = m\dot{\eta},\ p_\zeta = m\dot{\zeta})$ and (l_1, l_2, l_3) are the components of the internal angular momentum in the body system (i.e. the rigid body angular momentum with respect to the CM).

The configuration space for our problem is $K = \mathbf{R}^3 \times SO(3)$ parametrized by $(\xi, \eta, \zeta; A)$. The phase space will be the cotangent bundle T^*K, parametrized by $(\xi, \eta, \zeta, A; p_\xi, p_\eta, p_\zeta, l_1, l_2, l_3)$, so that the Hamiltonian of our problem is the function $H : T^*K \to \mathbf{R}$ given by:

$$H = \frac{p_\xi^2+p_\eta^2+p_\zeta^2}{2m} + \frac{l_1^2}{2I_1} + \frac{l_2^2}{2I_2} + \frac{l_3^2}{2I_3} \quad \begin{array}{l} -\frac{GMm}{r} + \frac{GM}{2r^3}[3(\frac{x^2}{r^2}I_1 + \frac{y^2}{r^2}I_2 + \frac{z^2}{r^2}I_3) - \\ -(I_1+I_2+I_3)] + \mathcal{O}(\frac{1}{r^4}) \end{array} \tag{7}$$

We break this expresion in three terms:

$$\begin{aligned} H_{01}(\xi,\eta,\zeta;p_\xi,p_\eta,p_\zeta) &= \frac{p_\xi^2+p_\eta^2+p_\zeta^2}{2m} - \frac{GMm}{r}\ , &(8)\\ H_{02}(l_1,l_2,l_3) &= \frac{l_1^2}{2I_1} + \frac{l_2^2}{2I_2} + \frac{l_3^2}{2I_3}\ , &(9)\\ H_{int}(\xi,\eta,\zeta,A) &= \frac{GM}{2r^3}[3(\frac{x^2}{r^2}I_1 + \frac{y^2}{r^2}I_2 + \frac{z^2}{r^2}I_3) - (I_1+I_2+I_3)] + \mathcal{O}(\frac{1}{r^4})\ ; &(10) \end{aligned}$$

each one describing one kind of problem: H_{01} describes the two-body problem, H_{02} describes the free rigid body problem and H_{int} describes the interaction-term for the "coupled"-problem, which is the object of our interest.

At this point one can immediately write the canonical equations using the above Hamiltonian (see for instance [AbraMa78] for how to do this). A system of 12 first order differential equations is obtained. The first idea that one can have is to consider this system as a perturbation problem with H_{int} as perturbation. If so, we get the classical two-body problem for the CM (with the Keplerian solution) and the free rigid body problem whose motion, in the angular momentum space, is given by intersection between the sphere of modulus of angular momentum (which is conserved) and the ellipsoid of kinetic energy (which is another first integral) - see [HolMar83]. In the case $I_1 > I_2 > I_3$ we have two saddle points and four heteroclinic orbits connecting these saddles in the rigid body angular momentum space. Next, when the interaction is introduced , one can ask if the heteroclinic orbits are preserved. For example, one might apply the Melnikov's method to see if the heteroclinic orbits become transversal heteroclinic orbits. The problem is that the complete equations of the heteroclinic orbits (in $T^*SO(3)$) are not explicitly known and this makes closed form calculations, at least now, impossible. We therefore seek another perturbative problem and make additional approximations and assumptions about the system.

First we shall suppose that the motion around the CM (i.e. the rotation motion) has no influence on the motion of the CM around the center of attraction (i.e. the revolution motion). More precisely, if we consider $T^*K \simeq T^*\mathbf{R}^3 \times T^*SO(3)$ then we shall assume that:

A1. The canonical equations in the first 6 coordinates $(\xi, \eta, \zeta, p_\xi, p_\eta, p_\zeta)$ are given by the Hamiltonian H_{01}.

This means that the motion of the CM is given by a Keplerian orbit unperturbed by the attitude motion, parametrized, for instance by:

$$\begin{aligned} \xi &= r\,cos\,v \\ \eta &= r\,sin\,v \\ \zeta &= 0\ , \end{aligned} \tag{11}$$

and:

$$\frac{1}{r} = \frac{1}{p}(1+\varepsilon cos\,v) \tag{12}$$

$$r^2\dot{v} = C \tag{13}$$

$$GMp = C^2\ . \tag{14}$$

Here v is the true anomaly (the planar angle measured between the position vector and the apocenter vector), p is the parameter of the orbit, ε the eccentricity and C the constant of areas. We shall consider only the cases $0 \leq \varepsilon < 1$, namely circular and elliptic orbits. Then, from (13) and (14) we see that we can invert the dependency $v = v(t)$ into $t = t(v)$ and obtain:

$$\frac{d}{dt} = \frac{C}{r^2}\frac{d}{dv}\ . \tag{15}$$

In fact, a similarity criterion for this approximation is given by:

$$s = \parallel \frac{H_{int}}{H_{01}} \parallel \sim \frac{I}{mr^2} = (\frac{R}{r})^2\ ,$$

where R is a characteristic dimension of the rigid body. Furthermore:

$$\parallel \frac{\partial H_{int}}{\partial x} \parallel \sim \frac{3GMxI_1}{r^5} + \frac{15GMx^3I_1}{r^7} + \frac{3GM}{r^4}I\frac{x}{r} \sim \frac{GMI}{r^4}$$

and

$$\| \frac{\partial H_{01}}{\partial x} \| \sim \frac{GMm}{r^2}\frac{x}{r} \sim \frac{GMm}{r^2} \ .$$

Then:

$$\| \frac{\partial H_{int}}{\partial x} \| \ / \ \| \frac{\partial H_{01}}{\partial x} \| \sim \frac{I}{mr^2} \sim s$$

Thus s measures the effect of the extent of the rigid body to the motion of the CM. Our approximation holds for $s \ll 1$.

The second approximation is less important than A1 but is made in order to avoid very messy equations. It concerns the higher order terms in the potential expansion (5):

A2. The higher order terms in the interaction Hamiltonian are negligible, i.e.:

$$H_{int} = \frac{3GM}{2r^3}(\frac{x^2}{r^2}I_1 + \frac{y^2}{r^2}I_2 + \frac{z^2}{r^2}I_3) - \frac{GM}{2r^3}(I_1 + I_2 + I_3) \tag{16}$$

Since the second term in H_{int} does not depend on $SO(3)$ variables, it will not affect the canonical equations on $T^*SO(3)$ and then it can be canceled from the interaction (it would perturb only the motion of the CM, but we have neglected these effects).

With these assumptions, we obtain a Hamiltonian system on $T^*SO(3)$ given by the following time-dependent Hamiltonian:

$$H_2(A; l_1, l_2, l_3; t) = \frac{l_1^2}{2I_1} + \frac{l_2^2}{2I_2} + \frac{l_3^2}{2I_3} + \frac{3GM}{2r^3}(\frac{x^2}{r^2}I_1 + \frac{y^2}{r^2}I_2 + \frac{z^2}{r^2}I_3) \tag{17}$$

where (x, y, z) are obtained using (1) and (11).

Now, even this Hamiltonian is too complicated because of the difficulty explained above. We still do not know the trajectory of heteroclinic orbits, so that we cannot see (17) as a perturbed Hamiltonian of (9). Therefore we shall consider only a particular case of the general rigid-body, namely the axially symmetric rigid body. Thus we shall assume that:

A3.
$$I_1 = I_2 \neq I_3 \ . \tag{18}$$

Now we can see that the Hamiltonian (17) is symmetric with respect to the rotations around the z-axis. Indeed, (17) can be rewritten as:

$$H_2 = \frac{l_1^2 + l_2^2}{2I_1} + \frac{l_3^2}{2I_3} - \frac{3GM}{2r^5}(I_1 - I_3)z^2 + \frac{3GM}{2r^3}I_1 \ . \tag{19}$$

As in the case of (16) we can cancel out the last term and we can see that the potential part depends only on r and z which are invariant under the rotations around the z-axis. Because of this symmetry we obtain that $l_3 = constant$ (an integral of motion) and we can reduce the system to the quotient space $SO(3)/SO(2) \simeq S^2$ and we get a Hamiltonian system on T^*S^2 with configuration space a 2-dimensional sphere. To express this, we use the Euler's angles and the Euler parametrization of $SO(3)$ - see fig.2. In this parametrization, the above Hamiltonian when the last term is dropped takes the form (after a little algebra using (11)):

$$\begin{array}{c} \tilde{H}_2(\Phi,\theta,\Psi;p_\Phi,p_\theta,p_\Psi;t) = \frac{(p_\Phi - p_\Psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\theta{}^2}{2I_1} + \frac{p_\Psi{}^2}{2I_3} - \\ -\frac{3GM}{2r^3}(I_1 - I_3)\sin^2\theta(\frac{1-\cos2(\Phi-v)}{2}) \ , \end{array} \tag{20}$$

where (Φ,θ,Ψ) are the Euler's angles and (p_Φ,p_θ,p_Ψ) are the associated canonical angular momenta (see [Gold80] for details).

