An Extension of Barbashin-Krasovski-LaSalle Theorem

RADU BALAN

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Abstract

In this paper we are given an extension of Barbashin-Krasovski-LaSalle Theorem) to the case when the kernel of the time derivative of the Liapunov candidate includes some trajectories. Our goal is to improve the sufficiency conditions for the asymptotic stability of the equilibrium. The starting point is the paper [ByMa94] where, especially, the case of zero-state observable dynamics is considered. We try to extend this result to the case of zero-state detectability (being motivated by the linear situation: Given (C, A) a detectable pair, if there exists a positive semidefinite matrix $P \geq 0$ such that: $A^T P + PA + C^T C = 0$ then A is Hurwitz -i.e. it has eigenvalues with negative real part).

In the first section we present the proof of the LaSalle's Invariance Principle and Barbashin-Krasovski Theorem using Barbălat's Lemma.

In the second section we present a geometric technique called the observability decomposition for nonlinear systems and we present the observability and zero-state detectability properties for nonlinear dynamics considered to be in the following form:

$$\begin{cases} \dot{x} &= f(x) \\ y &= h(x) \end{cases}$$

In the last section we suppose f to have only one equilibrium point, namely $\bar{x} = 0$ and under the assumption of detectability or certain equivalent condition we obtain the asymptotic stability of this equilibrium.

1 Barbashin-Krasovski-LaSalle's Theorems

1.1 Introduction and Statement of BKLS Theorems

We consider the following dynamical system:

$$\dot{x} = f(x) \quad , \; x \in U \subset \mathbf{R}^n \tag{1}$$

where $f \in C^1(D)$ is a vector field of class C^1 on U and U a domain in $\mathbf{R}^n$.

Definition Consider $x_0 \in U$ and $T = \{x(t) | 0 \leq t < t_{x_0}\}$ the positive trajectory initialized at x_0 (where $t_{x_0} \leq +\infty$ is the positive escaping time). We call $X_\omega \in U$ an *ω-limit point for T* if $t_{x_0} = +\infty$ and there is a sequence of positive real numbers $(t_k)_{k \in \mathbf{N}}$ such that:

$$\text{i)} \lim_{k \to \infty} t_k = \infty \qquad \text{and} \qquad \text{ii)} \lim_{k \to \infty} x(t_k) = x_\omega$$

We denote by $\Omega(x_0)$ or $\Omega(T)$ the set of ω-limit points of the trajectory T. From now on we consider those systems for which $t_x = +\infty$ for any $x \in D$. We shall denote by $x(t, x_0)$ the flow generated by the system (1) (i.e. the integral curve at the moment t when at $t_0 = 0$ the systems was in x_0). Thus $T = x(\mathbf{R}^+, x_0)$.

It is well-known (result due to Poincaré) that the set of ω-limit points is a closed and positive invariant set (see [HirSma74]). (By *positive invariant set* we mean a set S such that for any $x_0 \in S$, the positive trajectory starting from x_0 remains in S: $T \subset S$).

There is also another result for ω-limit points :

THEOREM 1 (Birkoff's Limit Sets Theorem - see [Birk12]) *Any bounded trajectory approaches its ω-limit set:*

$$\lim_{t \to \infty} d(x(t), \Omega(x_0)) = 0$$

where $d(x, M) = min_{y \in M} \parallel x - y \parallel$, is the distance from x to the closed set M.

Now we are able to state the LaSalle's Invariance Principle (see [LaSa60]):

THEOREM 2 (LaSalle's Invariance Principle) *Consider the dynamical system (1) with $U = \mathbf{R}^n$ and a function $V : D \to \mathbf{R}$ of class C^1 (D a domain included in U) with the following properties:*

i) It is bounded below (i.e. $\exists M \in \mathbf{R}$ such that $V(x) \geq M$, $\forall x \in D$)

ii) $\frac{dV}{dt} = L_f V \leq 0$, $\forall x \in D$

Consider $x_0 \in D$ and $T = \{x(t) | t \in \mathbf{R}^+, x(0) = x_0\}$ the positive trajectory starting from x_0 and completely included in D. Then one of the following holds:

1) The trajectory is unbounded;

2) The trajectory is bounded and its ω-limit set is included in $N = \{x \in D | \frac{dV}{dt}(x) = 0\}$ ($\Omega(x_0) \subset N$) or, equivalent, the distance from the trajectory to the maximal positive invariant set included in N tends to zero. □

Remark There is another variant of this theorem in which it is required D to be a positive invariant set. This variant ends to be a simple corollary of the above statement.

In the case of studying only the asymptotic behavior of an equilibrium, the previous theorem turns into the Barbashin-Krasovski statement:

THEOREM 3 (Barbashin-Krasovski Theorem) *Consider the dynamical system (1) with $\bar{x} = 0 \in U$ an equilibrium point (i.e. $f(0) = 0$). If there exists a function $V : D \to \mathbf{R}$ of class C^1 (D a neighborhood of the origin) such that:*

i) $V(x) > 0$, $x \neq 0$, $V(0) = 0$;

ii) $W = \frac{dV}{dt} = L_f V \geq 0$, $\forall x \in D$;

iii) In $N = \{x \in D | W(x) = 0\}$ there is not included any positive trajectory othere than the trivial solution $x(t) \equiv 0$;

then the equilibrium point $\bar{x} = 0$ is an asymptotical stable equilibrium. □

In the linear case (i.e. $f(x) = Ax$) the above statement turns into the following version:

THEOREM 4 *If there exists a solution $P \geq 0$ of the following Liapunov equation:*

$$A^T P + PA + C^T C = 0$$

with (C, A) an observable pair, then A is a hurwitz matrix (i.e. has eigenvalues with negative real part). □

In the next subsection we shall prove the Barbălat's Lemma which is a key tool in the proof of LaSalle's Invariance Principle that is presented in the third section. In the last section we shall prove Barbashin-Krasovski Theorem and the philosophy of the asymptotic stability in terms of this approach.

