A Note about Integrability of Distributions with Singularities

R. BALAN
Polytechnic Institute of Bucharest
Department of Automatic Control and Computers
313, Splaiul Independentei, Bucharest , Romania

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Abstract

In this paper we discuss Stefan's and Sussmann's papers about integrability of singular distributions. We point out some gaps and we give a different version of their results.

1 Introduction

Let M be a $\mathcal{C}^\infty$ finite-dimensional paracompact manifold; let $\mathcal{F}(M)$ denote the ring of the $\mathcal{C}^\infty$ real-valued functions defined on M and let $V^\infty(M)$ be the $\mathcal{F}(M)$-module of $\mathcal{C}^\infty$ vector fields on M. We put $n = \dim M$.

We call *distribution* on M , the mapping $L : x \in M \longrightarrow L(x) \subset T_xM$ where $L(x)$ is a vector subspace of the tangent space to M at x. The *dimension*(or *rank*) of the distribution is $\dim L(x)$ (it is punctually defined).

Let S be a set of $\mathcal{C}^\infty$ everywhere defined vector fields. The distribution generated by the set S is $L(x) = span_{\mathbf{R}}\{\, v|_x \ , v \in S \,\} \ \forall x \in M$.

We call $\mathcal{C}^\infty$*-distribution on* M , a distribution L generated by a set S of $\mathcal{C}^\infty$ vector fields.

The distribution L is called *integrable at* $x_0 \in M$ if there exists a submanifold $N_{x_0} \stackrel{i}{\hookrightarrow} M$ (i being the canonical inclusion) passing through x_0, such that $T_xN_{x_0} = L(x)$, for all $x \in N_{x_0}$ (more precisely, we have: $i_{*,x}(T_xN_{x_0}) = L(x)$, $\forall\ x \in N_{x_0}$, where $i_{*,x}$ is the differential of i in x). N_{x_0} is called an *integral manifold* of the distribution. From the definition it follows directly that $\dim N_{x_0} = \dim L(x_0)$ and L is also integrable at every $q \in N_{x_0}$.

The distribution is called *locally integrable*, or to have *the integral manifold property*, if for each point in M there is an integral manifold of the distribution L (namely if it is integrable at every point of M).

Let us consider the distribution L and a point $x_0 \in M$. If there exists a neighborhood of x_0 where the distribution has constant dimension then the point x_0 is called an *ordinary point* (or a *regular point*), otherwise it is called a *singular point*. If the distribution has singular points then we say that it is a *distribution with singularities.*

In §2 we discuss Stefan's and Sussmann's papers pointing out some gaps and we state a correct version of their results. In §3 we construct a splitting of a distribution and we prove some results about punctual integrability. In §4 we give the proof of our main result.

Since our study is punctual, we point out that the integral manifolds are always regular embedding submanifolds.

2 Discussion about Stefan's and Sussmann's papers and statement of the main result

If S is a set of vector fields everywhere defined on M then we denote by $S^{\#}$ the $\mathcal{F}(M)$-module generated by S (i.e. the smallest $\mathcal{F}(M)$-module which includes S). We observe that the distribution generated by S is the same as the distribution generated by $S^{\#}$.

2.1 Discussion about Stefan's and Sussmann's papers

In this section we are going to show by a counterexample that the implication $e \Rightarrow d$ of Theorem 4.2 from Sussmann's paper ([Su73]) and Theorem 4 from Stefan's paper ([St80]) do not hold .

We refer now to Stefan's paper and we begin by recalling the definition of local subintegrability. For a set S of $\mathcal{C}^\infty$ vector fields we denote by L the distribution generated by S. For every vector field X of S, the map $t \to X^t(x)$ denotes the integral curve of X passing through x at $t = 0$ and $dX^t(x)$ denotes the differential at x of the local diffeomorphism $X^t : M \to M$. The set S is called *locally subintegrable at* $x_0 \in M$ if there exists a neighborhood Ω of x_0 in M and a subset S^b of S which generates the distribution L^b and satisfies the following conditions :

(LS.1) $L^b(x_0) = L(x_0)$ and S^b is integrable on Ω

(LS.2) For every vector field X in S there exists $\varepsilon > 0$ such that

$$dX^t(x_0).L^b(x_0) = L^b(X^t(x_0)) \text{ for } |t| < \varepsilon$$

We remark that the choice of the subset S^b may depend on the point x_0.