Since Ψ is a cyclic variable and $p_\Psi = l_3 = constant$, the above Hamiltonian is actually defined on $T^*S^2 \times \mathbf{R}$. For the top case, the heteroclinic orbits of the free rigid body are degenerate to the equilibrium points and imply $p_\Psi = 0$. This suggests that we examine the case $p_\Psi = 0$. So, finally we shall restrict ourself to the particular case:

A4. $$p_\Psi = l_3 = 0 \tag{21}$$

which is the rigid body with no spin.

Then, under the 4 assumptions A1–A4, the reduced Hamiltonian describing our rigid body is defined on $T^*S^2 \times \mathbf{R}$ by:

$$H_{red}(\Phi,\theta;p_\Phi,p_\theta;t) = \frac{p_\Phi{}^2}{2I_1\sin^2\theta} + \frac{p_\theta{}^2}{2I_1} - \frac{3GM}{2r^3}(I_1 - I_3)\sin^2\theta\frac{1-\cos2(\Phi-v)}{2} \ . \tag{22}$$

We mention that the system has two singular points on the sphere, namely $\theta = 0$ and $\theta = \pi$ (the north and south poles), and this is due to the Euler's parametrization of the $SO(3)$. Furthermore, we see that (20) is invariant under the discrete transformation:

$$\begin{array}{l} \theta \longrightarrow \pi - \theta \\ \Phi \longrightarrow \Phi + \pi \\ p_\Phi \longrightarrow p_\Phi \\ p_\theta \longrightarrow -p_\theta \\ p_\Psi \longrightarrow -p_\Psi \end{array} \tag{23}$$

This symmetry reduces the system from the 2-dimensional sphere (with 2 singular points) to the 2-dimensional projective space $\mathbf{RP}^2$ (with one singular point); but we will not use this reduction subsequently.

3 The Analysis of the Hamiltonian System on the Sphere

As we have seen, our problem can be reduced to a system on $T^*S^2 \times \mathbf{R}$ with the Hamiltonian (22). The canonical equations can be written now as:

$$\begin{array}{l} \dot{\Phi} = \frac{\partial H}{\partial p_\Phi} = \frac{p_\Phi}{I_1 \sin^2\theta} \\ \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I_1} \\ \dot{p_\Phi} = -\frac{\partial H}{\partial \Phi} = \frac{3GM}{2r^3}(I_1 - I_3)\sin^2\theta \sin 2(\Phi - v) \\ \dot{p_\theta} = -\frac{\partial H}{\partial \theta} = \frac{{p_\Phi}^2 \cos\theta}{I_1 \sin^3\theta} + \frac{3GM}{2r^3}(I_1 - I_3)\sin 2\theta (\frac{1-\cos 2(\Phi - v)}{2}) \end{array} \quad (24)$$

We prefer to change the time variable to v using (15) and (12). At the same time, we make a change of variable: $\varphi = 2(\Phi - v)$. Then (24) is brought into the following non canonical form:

$$\begin{array}{l} \frac{d\varphi}{dv} = \frac{2r^2}{CI_1}\frac{p_\Phi}{sin^2\theta} - 2 \\ \frac{d\theta}{dv} = \frac{r^2}{CI_1}p_\theta \\ \frac{dp_\Phi}{dv} = \frac{3GM}{2rC}(I_1 - I_3)sin^2\theta \, \sin\varphi \\ \frac{dp_\theta}{dv} = \frac{r^2}{CI_1}\frac{{p_\Phi}^2\cos\theta}{sin^3\theta} + \frac{3GM}{2rC}(I_1 - I_3)\sin 2\theta \frac{1-\cos\varphi}{2} \end{array} \quad (25)$$

There are two zeros of the vector field (which are equilibria only in the circular motion) given by:

$$P_1 = (0, \frac{\pi}{2}, \frac{CI_1}{r^2}, 0) \quad , \quad P_2 = (\pi, \frac{\pi}{2}, \frac{CI_1}{r^2}, 0) \quad (26)$$

where $P = (\varphi, \theta, p_\Phi, p_\theta)$ is the parametrization of T^*S^2.

On the general case, when the CM is moving on an elliptic orbit, we have the following result:

LEMMA 3.1 *The following manifold:*

$$M_{inv} = \{\theta = \frac{\pi}{2}, p_\theta = 0\} \subset T^*S^2 \quad (27)$$

is an invariant manifold for (25). *Moreover, M_{inv} is diffeomorphically equivalent to T^*S^1.* □

This fact comes from a simple check of the second and fourth equations of (25). On the other hand, the above manifold is invariant for both circular and elliptic motions of the CM so that this provides important information about the flow. Also we can see that

$P_1, P_2 \in M_{inv}$. The motion restricted to M_{inv} represents the planar motion case of the rigid body.

Now we are able to define the unperturbed and perturbed systems.

The unperturbed system is given by (25) when the CM has a circular motion. The perturbed system is (25) when the CM has an elliptic motion. Thus, the eccentricity ε (defined in (12)) plays the rôle of a perturbation parameter. Using (12), the system (25) becomes:

$$x' = f(x) + \varepsilon g(x, v, \varepsilon) \ , \tag{28}$$

where $x^T = (\varphi, \theta, p_\Phi, p_\theta) \in T^*S^2$ is the state vector and f, g are vector fields given by:

$$f(x) = \begin{bmatrix} \frac{2p^2}{CI_1}\frac{p_\Phi}{sin^2\theta} - 2 \\ \frac{p^2}{CI_1}p_\theta \\ \frac{3GM}{2pC}(I_1 - I_3)\sin^2\theta \sin\varphi \\ \frac{p^2}{CI_1}\frac{{p_\Phi}^2\cos\theta}{sin^3\theta} + \frac{3GM}{2pC}(I_1 - I_3)\sin 2\theta \frac{1-\cos\varphi}{2} \end{bmatrix} \tag{29}$$

and:

$$g(x, v, \varepsilon) = \begin{bmatrix} -\frac{2p^2}{CI_1}\frac{p_\Phi \cos v(2+\varepsilon\cos v)}{sin^2\theta(1+\varepsilon\cos v)^2} \\ -\frac{p^2}{CI_1}\frac{p_\theta \cos v(2+\varepsilon\cos v)}{(1+\varepsilon\cos v)^2} \\ \frac{3GM}{2pC}(I_1 - I_3)sin^2\theta \cos v \sin\varphi \\ -\frac{p^2}{CI_1}\frac{{p_\Phi}^2\cos\theta\cos v(2+\varepsilon\cos v)}{sin^3\theta(1+\varepsilon\cos v)^2} + \frac{3GM}{2pC}(I_1 - I_3)\sin 2\theta \cos v \frac{1-\cos\varphi}{2} \end{bmatrix} . \tag{30}$$

Now we analyze the unperturbed system. Suppose that the CM is moving on a circular orbit given by $r = p$. First we change canonical the variables as follows:

$$x_1 = \varphi \qquad x_2 = \theta$$

$$p_1 = \frac{r^2}{2CI_1}p_\Phi \qquad p_2 = \frac{r^2}{CI_1}p_\theta$$

by which (25) is brought into the following form (recal that $GMp = C^2$ from (14)):

$$\begin{aligned} \frac{dx_1}{dv} &= \frac{4p_1}{\sin^2 x_2} - 2 \\ \frac{dx_2}{dv} &= p_2 \\ \frac{dp_1}{dv} &= \frac{3}{4}\frac{I_1 - I_3}{I_1}\sin^2 x_2 \sin x_1 \\ \frac{dp_2}{dv} &= \frac{4p_1^2\cos x_2}{\sin^3 x_2} + \frac{3}{2}\frac{I_1 - I_3}{I_1}\sin 2x_2 \frac{1-\cos x_1}{2}. \end{aligned} \tag{31}$$

This is a Hamiltonian system with Hamiltonian $\bar{H} : T^*S^2 \to \mathbf{R}$:

$$\bar{H}(x_1, x_2; p_1, p_2) = \frac{2p_1^2}{\sin^2 x_2} + \frac{p_2^2}{2} - \frac{3}{2}\frac{I_1 - I_3}{I_1}\sin^2 x_2 \frac{1 - \cos x_1}{2} - 2p_1 \ . \tag{32}$$

The zeros P_1, P_2 given by (26) become relative equilibria for the system. In the new variables:

$$Q_1 = (0, \frac{\pi}{2}, \frac{1}{2}, 0) \qquad Q_2 = (\pi, \frac{\pi}{2}, \frac{1}{2}, 0) \ .$$

The differential of the vector field (25) (i.e. of the right-hand side) has the form:

$$Df|_{(x_1,x_2,p_1,p_2)} = \begin{bmatrix} 0 & -\frac{8p_1 \cos x_2}{\sin^3 x_2} & \frac{4}{\sin^2 x_2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{3}{4}\frac{I_1 - I_3}{I_1}\sin^2 x_2 \cos x_1 & \frac{3}{4}\frac{I_1 - I_3}{I_1} \sin 2x_2 \sin x_1 & 0 & 0 \\ \frac{3}{4}\frac{I_1 - I_3}{I_1} \sin 2x_2 \sin x_1 & * & \frac{8p_1 \cos x_2}{\sin^3 x_2} & 0 \end{bmatrix} \tag{33}$$

where:

$$* = -\frac{4p_1^2}{\sin^2 x_2} - \frac{12p_1^2 \cos^2 x_2}{\sin^4 x_2} + 3\frac{I_1 - I_3}{I_1} \cos 2x_2 (\frac{1 - \cos x_1}{2}) \ .$$

At Q_1 this becomes:

$$Df(Q_1) = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{3}{4}\frac{I_1 - I_3}{I_1} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} , \tag{34}$$

while at Q_2 we get:

$$Df(Q_2) = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{3}{4}\frac{I_1 - I_3}{I_1} & 0 & 0 & 0 \\ 0 & -1 - 3\frac{I_1 - I_3}{I_1} & 0 & 0 \end{bmatrix} . \tag{35}$$

The characteristic polynomials are:

$$\begin{array}{l} p_{Q_1}(s) = (s^2 + 1)(s^2 - 3\frac{I_1 - I_3}{I_1}) \\ p_{Q_2}(s) = (s^2 + 3\frac{I_1 - I_3}{I_1})(s^2 + 1 + 3\frac{I_1 - I_3}{I_1}) \end{array} \tag{36}$$

Now we see that, depending on the value of $a = 3\frac{I_1 - I_3}{I_1}$, we have different types of equilibria:

· for $a > 0$, Q_1 is center-saddle and Q_2 is center;

· for $-1 < a < 0$, Q_1 is center and Q_2 is center-saddle;
· for $a < -1$, Q_1 is center and Q_2 is saddle;

We turn now to the restricted system on the invariant manifold given in Lemma 3.1. We shall prove that for any value of $a \neq 0$ we have two homclinic orbits to a saddle point and that these connections are preserved under the perturbation. Then we shall return to the full system on T^*S^2 (31) and we shall prove that the stable and unstable manifolds to the saddle point (Q_2, when $a < -1$) are candidates for a hyperbolic structure and consequently for a global chaotical motion. This will imply the analytic nonintegrability of the Hamiltonian system.

Now let us consider the restriction of (25) to the invariant manifold M_{inv} given in Lemma 3.1. We set $\theta = \frac{\pi}{2}$ and $p_\theta = 0$ and we obtain:

$$\begin{array}{rcl} \frac{dp_\Phi}{dv} & = & \frac{3GM}{2rC}(I_1 - I_3)\sin\varphi \\ \frac{d\varphi}{dv} & = & \frac{2r^2}{CI_1}p_\Phi - 2 \end{array} \tag{37}$$

which is equivalent to the following second order differential equation:

$$\frac{d^2\varphi}{dv^2} - 3\frac{GMr}{C^2}\frac{I_1 - I_3}{I_1}\sin\varphi = \frac{2}{r}\frac{dr}{dv}(2 + \frac{d\varphi}{dv}) \tag{38}$$

This system is still a Hamiltonian system; it may be obtained from (22) by setting $\theta = \frac{\pi}{2}$ and $p_\theta = 0$. We rewrite (38) as a system of 2 first differential equations with the following state variables: if $I_1 < I_3$ $y_1 = \varphi$ else $y_1 = \varphi + \pi$ and $y_2 = \frac{d\varphi}{dv}$. This yields:

$$\begin{array}{rcl} \frac{dy_1}{dv} & = & y_2 \\ \frac{dy_2}{dv} & = & -3\frac{GMr}{C^2}|\frac{I_1 - I_3}{I_1}|\sin y_1 + \frac{2}{r}\frac{dr}{dv}(2 + y_2) \end{array} \tag{39}$$

We next use the decomposition given in (28), using the definitions of (12) and (14). We get:

$$\begin{array}{rcl} \frac{dy_1}{dv} & = & y_2 \\ \frac{dy_2}{dv} & = & -3|\frac{I_1 - I_3}{I_1}|\sin y_1 + \varepsilon(3|\frac{I_1 - I_3}{I_1}|\sin y_1\frac{\cos v}{1+\varepsilon\cos v} + 2(2 + y_2)\frac{\sin v}{1+\varepsilon\cos v}) \end{array} \tag{40}$$

Letting $\Omega^2 = 3|\frac{I_1 - I_3}{I_1}| > 0$, we see that the unperturbed system corresponds to $\frac{d^2y_1}{dv^2} + \Omega^2\sin y_1 = 0$ which is a pendulum equation. This equation has two homoclinic connections to $y_1 = \pi,\ y_2 = 0$, given by:

$$\begin{array}{rcl} y_1^0(v) & = & \pm 2\, arctan(sinh(\Omega v)) \\ y_2^0(v) & = & \pm 2\Omega\, sech(\Omega v)\ , \end{array} \tag{41}$$

where $+$ stands for the upper homoclinic connection (in the y_1, y_2 plane) and $-$ corresponds to the lower branch.

Now we ask if these homoclinic connections are preserved under the perturbation. The answer is given by the following result:

LEMMA 3.2 *For any $\Omega > 0$ the perturbed system* (40) *has infinitely many transversal homoclinic orbits for any $\varepsilon \in (0,1)$ excepting, at most, for a finite number of values.* $\square$

The proof uses Melnikov's function and Smale-Birkhoff Theorem and is presented in the section 5. This result has been also proved in [TeoGra92] and [Bur87] (conform to [TeoGra92]).

Now we can return to (31) which describes the unperturbed two degree of freedom system on T^*S^2. We consider only the case $a < -1$, or equivalently $I_3/I_1 > 4/3$. The unperturbed system has at Q_2 a saddle point and then two invariant 2-dimensional manifolds pass through Q_2, the stable and unstable manifolds. From the above discussion we know there exist two homoclinic connections. Thus, the intersection of the stable and unstable manifolds is non empty: it contains the fixed point Q_2 and two 1-dimensional curves asymptotic to it. In order to obtain transversal intersection of these manifolds for the perturbed system, we need to prove that the intersection is precisely of dimension 1 and this is achieved by the following Lemma:

LEMMA 3.3 *For the unperturbed system* (31) *(i.e. $\varepsilon = 0$) with $I_3/I_1 > 4/3$ consider a point q^0 on the homoclinic connections* (41) *away from Q_2. Let us denote by $W^{s,u}$ the stable/unstable manifolds passing through Q_2. Then $q^0 \in W^s \cap W^u$ and* $\dim(T_{q^0}W^s + T_{q^0}W^u) = 3$ *for all values of Ω excepting at most 1 value.* $\square$

The proof given in section 6, is based on two steps: firstly we find the first correction to the stable and unstable manifold around the relative equilibrium Q_2; then we solve asymptotically the first variational system that transports the tangent vectors along the homoclinic orbits. Note that the condition $I_3/I_1 > 4/3$ (or $a < -1$) is required for Q_2 to be a saddle point with two dimensional stable and unstable manifolds. The critical value of Ω is found to be $\Omega_c = 1.70557$ and this happens only for the upper branch of the homoclinic orbits. The details are presented in section 6.

4 Statement of the Main Results

In this section we present the conclusion of the Lemmas 3.2 and 3.3 from the previous section.