The gap we are refering to occurs in the following:

"**STATEMENT** (*Theorem4 from[St80]*) *A set S of $\mathcal{C}^\infty$ vector fields is integrable if and only if the set $S^\#$ is locally subintegrable on M.* "$\square$
This is in turn a

COUNTEREXAMPLE *Let $M = \mathbf{R}^2$ and let S be the set of all vector fields of the form:*

$$\frac{\partial}{\partial x} + \Phi(x,y)\frac{\partial}{\partial y}$$

where Φ is an arbitrary smooth (i.e. $\mathcal{C}^\infty$) function which satisfies two requirements:
1) $\Phi(0,0)=0$
2) $\frac{\partial \Phi}{\partial x} \equiv 0$ in some neighborhood of the origin depending on Φ. $\diamond$

The distribution L generated by S is defined as:

$$L(x) = \begin{cases} T_x\mathbf{R}^2 & , x \neq (0,0) \\ span_{\mathbf{R}}\{\frac{\partial}{\partial x}|_{(0,0)}\} & , x = (0,0) \end{cases}$$

and its dimension is given by :

$$dim L(x) = \begin{cases} 2 & , x \neq (0,0) \\ 1 & , x = (0,0) \end{cases}$$

It is clear now that L is not integrable at the origin. We will prove that $S^\#$ is locally subintegrable on $M = \mathbf{R}^2$.

Let $x_0 \in \mathbf{R}^2$, $x_0 \neq (0,0)$.Let Ω be a neighborhood of x_0 such that $O(0,0) \notin \overline{\Omega}$ ($\overline{\Omega}$ denotes the closure of Ω). Then there exists a function Φ so that $\Phi(q) \neq 0, \forall q \in \Omega$ and $\Phi(x,y)\frac{\partial}{\partial y} \in S^\#$. Let $\Psi : \mathbf{R}^2 \to \mathbf{R}$ be a smooth function such that $\Psi(q) \neq 0, \forall q \in \mathbf{R}^2$ *and* $\Psi(q) = \Phi(q)\ \forall q \in \Omega$ (Ψ can be found , possibly by reducing the neighborhood Ω and using the partition of unit) and let $Y_2 = \Psi^{-1}\Phi\frac{\partial}{\partial y} \in S^\#$. On Ω we have $Y_2|_\Omega = \frac{\partial}{\partial y}|_\Omega$. Let $Y_1 = \frac{\partial}{\partial x} \in S^\#$ and $S^b = \{Y_1, Y_2\}$. The condition $(LS.1)$ is fulfilled by this S^b. For every x in S there exists $\varepsilon > 0$ such that $X^t(x_0) \in \Omega$ with $|t| < \varepsilon$. Since X^t is a local diffeomorphism and $\dim L^b(x) = 2$, for every $x \in M$ the second condition $(LS.2)$ is also fulfilled.

Let us take now $S^b = \{\frac{\partial}{\partial x}\}$ and let denote by $Y = \frac{\partial}{\partial x}$. We are going to verify that S is locally subintegrable at $x_0 = (0,0)$. This S^b fulfills the condition $(LS.1)$. Let $X \in S^\#$. Then X is of the form $X = f_1\frac{\partial}{\partial x} + f_2\Phi\frac{\partial}{\partial y}$, where $f_1, f_2 : \mathbf{R}^2 \to \mathbf{R}$ are arbitrary smooth functions and Φ satisfies the two requirements. Remark that $\exists \mu > 0$ such that $\Phi(x,0) = 0$, for all $|x| < \mu$.
We find the integral curve of the vector field X passing through the origin. We have the system:

$$\begin{cases} \dot{x} = f_1(x,y) & , x(0) = 0 \\ \dot{y} = f_2(x,y)\Phi(x,y) & , y(0) = 0 \end{cases}$$