As we have said, Lemma 3.2 has been proved in some other papers (see [TeoGra92] and [Bur87]). We give here just a briefly interpretation of the symbolic dynamics associated to the chaotic motion that occurs due to the existence of transversal homoclinic points. This idea is taken from a lecture given by Professor P. Holmes at Princeton University.

THEOREM 4.1 *For any $\Omega > 0$ and almost any $\varepsilon \in (0,1)$ (except, at most, a finite number of values) the planar attitude motion of the rigid body (i.e. the motion restricted to M_{inv}) has chaotic behaviour in the following sense: for any sequence of integers $s = (s_k)_{k\in\mathbf{Z}}$, $s_z \in \mathbf{Z}$ there exists a sequence of increasing numbers $(t_k)_{k\in\mathbf{Z}}$, $t_k < t_{k+1}$, $t_k \in \mathbf{R}$ and a trajectory of (37) such that : $\varphi(t_k) = 2\pi s_k$, for any k.* □

This means that the rigid body can rotate for an arbitrary number of times in one sense, then rotate in the opposite sense for another arbitrary number of times and so on. Orbits associated with such arbitrary sequences are found using the Markov partition construction (the "horseshoe") associated to the transversal homoclinic points. This standard construction is presented in many papers; we refer the reader, for instance, to [Moser73],[GucHol93] or [Xia92].

From Lemmas 3.2 and 3.3 we conclude the transversal intersection of the stable and unstable manifolds of the Poincaré mapping of the perturbed system (28). This transversality gives us the analytic nonintegrability of the system (we refer the reader to [Kozlov83] for an extensive survey on nonintegrability of Hamiltonian systems). Here we shall state a result about non-existence of two analytic, independent first integrals.

Suppose we have a periodic and analytic Hamiltonian $H_\varepsilon : \mathbf{R}^{2n} \times \mathbf{R} \to \mathbf{R}$ dependent analytic on a small parameter $\varepsilon > 0$ ($H_\varepsilon(x,p,t+T) = H_\varepsilon(x,p,t)$). Consider the Hamiltonian system:

$$\begin{array}{rcl} \dot{x} & = & \frac{\partial H_\varepsilon}{\partial p} \\ \dot{p} & = & -\frac{\partial H_\varepsilon}{\partial x} \end{array} \tag{42}$$

and associate to it the Poincaré returning map:

$$P^\varepsilon_{t_0} : (x_1,p_1) \mapsto P^\varepsilon_{t_0}(x_1,p_1) = (x_2,p_2)$$

where x_2, p_2 is the solution of (42) at t_0+T when at t_0 $(x,p) = (x_1,p_1)$ $(x_1,x_2,p_1,p_2 \in \mathbf{R}^n)$.

A function $F^\varepsilon : \mathbf{R}^{2n} \times \mathbf{R} \to \mathbf{R}$ periodic in time ($F^\varepsilon(x,p,t+T) = F^\varepsilon(x,p,t)$) and depending on ε as a formal power series:

$$F^\varepsilon(x,p,t) = \sum_{i\geq 0} \varepsilon^i F^i(x,p,t)$$

is said to be an *analytic first integral* if:

1) $F^i : \mathbf{R}^{2n} \times \mathbf{R} \to \mathbf{R}$ are analytic ;

2) $F^\varepsilon(P^\varepsilon_{t_0}(x,p),t_0) = F^\varepsilon(x,p,t_0)$, for any $(x,p) \in \mathbf{R}^{2n}$ and $t_0 \in [0,T]$

A set of n analytic first integrals $F_1^\varepsilon, \ldots, F_n^\varepsilon : \mathbf{R}^{2n} \times \mathbf{R} \to \mathbf{R}$ is said to be *independent* if the level set:

$$M_c(t_0) = \{(x,p) \in \mathbf{R}^n \mid F_k^\varepsilon(x,p,t_0) = c_k \ , \ 1 \le k \le n \ \}$$

does not include any manifold of dimension higher than n. In fact, the set $M_c(t_0)$ is an analytic set and, because of Lojaciewicz's result, that we shall state and use in a moment, it can be written as local finite union of analytic manifolds. Now we can state our nonintegrability result:

THEOREM 4.2 *Consider a spinless, axialsymmetric rigid body lying in a central gravitational field, whose attitude motion dynamics is given by* (25). *If* $I_3/I_1 > 4/3$ *then there do not exist 2 analytic, independent first integrals, and the system is analytically nonintegrable.* □

We shall give a straightforward proof of this result (as well as for any system in which there is a transversal intersection of the stable and unstable manifolds to a periodic orbit) based on the λ-Lemma and Lojaciewicz's Structure Theorem for Real Analytic Manifolds. Another proof can be done using the [Kozlov83] paper, by noting that the union of stable and unstable manifolds $W_\varepsilon^s \cup W_\varepsilon^u$ is a key set, in the terminology of the aforementioned paper. We recall now the two results; from the Lojaciewicz's Structure Theorem we present only the result that we are using.

λ-Lemma (see [Palis69])*Let* f *be a* $\mathcal{C}^1$ *diffeomorphism of* $\mathbf{R}^n$ *with a hyperbolic fixed point* p *having* s *and* u *dimensional stable and unstable manifolds* ($s+u=n$), *and let* D *be a* u*-dimensional disk in* $W^u(p)$. *Let* Δ *be a* u*-dimensional disk meeting* $W^s(p)$ *transversely at some point* q. *Then* $\bigcup_{n\ge 0} f^n(\Delta)$ *contains* u*-dimensional disks arbitrarily close to* D. ◇

Lojaciewicz's Structure Theorem for Real Analytic Manifolds (see [KraPar92] for the complete statement, pp.154) *Let* $\Phi(x_1, \ldots, x_n)$ *be a real nontrivial analytic function in a neighborhood of the origin. Then there exist numbers* $\delta_j > 0, j = 1, ..., n$ *so that the set:*

$$Z = \{x \in \mathbf{R}^n | \ |x_j| < \delta_j, \forall j \ \text{and} \ \Phi(x) = 0 \ \}$$

has a decomposition:

$$Z = V^{n-1} \cup \cdots \cup V^0 \ .$$

The set V^0 *is either empty or consists of the origin alone. For* $1 \le k \le n-1$ *we may write* V^k *as a finite, disjoint union of* k*-dimensional submanifolds (in the full statement, an explicit description of these manifolds is given).* ◇

Now we prove Theorem 4.2. Suppose there are 2 analytic, independent first integrals, say F_1^ε and F_2^ε. Suppose we have fixed t_0 and denote by $P_{t_0}^\varepsilon$ the Poincaré map. Then, on stable manifold they must be constant. The same thing happens on the unstable manifold. Because the stable and unstable manifolds intersect, the values of F_1^ε, respectively F_2^ε, must be the same on these manifolds, that is:

$$F_1^\varepsilon(W^u) = F_1^\varepsilon(W^s) = c_1 \quad ; \quad F_2^\varepsilon(W^u) = F_2^\varepsilon(W^s) = c_2 \ .$$

Now, pick a point $s_0 \in W^u$ and consider q a transversal intersection point between W^s and W^u, different from the fixed point of $P_{t_0}^\varepsilon$ (such a point exists because of Lemmas 3.2 and 3.3). Let Δ be a 2-dimensional disk in W^u containing q, as in λ-Lemma. Then, for any neighborhood of s_0 there exists an integer $n > 0$ such that $(P_{t_0}^\varepsilon)^n(\Delta)$ intersects nonempty the neighborhood. Now we apply the Lojaciewicz's Theorem to:

$$\Phi(x) = (F_1^\varepsilon(x+s_0) - c_1)^2 + (F_2^\varepsilon(x+s_0) - c_2)^2$$

Denote by

$$Z_\delta = \{x \in T^*S^2 | \parallel x \parallel \leq \delta \ , \ \Phi(x) = 0\}$$

which is the intersection between the level set $Z = \{x \in T^*S^2 | \Phi(x) = 0\}$ and the ball $B_\delta(s_0) = \{x \in T^*S^2 | \parallel x - s_0 \parallel \leq \delta\}$. Then $W^u \cap B_\delta(s_0)$ and $W^s \cap B_\delta(s_0)$ are both included in Z_δ. Particularly we are interested in the inclusion $W^u \cap B_\delta(s_0) \subset Z_\delta$. Now we have a decomposition of Z_δ into a union of manifolds of dimension 0 (the point s_0), 1 and 2 (dimensions higher than 2 are forbidden by the condition that F_1^ε and F_2^ε are independent). Now, if we look to the union of manifolds of dimension 2 we see that here must lie an infinite sequence of submanifolds of the form $(P_{t_0}^\varepsilon)^n(\Delta) \cap B_\delta(s_0)$, for some n. Then we conclude that Z_δ is not a local finite union of manifolds and this proves the contradiction. So, our assumption of the existence of 2 analytic, independent first integrals is false.

5 Proof of Lemma 3.2

For a system of the form (28), the Melnikov's function is given by (see [GucHol93]):

$$M(v_0) = \int_{-\infty}^{\infty} f(y^0(v)) \wedge g(y^0(v), v + v_0, \varepsilon) dv$$

where $y^0(v)$ is the parametrization of the homoclinic orbit and the wedge product $\wedge$ is defined as $f \wedge g = f_1 g_2 - f_2 g_1$ (f_1, f_2 and g_1, g_2 are respectively, the components of the vector fields f and g). The Melnikov's function measures the distance between the stable and unstable manifolds to the cycle that is born from an unperturbed saddle point, under the periodic perturbation. If this function has a simple zero for some v_0 then the two invariant manifolds (in the extended space) intersect transversally and we obtain, via the Poincaré-Smale-Birkhoff theorem the existence of infinitely many periodic orbits.

For the system (40) the Melnikov's function takes the form:

$$M(v_0) = \int_{-\infty}^{\infty} y_2^0(v)[\Omega^2 \sin y_1^0(v) \frac{\cos(v+v_0)}{1+\varepsilon\cos(v+v_0)} + 2(2+y_2^0(v))\frac{\sin(v+v_0)}{1+\varepsilon\cos(v+v_0)}]dv \ .$$

We see that from (41) that $y_1^0(-v) = -y_1^0(v)$ and $y_2^0(-v) = y_2^0(v)$. Then, by an oddness argument it follows that $M(0) = 0$. The only problem is now to prove that $v_0 = 0$ is a simple zero. For this, we compute $M'(0)$:

$$\frac{dM}{dv_0}|_{v_0=0} = \int_{-\infty}^{\infty} y_2^0(v)[-\Omega^2 \sin y_1^0(v)\frac{\sin v}{(1+\varepsilon\cos v)^2} + 2(2+y_2^0(v))\frac{\varepsilon+\cos v}{(1+\varepsilon\cos v)^2}]dv \ .$$

We fix $\Omega > 0$ and then the above function becomes an ε-dependent function. We shall prove that, for small ε, it is not zero. It is sufficient to set $\varepsilon = 0$ above, to obtain:

$$\frac{dM}{dv_0}(v_0 = 0, \varepsilon = 0) = \int_{-\infty}^{\infty}(-\Omega^2 y_2^0(v)\sin y_1^0(v)\sin v + 2y_2^0(v)(2+y_2^0(v))\cos v)dv \ .$$

We use the explicit expressions for $y_1^0(v)$ and $y_2^0(v)$ given in (41) and we obtain:

$$\frac{dM}{dv_0}(v_0 = 0, \varepsilon = 0) = -4\Omega^3 F_1 \pm 8\Omega F_2 + 8\Omega^2 F_3$$

where F_1, F_2, F_3 are integrals that we evaluate by the method of residus as:

$$F_1 = \int_{-\infty}^{\infty} sech^2(\Omega v) tanh(\Omega v)\sin v \ dv = \frac{\pi}{2\Omega^3}\frac{1}{sinh\frac{\pi}{2\Omega}} \ ,$$

$$F_2 = \int_{-\infty}^{\infty} sech(\Omega v)\cos v \ dv = \frac{\pi}{\Omega}\frac{1}{cosh\frac{\pi}{2\Omega}} \ ,$$

$$F_3 = \int_{-\infty}^{\infty} sech^2(\Omega v)\cos v \ dv = \frac{\pi}{\Omega^2}\frac{1}{sinh\frac{\pi}{2\Omega}} \ .$$

The final result is:

$$\frac{dM}{dv_0}|_{v_0=0,\varepsilon=0} = \frac{6\pi}{sinh\frac{\pi}{2\Omega}} \pm \frac{8\pi}{cosh\frac{\pi}{2\Omega}} \ , \tag{43}$$

which, for the upper branch is always positive and has no zeros, whereas for the lower branch we have a zero at $\Omega_0 = \frac{\pi}{ln\, 7} = 1.61446$.

By analyticity of $\frac{dM}{dv_0}(v_0 = 0)$ as function of ε we conclude that for any $\Omega > 0$, $\Omega \neq \Omega_0$ we have at most a finite number of zeros and the proof is finished.

6 Proof of Lemma 3.3

The situation is now the following: we have the unperturbed system given by (31) and we are in the case when $a = 3\frac{I_1 - I_3}{I_1} < -1$. This means that Q_2 is a saddle point and it is a hyperbolic equilibrium in T^*S^2 for (31). We know from the Stable Manifold Theorem (see [Kelley67]) that two invariant 2-dimensional manifolds pass through Q_2 tangent to, respectivelly, the stable space and unstable eigenspace of the linearized system (35). Each of them contains the homoclinic connections (41) so that their intersection, excluding Q_2, is not empty. We want to prove that, along of these homoclinic orbits there are three independent vectors tangent to the union of the manifolds (i.e. two of them tangent to one manifold and the third vector to the other manifold). We choose one of the three independent vectors to be the tangent vector to the homoclinic orbits at that point (q^0). This is tangent to both invariant manifolds. We shall prove that taking two other vectors tangent to the unstable, respectively, stable manifolds near Q_2, they are transported by the flow on the homoclinic orbit forward, respectively, backward at the same point into two independent vectors. For, we need two facts: firstly we have to know the first correction of the tangent spaces to the stable/unstable manifolds near Q_2 and secondly, we have to find an asymptotic approximation for the transports along the homoclinic orbits of a tangent vector (i.e. an asymptotic expansion of the solution of the first variational equation).

To simplify the calculus we translate the equilibrium point Q_2 into the origin by changing variables as follows:

$$\xi_1 = x_1 - \pi \ , \ \xi_2 = x_2 - \frac{\pi}{2} \ , \ \xi_3 = p_1 - \frac{1}{2} \ , \ \xi_4 = p_2 \ .$$

Then, the system (31) becomes:

$$\begin{array}{rcl} {\xi_1}' & = & \frac{4\xi_3+2}{\cos^2\xi_2} - 2 \\ {\xi_2}' & = & \xi_4 \\ {\xi_3}' & = & \frac{1}{4}\Omega^2\cos^2\xi_2 \sin\xi_1 \\ {\xi_4}' & = & -\frac{4(\xi_3+\frac{1}{2})^2\sin\xi_2}{\cos^3\xi_2} + \frac{1}{2}\Omega^2\sin(2\xi_2)\frac{1+\cos\xi_1}{2} \ , \end{array} \tag{44}$$

whose Hamiltonian is:

$$\begin{array}{l} H(\xi_1,\xi_2,\xi_3,\xi_4) = \bar{H}(\xi_1+\pi, \xi_2+\frac{\pi}{2}, \xi_3+\frac{1}{2}, \xi_4) = \\ \frac{2{\xi_3}^2+2\xi_3+\frac{1}{2}}{\cos^2\xi_2} + \frac{{\xi_4}^2}{2} - 2\xi_3 - 1 + \frac{1}{2}\Omega^2\cos^2\xi_2\frac{1+\cos\xi_1}{2} \ . \end{array} \tag{45}$$

and the equilibrium point is now the origin $(\xi_1,\xi_2,\xi_3,\xi_4) = (0,0,0,0)$.

We compute now the first correction to the tangent spaces to the invariant manifolds. For, we use a very nice result about these manifolds, proved in [Schaft91] or see also [Kozlov83]. The result says that both the stable and unstable manifolds are Lagrange submanifolds (see [AbraMa78] for details on Lagrange submanifolds). Then, there exist two analytic scalar functions $V^{s,u} : D \subset \mathbf{R}^2 \to \mathbf{R}$, $(\xi_1, \xi_2) \mapsto V^{s,u}(\xi_1, \xi_2)$ defined on a neighborhood of the origin that satisfy the Hamilton-Jacobi equation:

$$H(\xi_1, \xi_2, \nabla V^{s,u}) = H(0) \tag{46}$$

and the graphs of the gradient of these functions are exactly the local stable and , respectively, unstable manifolds of (44), providing that these two manifolds can be parametrized by using the first two coordinates ξ_1 and ξ_2 (this is the disconjugacy condition of the Hamiltonian system with respect to the stable and unstable solutions of (46)). We shall use the equation (46) to find the first correction to the quadratic terms of $V^{s,u}$ (i.e. the third order terms).