We obtain a solution x=x(t) at least continuous. We choose $\varepsilon > 0$ such that we have: $|x(t)| < \mu$, for all $|t| < \varepsilon$. Then $\Phi(x(t), 0) = 0$.
Since $y(t) = 0$, $|t| < \varepsilon$ is a particular solution of the second equation and using the theorem of existence and unicity of the Cauchy problem we obtain the system solution: $x = x(t)$, $y = 0$ for $|t| < \varepsilon$. The flow associated to the vector field X is defined by :

$$X^t(x, y) = (\phi(t, x, y), \psi(t, x, y))$$

for x, y small enough, in a neighborhood of the origin, and:

$$dX^t(x_0).Y(x_0) = \frac{\partial \phi}{\partial x}(t, 0, 0)\frac{\partial}{\partial x}|_p + \frac{\partial \psi}{\partial x}(t, 0, 0)\frac{\partial}{\partial y}|_p$$

where $p = X^t(x_0) = (x(t), 0)$. But $\frac{\partial \psi}{\partial t} = f_2 \Phi$ then:

$$\frac{\partial}{\partial t}(\frac{\partial \psi}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial \psi}{\partial t}) = \frac{\partial f_2}{\partial x}\Phi + f_2 \frac{\partial \Phi}{\partial x}$$

Let denote by $g(t) = \frac{\partial \psi}{\partial x}(t, 0, 0)$. Since $\psi(0, x, y) = y$ we have $g(0) = 0$. In the neighbourhood of the origin where $\frac{\partial \Phi}{\partial x}(p) = 0$ and $\Phi(p) = 0$ with p as above, we obtain: $\frac{dg}{dt}(t) = 0$, for $|t| < \varepsilon$. Then $g(t) = 0$ and :

$$dX^t(x_0).Y(x_0) = \frac{\partial \phi}{\partial x}(t, 0, 0)Y|_p \in L^b(X^t(x_0))$$

Since $\frac{\partial \phi}{\partial x}(t, 0, 0) \neq 0$ (the flow is a local diffeomorphism) we conclude that:

$$dX^t(x_0).L^b(x_0) = L^b(X^t(x_0)) \quad (i.e.\ (LS.2))$$

Then $S^{\#}$ is locally subintegrable on $M = \mathbf{R}^2$.

The gap (in [St80]) occurs in the proof of the *Lemma*(6.2). It affects *Theorem* 4, that uses this Lemma, and ,implicitly, the proof of *Theorem* 5, that uses *Theorem* 4.

The solution that we propose in subsection 2.2 is to reformulate the condition of the existence of ε (see the condition (LS.2)) in such a way that it becomes independent of every other conditions (that means there exists an $\varepsilon > 0$ "good" for all vector fields). This happens , for example, in the case when $S^{\#}$ is finitely generated, because we choose $\varepsilon = \min_i \varepsilon_{X_i}$, where $\{X_i\}_{i=\overline{1,p}}$ spans the module. From here we obtain *Theorem* 5 ([St80]).

Now we turn to Sussmann's paper. Even though the implication e⇒d is false, the other equivalences are true. We prove this directly on the Sussmann's proof (for this we suppose that the reader is familiar with the Sussmann's paper — [Su73]): We will prove that from (a) it results (d) (in Theorem 4.2).

The implication (a)$\Rightarrow$(e) is true (for example it is included in Theorem 2.1 of this paper) and from both (a) and (e) we will obtain (d). We have that $W^1(t), \ldots, W^k(t) \in \Delta(X_t(m))$ are independent. Since $X_t(m)$ belongs to the integral manifold of Δ passing through m it results $\dim \Delta(X_t(m)) = \dim \Delta(m)$ and so $W^1(t), \ldots, W^k(t)$ form a basis for $\Delta(X_t(m))$. Now the proof is complete.

2.2 Statement of the main result

Inspired by the previous discussion, we state now the main result of this paper.