Firstly we check the disconjugacy. For we recall $Df(Q_2)$ given in (35). We know from (36) that the spectrum of the linearized system is given by $Spec = \{\Omega, -\Omega, \sqrt{\Omega^2-1}, -\sqrt{\Omega^2-1}\}$. The corresponding eigenvectors are:

· the unstable space:

$$\lambda_1 = \Omega \quad , \quad v_1^T = (1, 0, \frac{\Omega}{4}, 0)$$

$$\lambda_2 = \sqrt{\Omega^2-1} \quad , \quad v_2^T = (0, 1, 0, \sqrt{\Omega^2-1}) \ ;$$

· the stable space:

$$\lambda_3 = -\Omega \quad , \quad v_3^T = (1, 0, -\frac{\Omega}{4}, 0)$$

$$\lambda_4 = -\sqrt{\Omega^2-1} \quad , \quad v_4^T = (0, 1, 0, -\sqrt{\Omega^2-1}) \ .$$

Now it is obvious that the projections of both E^u and E^s, the unstable and stable spaces spanned by $e_1^T = (1,0,0,0)$ and $e_2^T = (0,1,0,0)$, are of dimensions 2. Even more, from the geometric theory of the Algebraic Riccati Equations (see [Shay83] for details) we know that the quadratic terms in $V^{s,u}$ are given by:

$$X^s = \begin{bmatrix} -\frac{\Omega}{4} & 0 \\ 0 & -\sqrt{\Omega^2-1} \end{bmatrix} \quad , \quad X^u = \begin{bmatrix} \frac{\Omega}{4} & 0 \\ 0 & \sqrt{\Omega^2-1} \end{bmatrix} \tag{47}$$

Now, if we keep up to the third term in $V^{s,u}$ we obtain:

$$\begin{aligned} V^s_{\leq 3}(\xi_1, \xi_2) &= \tfrac{1}{2}\xi^T X^s \xi + third\_order\_terms = \\ &= -\tfrac{\Omega}{8}{\xi_1}^2 - \tfrac{\sqrt{\Omega^2-1}}{2}{\xi_2}^2 + b_1{\xi_1}^3 + b_2{\xi_1}^2\xi_2 + b_3\xi_1{\xi_2}^2 + b_4{\xi_2}^3 \end{aligned}$$

and analogously for $V^u_{\leq 3}(\xi_1, \xi_2)$. We have now to introduce in (46) and identify b_1, b_2, b_3, b_4 by expanding up to the third order. The expansion of H up to the third order has the form:

$$H_{\leq 3}(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\Omega^2 - 1}{2} - \frac{1}{8}\Omega^2 {\xi_1}^2 - \frac{\Omega^2 - 1}{2}{\xi_2}^2 + 2{\xi_3}^2 + \frac{1}{2}{\xi_4}^2 + 2{\xi_2}^2\xi_3 \ ,$$

and the solutions for $V^s_{\leq 3}$ and $V^u_{\leq 3}$ are:

$$V^u(\xi_1, \xi_2) = \frac{\Omega}{8}{\xi_1}^2 + \frac{1}{2}(\sqrt{\Omega^2 - 1}){\xi_2}^2 - \frac{1}{2}\frac{\Omega}{\Omega + 2\sqrt{\Omega^2 - 1}}\xi_1{\xi_2}^2 \ ,$$

$$V^s(\xi_1, \xi_2) = -\frac{\Omega}{8}{\xi_1}^2 - \frac{1}{2}(\sqrt{\Omega^2 - 1}){\xi_2}^2 - \frac{1}{2}\frac{\Omega}{\Omega + 2\sqrt{\Omega^2 - 1}}\xi_1{\xi_2}^2 \ .$$

Now, the invariant manifolds are given by:

$$(\xi_1, \xi_2) \longrightarrow (\xi_1, \xi_2, \frac{\partial V^{u,s}}{\partial \xi_1}, \frac{\partial V^{u,s}}{\partial \xi_2}) \ ,$$

which are approximated by the following 2-dimensional manifolds:

· the unstable manifold:

$$(\xi_1, \xi_2) \longrightarrow (\xi_1, \xi_2, \frac{\Omega}{4}\xi_1 - \frac{1}{2}\frac{\Omega}{\Omega + 2\sqrt{\Omega^2 - 1}}{\xi_2}^2, (\sqrt{\Omega^2 - 1})\xi_2 - \frac{\Omega}{\Omega + 2\sqrt{\Omega^2 - 1}}\xi_1\xi_2)$$

· the stable manifold:

$$(\xi_1, \xi_2) \longrightarrow (\xi_1, \xi_2, -\frac{\Omega}{4}\xi_1 - \frac{1}{2}\frac{\Omega}{\Omega + 2\sqrt{\Omega^2 - 1}}{\xi_2}^2, -(\sqrt{\Omega^2 - 1})\xi_2 - \frac{\Omega}{\Omega + 2\sqrt{\Omega^2 - 1}}\xi_1\xi_2)$$

These expressions hold only for $|\xi_1| + |\xi_2|$ small enough.

The tangent vectors to these approximating manifolds, computed on the homoclinic orbits (where $\xi_2 = 0$) are given by:

· for the unstable manifold:

$$\begin{gathered} X_1 = \frac{\partial}{\partial \xi_1} + \frac{\Omega}{4}\frac{\partial}{\partial \xi_3} \\ X_2 = \frac{\partial}{\partial \xi_2} + (\sqrt{\Omega^2 - 1} - \frac{\Omega}{\Omega + 2\sqrt{\Omega^2 - 1}}\xi_1)\frac{\partial}{\partial \xi_4} \ ; \end{gathered} \qquad (48)$$

· for the stable manifold:

$$\begin{gathered} X_3 = \frac{\partial}{\partial \xi_1} - \frac{\Omega}{4}\frac{\partial}{\partial \xi_3} \\ X_4 = \frac{\partial}{\partial \xi_2} - (\sqrt{\Omega^2 - 1} - \frac{\Omega}{\Omega + 2\sqrt{\Omega^2 - 1}}\xi_1)\frac{\partial}{\partial \xi_4} \ . \end{gathered} \qquad (49)$$

Now it is straightforward to see that X_1 and X_3 are tangent to the homoclinic orbits at the origin. Thus, what we have to do is to prove that X_2 is not transported along the homoclinic orbits into X_4.

It is known that a tangent vector is transported along a curve via the first variational system which is a linear time-varying system of the form:

$$z' = Df|_{\varphi(v)} z \tag{50}$$

For our system (31), the differential of the vector field along the homoclinic orbits has the form (recall $x_2 = \frac{\pi}{2}$ and $p_2 = 0$):

$$Df|_{(x_1(v),\frac{\pi}{2},p_1(v),0)} = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{4}\Omega^2 \cos x_1 & 0 & 0 & 0 \\ 0 & -4p_1^2 + \Omega^2(\frac{1-\cos x_1}{2}) & 0 & 0 \end{bmatrix}$$

. We see that, if $z^T = (z_1, z_2, z_3, z_4)$ then (50) decomposes into two 2-dimensional systems:

$$\begin{aligned} z_1' &= 4z_3 \\ z_3' &= -\tfrac{1}{4}\Omega^2 \cos x_1\, z_1 \end{aligned} \tag{51}$$

and:

$$\begin{aligned} z_2' &= z_4 \\ z_4' &= (-4p_1^2 + \Omega^2 \tfrac{1-\cos x_1}{2}) z_2 \ . \end{aligned} \tag{52}$$

The initial condition for the forward transport is given by X_2: $z_1 = 0$, $z_2 = 1$, $z_3 = 0$, $z_4 = \sqrt{\Omega^2 - 1} - \frac{\Omega}{\Omega + 2\sqrt{\Omega^2-1}}\xi_1$ and then only (52) is the system to be analyzed (actually (51) gives the transport of the tangent vector to the homoclinic orbits along the homoclinic orbits and the solution is obvious $z_1 = x_1'(v)$, $z_3 = p_1'(v)$). We rewrite (52) as a second-order differential equation:

$$z_2'' + (4p_1^2 - \Omega^2 \frac{1 - \cos x_1}{2}) z_2 = 0 \ .$$

We use the explicit form (41) of the homoclinic orbits (recall now $I_1 < I_3$ and then $x_1 = y_1^0$, $p_1 = \frac{1}{2} + \frac{1}{4}y_2^0$) and we get:

$$z_2'' + h(v) z_2 = 0 \qquad \text{where } h(v) = 1 \pm 2\Omega sech(\Omega v) + \Omega^2(1 - 2tanh^2(\Omega v)) \ . \tag{53}$$

Now we change the variable $v \to u = tanh(\frac{\Omega v}{2})$. Then (53) becomes:

$$\frac{\Omega^2}{4}(1-u^2)^2 \frac{d^2 z_2}{du^2} - \frac{\Omega^2}{2} u (1-u^2) \frac{dz_2}{du} + k(u) z_2 = 0 \tag{54}$$

with:

$$k(u) = 1 \pm 2\Omega \frac{1-u^2}{1+u^2} + \Omega^2 \frac{1-6u^2+u^4}{(1+u^2)^2} \tag{55}$$

and the interval of analysis is $(-1,1)$. The initial condition, which is given by the tangent vector to the unstable manifold, corresponds to $u \to -1$. Let's consider $u = -1+\varepsilon$ and try to evaluate z_4 up to order ε. Firstly, we have to find ξ_1. We have:

$$\xi_1 = x_1 - \pi = (\pm 2 arctan(sinh(\Omega v)) - \pi) mod\ 2\pi \ .$$

If we consider $tanh \frac{\Omega v}{2} = -1+\varepsilon$ and expand ξ_1 we get:

$$\xi_1 = \pm 2\varepsilon + \mathcal{O}(\varepsilon^2) \ .$$

Now, we need $\frac{dz_2}{du}$. We know that $\frac{dz_2}{dv} = z_4$, then:

$$\frac{dz_2}{du} = \frac{dv}{du}\frac{dz_2}{dv} = \frac{2}{\Omega}\frac{1}{1-u^2} z_4 \ ,$$

and using the initial condition for z_4, at $u = -1+\varepsilon$ we obtain:

$$\frac{\frac{dz_2}{du}}{z_2}\Big|_{u=-1+\varepsilon} = \frac{\sqrt{\Omega^2-1}}{\Omega}\frac{1}{\varepsilon} \mp \frac{2}{\Omega + 2\sqrt{\Omega^2-1}} + \mathcal{O}(\varepsilon) \ . \tag{56}$$

Now we analyze the asymptotic solution of (54) near to $u_0 = -1$. Firstly we see that both $u_0 = -1$ and $u_1 = 1$ are regular singular points (see [BenOrs78] for a general tratement of asymptotic approximations). We look for an asymptotic of the form $z_2 \sim (1+u)^\alpha$ (near u_0). Substituting into (54) and setting $u = -1$ we get for α the equation:

$$\alpha^2\Omega^2 = -k(-1) \ \Rightarrow \ \alpha = \pm\frac{1}{\Omega}\sqrt{-k(-1)} = \pm\frac{\sqrt{\Omega^2-1}}{\Omega}$$

The Frobenius solution of the equation has then the form: $z_2 = (1+u)^\alpha P(u)$ where $P(u)$ is a polynomial in u. Keeping only the first two terms from $P(u)$, we get:

$$z_2 \sim (1+u)^\alpha (C_0 + C_1 u) \tag{57}$$

We compute C_0 and C_1 by requiring the initial condition (56). We obtain:

$$\frac{\frac{dz_2}{du}}{z_2} = \frac{\alpha}{1+u} + \frac{C_1}{C_0 + C_1 u} \overset{u=-1+\varepsilon}{\sim} \frac{\alpha}{\varepsilon} + \frac{C_1}{C_0 - C_1} + \mathcal{O}(\varepsilon) \ . \tag{58}$$

By comparing (58) with (56) we get that $\alpha = \frac{\sqrt{\Omega^2-1}}{\Omega}$ and:

$$K = \frac{C_1}{C_0} = \mp \frac{2}{\Omega \mp 2 + 2\sqrt{\Omega^2 - 1}} \ . \tag{59}$$

Then, at $u = 0$ we get:

$$\frac{\frac{dz_2}{du}}{z_2}|_{u=0} = \alpha + K \ . \tag{60}$$

Similarly we can compute the transport of X_2 backward in time, from $u = 1-\varepsilon$ to $u = 0$. We get the following approximations:

$$\xi_1 = \mp 2\varepsilon + \mathcal{O}(\varepsilon^2)$$

$$\frac{\frac{dz_2}{du}}{z_2}|_{u=1-\varepsilon} = -\frac{\sqrt{\Omega^2-1}}{\Omega}\frac{1}{\varepsilon} \mp \frac{2}{\Omega + 2\sqrt{\Omega^2-1}} + \mathcal{O}(\varepsilon)$$

$$z_2 = (1-u)^{\beta}(D_0 + D_1) \ \ , \ \ \beta = \pm \frac{\sqrt{\Omega^2-1}}{\Omega}$$

and then:

$$\frac{\frac{dz_2}{du}}{z_2} = -\alpha - K \ , \tag{61}$$

with the same expressions for α and K as above.

Thus, the condition that at $u = 0$ to have three independent vectors is that (60) and (61) do not coincide, that is:

$$\alpha + K \neq -\alpha - K \tag{62}$$

If the above condition is fulfilled, then the tangent vector X_2 is everywhere independent of X_4 and this proves the Lemma.

For the lower branch, the condition (62) takes the form:

$$\frac{\sqrt{\Omega^2-1}}{\Omega} + \frac{2}{\Omega + 2 + 2\sqrt{\Omega^2-1}} \neq 0 \tag{63}$$

which is always true for $\Omega > 1$. For the upper branch the condition (62) becomes:

$$\frac{\sqrt{\Omega^2-1}}{\Omega} - \frac{2}{\Omega - 2 + 2\sqrt{\Omega^2-1}} \neq 0 \tag{64}$$

which has a root at $\Omega \simeq 1.70557$. To completely solve the problem for the upper branch, one must go to higher order approximations for $V^{s,u}$ in (46) and $z_2(u)$ in (57), but, for genericity, this result is enough.

7 Conclusions

In this paper we have obtained an analytic nonintegrability result for a Hamiltonian system modelling the rotation motion of a rigid body in a central gravitational field. To obtain the result, four assumptions were made.

The first assumption concerns the motion of the center of mass of the rigid body, which is supposed to be undisturbed by the rotation motion. This is reasonable if the ratio between the dimension of the rigid body and the distance to the center of attraction is much less than 1.

The second assumption is less critical but is made in order to avoid messy calculus. Under this assumption we neglect the higher-order terms in the interaction Hamiltonian.

The third assumption, namely the axialsymmetry of the rigid body, is made in order to progress in description. The condition $I_3/I_1 > 4/3$ is essential for the hyperbolicity of P_2 and for the transversal intersection of the stable/unstable manifolds of the perturbed system.

The fourth assumption, i.e. to consider a spinless top, is a technical one. Assuming a spinless top we are able to find analytic expressions for the homoclinic orbits and then to construct the horseshoes.

Under these assumptions we have proved that our problem gives rise to a time-varying Hamiltonian system on a two-dimensional sphere. The eccentricity of the orbit of the CM plays the rôle of a perturbation parameter. The unperturbed system (i.e. corresponding to a circular orbit of the CM) has a hyperbolic saddle point whose stable and unstable manifolds intersect along the homoclinic connections. The perturbation preserves the homoclinic connections, which become transversal homoclinic orbits. Then the stable and unstable manifolds intersect transversaly in T^*S^2. This is immediately connected with chaotic behaviour of the flow and, especially, with analytic nonintegrability of the system.

Acknowledgements

I wish to express my gratitude for Eng. Toma Manciu who told me about this problem four years ago and I also wish to thank Professor P. Holmes for his helpful comments and suggestions.