THEOREM 2.1 *Let $\mathcal{L}$ be a $\mathcal{F}(M)$-module of $\mathcal{C}^\infty$ vector fields on M and let L denote the associated distribution. Let $x_0 \in M$ and $k = \dim L(x_0)$. Then L is integrable at x_0 if and only if there exist $\varepsilon > 0$, vector fields $a_1, \ldots, a_k \in \mathcal{L}$ and a neighborhood $\mathcal{U}$ of x_0 that satisfy the following conditions:*
1) At the point x_0 $a_1|_{x_0}, \ldots, a_k|_{x_0}$ span $L(x_0)$
2) For all smooth vector field $Z \in \mathcal{L}$, there exist smooth functions $\lambda_i^j : (-\mu_Z, \mu_Z) \to \mathbf{R}$ such that for all $t \in (-\mu_Z, \mu_Z)$ and $1 \le i \le k$ we have:

$$[Z, a_i]|_{\exp tZ.x_0} = \sum_{j=1}^{k} \lambda_i^j(t) a_j|_{\exp tZ.x_0} \tag{1}$$

where: $\mu_Z \stackrel{\text{def}}{=} \sup\{\nu | \nu \le \varepsilon$ and $\exp tZ.x_0 \in \mathcal{U}$ for all $|t| < \nu\}$ $\square$

3 Split of the distribution and some punctual results

Let $\mathcal{L}$ be a $\mathcal{F}(M)$-module of $\mathcal{C}^\infty$ vector fields and let L denote the associated distribution. Let $x_0 \in M$ be a fixed point. Let $k = \dim L(x_0) \le n = \dim M$. Then there exist k vector fields $a_1, \ldots, a_k \in \mathcal{L}$ such that $a_1|_{x_0}, \ldots, a_k|_{x_0}$ span $L(x_0)$ and in a chart around x_0 ,$(\mathcal{U}, \varphi)$, we have:

$$a_i = \frac{\partial}{\partial x^i} + \sum_{j=k+1}^{n} a_{ji} \frac{\partial}{\partial x^j} \tag{2}$$

From now on we will agree implicitly that $x \in \mathcal{U}$. We associate to (a_i) the family $\mathcal{F}_\varepsilon$ defined by:

$$\mathcal{F}_\varepsilon = \{a_\alpha \in V^\infty(M) \mid a_\alpha \stackrel{\text{def}}{=} \sum_{i=1}^{k} \alpha_i a_i \ , \ \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbf{R}^k, |\alpha| \stackrel{\text{def}}{=} \sum_{i=1}^{k} < \varepsilon\}$$

$\mathcal{F}_\varepsilon$ can be identified with a ball in a k-dimensional space. For $\varepsilon > 0$ small enough we know that $\exp : \mathcal{F}_\varepsilon \to M$ is a regular embedding. So $\exp \mathcal{F}_{\varepsilon.x_0} \subset M$

is a submanifold in M of dimension k ($\exp \mathcal{F}_{\varepsilon . x_0} \stackrel{\text{def}}{=} \{\exp a_\alpha . x_0 |\ a_\alpha \in \mathcal{F}_\varepsilon\}$ and $\exp a_\alpha . x_0$ denotes $x(1)$ where $x(t)$ is the solution of the differential system $\dot{x}(t) = a_\alpha|_{x(t)}$ with the initial condition $x(0) = x_0$).

LEMMA 3.1 *If L is integrable at x_0, then $\mathcal{N}_{\varepsilon . x_0} \stackrel{\text{def}}{=} \exp \mathcal{F}_{\varepsilon . x_0}$ is an integral manifold of L passing through x_0.* $\square$

Let

$$\mathcal{G} \stackrel{\text{def}}{=} \{b \in \mathcal{L} | b = \sum_{j=k+1}^{n} b^j(x) c_j(x), \textit{where } b^j(x) \in \mathcal{F}(M) \textit{ and } c_j(x)|_{\mathcal{U}} = \frac{\partial}{\partial x^j}\}$$

$$L_{(-1)} \stackrel{\text{def}}{=} \{a_\alpha \in V^\infty(M) \mid \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbf{R}^k\}$$