References

[AbraMa78] R. Abraham and J.E. Marsden, **Foundations of Mechanics**, 2nd edition , The Benjamin/Cummings Publishings Company, Reading, Ma, 1978

[Belets66] V.V. Beletskii, **Motion of an Artificial Satellite about its Center of Mass**, Israel Program for Scientific Translations, Jerusalem, 1966

[BenOrs78] C. Bender & S. Orszag, **Advanced Mathematics Methods for Scientists and Engineers** , McGraw-Hill Inc., 1978

[Bur87] A. Burov, *Non integrable planar oscillations of a satellite on an elliptic orbit*, Vestnik Moskovskogo Un., Ser. I, pp.71–73, (1987) (in Russian)

[CelFal92] A. Celletti & C. Falconi, *Construction of Invariant Tori for the Spin-Orbit Problem in the Mercury-Sun System*, Celestial Mechanics and Dynamical Astronomy 53, pp.113–127, 1992

[CidEl85] R. Cid & A. Elipe, *On the Motion of Three Rigid Bodies: Central Configurations*, Celestial Mechanics 37, pp.113–126, 1985

[Dubo58] G.N. Duboshin, *Differential Equations of the Translational-Rotational Motion of Mutual Attracting Bodies*, Astronomicheskii Zhurnal, vol.35, no.2, pp.265–276, 1958

[Erem83] E.N. Eremenko, *Stationary Motions in a Restricted Problem of Three Rigid Bodies*, Celestial Mechanics, vol.31, pp.339–362, 1983

[Gold80] H. Goldstein ,**Classical Mechanics**, 2nd edition, Addison-wesley, Reading, Ma, 1980

[GucHol93] J. Guckenheimer and P. Holmes ,**Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields**, Applied Mathematical Sciences, vol.42, 4th printing, Springer-Verlag, New York, 1993

[Gray92] G. Gray D. Kammer and I. Dobson , *Chaos in an attitude manoever of a damped stallite due to time-periodic perturbations*, in SpaceFlight Mechanics 1992, vol.79, part 1, Advances in the Astronautical Sciences, Proceedings of the AASIAIAA Spaceflight Mechanics Meeting held February 24-26, 1992, Colorado, pp.593–612

[HolMar83] P.J. Holmes and J.E. Marsden ,*Horseshoes and Arnold Diffusion for Hamiltonian Systems on Lie Groups*, Indiana University Mathematics Journal, vol.32, no.2, pp.273–309, (1983)

[Kelley67] Al. Kelley, *The Stable, Center-Stable, Center, Center-Unstable, Unstable Manifolds*, Journal of Differential Equations 3, pp. 546–570, (1967)

[Kozlov83] V.V. Kozlov, *Integrability and non-integrability in Hamiltonian mechanics*, Uspekhi Mat.Nauk 38:1, pp.3–67, (1983); English transl.: Russian Math. Surveys 38:1, pp.1–76, (1983)

[KraPar92] S.G. Krantz & H.R. Parks, **A Primer of Real Analytic Functions**, A Series of Advanced Textbooks in Mathematics, vol.4, Birkhäuser Verlag, 1992

[Manciu95] T. Manciu & R. Balan, *On the Angular Momentum Conservation Law in Newtonian Field*, Romanian Astronomical Journal, to appear

[Moser73] J. Moser,**Stable and Random Motions in Dynamical Systems**, Princeton University Press and University of Tokyo Press, Princeton 1973

[Palis69] J. Palis, *On Morse-Smale dynamical systems*, Topology, 8, pp.385–405, (1969)

[Poin899] H. Poincaré ,**Les Méthodes Nouvelles de la Méchanique Céleste**, tome 3, Gauthiers-Villars, Paris , 1899

[Schaft91] A.J.van der Schaft, *On a state space approach to nonlinear H_∞ control*, Systems & Control Letters 16, pp.1–8, (1991)

[Shay83] M.A. Shayman, *Geometry of the algebraic Riccati equation, Part I & II* , SIAM J. Control and Optimization, vol.21, no.3, pp.375–409, (1983)

[TeoGra92] P. Teofilatto & F. Graziani, *Chaotic motion of satellites*, in Dynamical Systems, Theory and Applications, S.I. Anderson, A.E. Anderson & U.Ottoson eds., World Scientific, pp.233–266, 1992

[Tiss889] F. Tisserand ,**Traité de mécanique céléste** , vol.1–4, Gauthier-Villars , Paris 1889-1894

[Xia92] Z. Xia ,*Melnikov Method and Transversal Homoclinic Points in the Restricted Three-Body Problem*, Journal of Differential Equations 96, pp.170–184, (1992)

[Xia93] Z. Xia ,*Arnold Diffusion in the Elliptic Restricted Three-Body Problem*, Journal of Dynamics and Differential Equations, vol.5, no.2, pp.213–240, (1993)

[WaMaKr92] L.-S. Wang, J.H. Maddocks & P.S. Krishnaprasad, *Steady Rigid-Body Motions in a Central Gravitational Field*, The Journal of the Astronautical Sciences, vol.40, no.4, pp.449–478, 1992

[Wigg88] S. Wiggins, **Global Bifurcations and Chaos: Analytical Methods**, Applied Sciences Series, vol.73, Springer-Verlag, New York, 1988

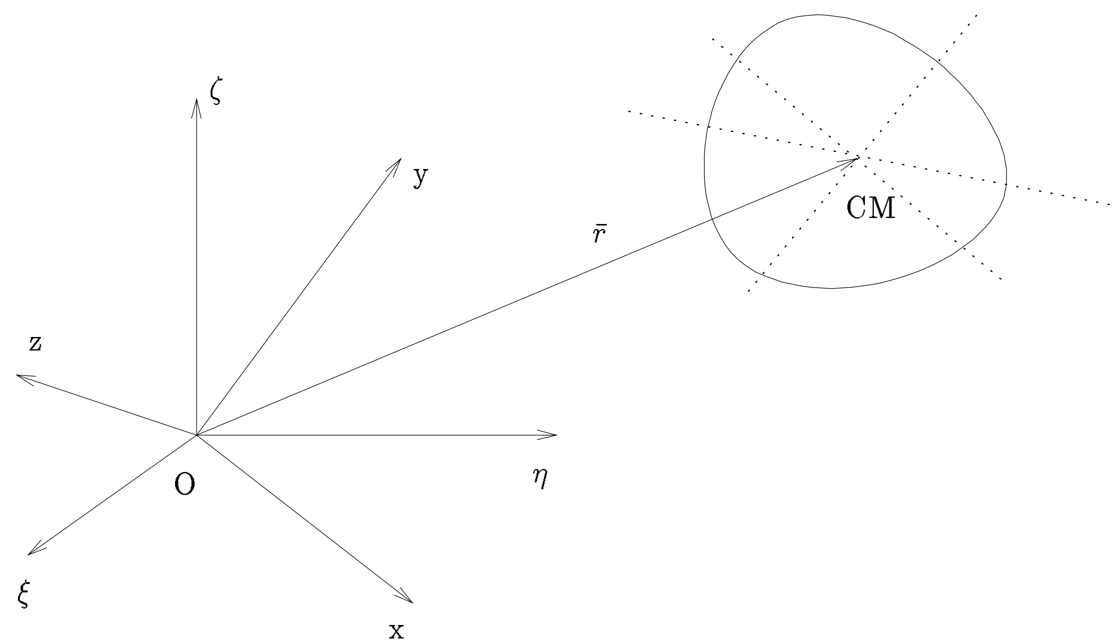

Figure 1: The fixed and rigid body frames

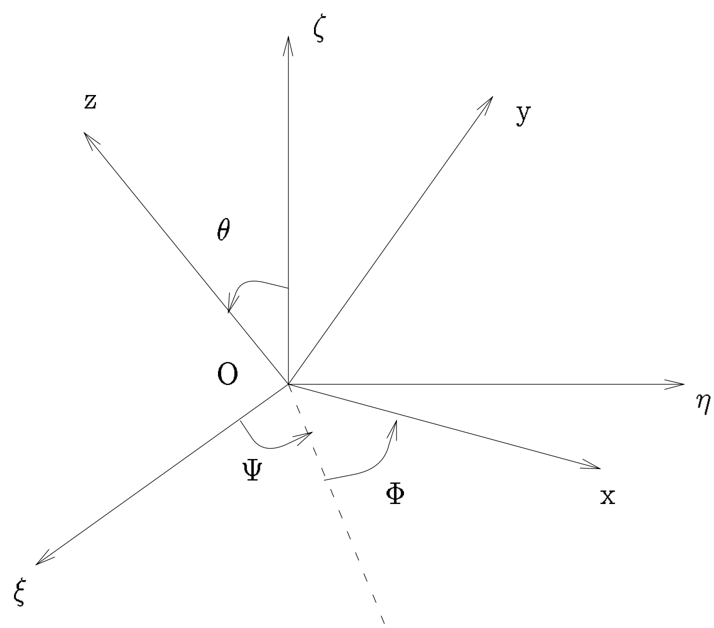

Figure 2: The Euler's angles