It is very easy to prove the following lemma:

LEMMA 3.2 *The distribution generated by $\mathcal{G} \oplus L_{(-1)}$ coincides locally with L. That means: $L(x) = \mathcal{G}|_x \oplus L_{(-1)}|_x$, for all $x \in \mathcal{U}$ ($\oplus$ denotes a direct sum).* $\square$

Clearly, $\dim L(x_0) = \dim L_{(-1)}|_{x_0}$ and: $\mathcal{G}|_{x_0} = \{0\}$.
We have obtained two algebraic structures which generate locally the distribution: $L_{(-1)}$, which is a k-dimensional $\mathbf{R}$-vector subspace, and $\mathcal{G}$, which is a $\mathcal{F}(M)$-module and we say that $(L_{(-1)}, \mathcal{G})$ is a *splitting* of the distribution generated by $\mathcal{L}$.

Directly from the definition and previous lemma it results:

LEMMA 3.3 *The distribution L is integrable at x_0 if and only if we have the relations:*
R_1. $L_{(-1)}|_x = T_x \mathcal{N}_{\varepsilon . x_0}$, for all $x \in \mathcal{N}_{\varepsilon . x_0}$
R_2. $\mathcal{G}|_{\mathcal{N}_{\varepsilon . x_0}} = 0$ (that means $\mathcal{G}|_x = 0$, for all $x \in \mathcal{N}_{\varepsilon . x_0}$). $\square$

Following the proof of Nagano's theorem we have the next lemma (see [Na66] for proof):

LEMMA 3.4 *The distribution L is integrable at x_0 if and only if:*
1) $[u, v]|_{\exp tv . x_0} = 0$, for all $u, v \in L_{(-1)}$ and $|t| < \varepsilon$, ε depending on v.
2) $\mathcal{G}|_{\mathcal{N}_{\varepsilon . x_0}} = 0$ $\square$

In order to prove the main result we need the following lemma:

LEMMA 3.5 *Let $a_1, \ldots, a_k \in V^\infty(\mathcal{U})$ ($\mathcal{U}$ being an open neighborhood of x_0) be smooth vector fields and let $Q = span_{\mathcal{F}(\mathcal{U})}\{a_1, \ldots, a_k\}$. Let $Z \in V^\infty(\mathcal{U})$ and $\{b_1, \ldots, b_k\} \subset Q$ such that $b_i = \sum_{j=1}^k f_{ij} a_j$ and $a_i = \sum_{j=1}^k g_{ij} b_j$ where $f_{ij}, g_{ij} : \mathcal{U} \to \mathbf{R}$ are smooth functions.*
If there exist $\mathcal{C}^\infty$ functions $\lambda_i^j : (-\varepsilon, \varepsilon) \to \mathbf{R}$, $i, j = \overline{1, k}$ such that:

$$[Z, a_i]|_{\exp tZ . x_0} = \sum_{j=1}^{k} \lambda_i^j(t) a_j|_{\exp tZ . x_0}$$

then there exist $\mathcal{C}^\infty$ functions $\mu_i^j : (-\varepsilon, \varepsilon) \to \mathbf{R}$, $i, j = \overline{1,k}$ such that:

$$[Z, b_i]|_{\exp tZ.x_0} = \sum_{j=1}^{k} \mu_i^j(t) b_j|_{\exp tZ.x_0}$$

Proof

We obtain:

$$[Z, b_i]|_{\exp tZ.x_0} = \sum_{l=1}^{k}[\sum_{j=1}^{k} g_{jl} Z(f_{ij}) + \sum_{j,s=1}^{k} g_{sl}\lambda_j^s f_{ij}] b_l|_{\exp tZ.x_0} = \sum_{l=1}^{k} \mu_i^l b_l|_{\exp tZ.x_0}$$

Q.E.D. $\square$

4 Proof of the main result

Lemma 3.5 shows that the condition (2) from Theorem 2.1 is invariant under a change of the basis. Then we choose for $\{a_i\}$ the vector fields which form the basis of $L_{(-1)}$ obtained by a splitting of L. Moreover, let ε be as in the definition of $\mathcal{N}_{\varepsilon.x_0}$.

Suppose L integrable. We choose $\mathcal{U}$ as in §3.

1) It is checked by the construction of vector fields $\{a_i\}$

2) Let $Z \in \mathcal{L}$. Then $Z = \sum_{j=1}^{k} f_j a_j + b$ where $b \in \mathcal{G}$ and $f_j \in \mathcal{F}(M)$. We obtain:

$$[Z, a_i] = \sum_{j=1}^{k} f_j [a_j, a_i] + [b, a_i] - \sum_{j=1}^{k} a_i(f_j) a_j$$

Since L is integrable at x_0 and $tZ|_x \in L(x), \forall x \in \mathcal{N}_{\varepsilon.x_0}$ we have $x_t = \exp tZ.x_0 \in \mathcal{N}_{\varepsilon.x_0}$ and we obtain: $[Z, a_i]|_{x_t} = \sum_{j=1}^{k} \lambda_i^j(t) a_j|_{x_t}$ for all $|t| < \mu_Z$.

To prove the converse, we apply Lemma 3.4

a) We show that for all $a_1, a_2 \in L_{(-1)}$, $[a_1, a_2]|_{\exp ta_1.x_0} = 0$, with $|t| < \varepsilon = \mu_{a_1}$.

We write the given relation for $Z = a_1$ and $a_i = a_2$.

On one hand we have: $[a_1, a_2] = \sum_{j=k+1}^{n} \pi_j \frac{\partial}{\partial x^j}$

on the other hand: $\sum_{j=1}^{k} \lambda_2^j(t) a_j = \lambda_2^1(t)\frac{\partial}{\partial x^1} + \cdots + \lambda_2^k(t)\frac{\partial}{\partial x^k} + \sum_{s=k+1}^{n} \Theta_2^s(t)\frac{\partial}{\partial x^s}$

From $[Z, a_i]|_{\exp tZ.x_0} = \sum_{j=1}^{k} \lambda_i^j(t) a_j|_{\exp tZ.x_0}$ we obtain: $[a_1, a_2]|_{\exp tZ.x_0} = 0$, $|t| < \varepsilon$.

b) We show that $\mathcal{G}|_{\mathcal{N}_{\varepsilon.x_0}} = 0$. Let $X \in \mathcal{G}$. We put $Z_i = X + a_i$ and write: $[Z_i, a_i] = [X, a_i]$. Then, as above, we obtain: $[X, a_i]|_{\exp tZ_i.x_0} = 0$, $|t| < \mu_{Z_i}$.

Obviously: $[X, X]|_{\exp tZ_i.x_0} = 0$. Then:

$$[X, (X + a_i)]|_{\exp t(X+a_i).x_0} = 0 \text{ or } [Z_i, X]|_{\exp tZ_i.x_0} = 0, \ |t| < \mu_{Z_i}$$

We can apply a formula from 3.2 ([St80]) and we obtain:

$$\frac{d}{dt} X(x_t) = DZ_i \circ X|_{x_t}$$

(where $x_t = \exp tZ_i.x_0$ and DZ_i is the jacobian matrix of Z_i) with the initial condition: $X(u(0)) = X(x_0) = 0$ (recall that $X \in \mathcal{G}$). Using the theorem of existence and unicity of the solution of the Cauchy problem, we obtain: $X|_{\exp tZ_i.x_0} = 0$. But then $Z_i|_{\exp tZ_i.x_0} = (a_i + X)|_{\exp tZ_i.x_0} = a_i|_{\exp tZ.x_0}$. So: $\exp tZ_i.x_0 = \exp ta_i.x_0$. That means: $X|_{\exp ta_i.x_0} = 0$, and $|t| < \mu_{Z_i} = \mu_{X+a_i} = \mu_{a_i} = \varepsilon$. So: $\mathcal{G}|_{\mathcal{N}_{\varepsilon.x_0}} = 0$

Q.E.D. □

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