1.2 The Barbălat's Lemma

LEMMA 5 (Barbălat's Lemma) *Consider a function $f : [0, \infty) \to \mathbf{R}$ with the following properties:*

i) It is an uniform continuous function (i.e. $\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0\ \forall x, y \in [0, \infty)$, $|x - y| < \delta_\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$)

ii) There exists and it is finite:

$$\lim_{t\to\infty} \int_0^\infty f(\tau) d\tau$$

Then $\lim_{t\to\infty} f(t) = 0$.

Proof: *We shall prove that $\lim_{t\to\infty} f(t) = 0$ by contradiction. Suppose 'ex absurdo' that $f(t)$ does not converge to zero as t tends to infinity. This means that there exist $M > 0$ and a sequence $(t_k)_{k\geq 0}$, $t_k \to \infty$ such that $|f(t_k)| \geq M$. We set $\varepsilon = M/2$ and choose $\delta > 0$ such that for any $\tau \in [o, \delta]$ and $t \geq 0$ we have $|f(t) - f(t + \tau)| < M/2$. Then $|f(t_k) - t(t_k + \tau)| < M/2$ that implies $|f(t_k + \tau)| > M/2$, for any $k \geq 0$. We obtain:*

$$|\int_{t_k}^{t_k+\delta} f(\tau) d\tau| = \int_{t_k}^{t_k+\delta} |f(\tau)| d\tau > \frac{1}{2} M\delta \tag{2}$$

We have used that f is continuous and then it keeps constant sign on $[t_k, t_k + \delta]$.

We build up a sequence $(s_n)_{n\geq 1}$ in the following manner:

$$s_{2k-1} = t_k \qquad s_{2k} = t_k + \delta \quad , \ n \geq 1$$

Then $\lim_{n\to\infty} s_n = \infty$. *The inequality (2) shows that the sequence:*

$$I_n = \int_0^{s_n} f(\tau)d\tau \qquad , \ n \geq 1$$

is not a Cauchy chain and then does not converge. This is a contradiction with the hypothesis. The contradiction comes from our assumption that $f(t)$ *does not converge to zero as* t *tends to infinity. Now the proof is complete.* □

1.3 The Proof of LaSalle's Invariance Principle

1) If the trajectory is unbounded we have nothing to prove.

2) If the trajectory is bounded, let B denote a compact set such that $T \subset B$. We have the following relations:

$$\frac{dV}{dt} = W(x(t))$$

$$V(x(t)) - V(x_0) = \int_0^t W(x(\tau))d\tau \tag{3}$$

Now, because $\dot{x} = f(x)$ and $x(t)$ is bounded we obtain that $x(t)$ is uniform Lipschitz and also uniform continuous function on $[0,\infty)$. On the other hand, since W is continuous and B compact, then $W|_B$ is uniform continuous. Then $W(x(.)) : [0,\infty) \to \mathbf{R}$ is uniform continuous. Since $W(x(t)) \leq 0$ then $\int_0^t W(x(\tau))d\tau$ is monotone decreasing function. Using (3), boundedness of the trajectory and the lower boundedness of V we obtain that there exists:

$$\lim_{t\to\infty} \int_0^t W(x(\tau))d\tau$$

Now, using Barbălat's Lemma we get:

$$\lim_{t\to\infty} W(x(t)) = 0 \tag{4}$$

Let x_ω be an ω-point for our trajectory. Then there exists $t_k \to \infty$ such that $\lim_{k\to\infty} x(t_k) = x_\omega$. From (4) we get:

$$\lim_{k\to\infty} W(x_k) = 0$$

and because of continuity of W: $W(x_\omega) = 0$. This proves that the ω-point set is included in the kernel of W: $\Omega(x_0) \subset N$. □

1.4 Consequences

Firstly we recall the Liapunov stability theorem:

THEOREM 6 (Liapunov Theorem) *Consider the system (1) and* $\bar{x} = 0 \in U$ *an equilibrium point. If there exists a function* $V : U \to \mathbf{R}$ *such that:*

i) $V(x) > 0$, $x \neq 0$, $V(0) = 0$

ii) $\frac{dV}{dt} \leq 0$, $\forall x \in U$

then the equilibrium point $\bar{x} = 0$ *is stable.* □

whose proof is given, for instance, in [Khal93].

Now the Barbashin-Krasovski Theorem (Theorem 3) appears as a corollary of the previous results:

1) The stability of the equilibrium is implied by the Liapunov theorem stated above;

2) The attractivity of the equilibrium is obtained from the LaSalle's Invariance Principle because $N = \{0\}$;

and the proof of Theorem 3 is complete.

This construction suggests that in order to prove the asymptotic stability of an equilibrium we need to follow two steps with two different methods:

1) First we need to prove the stability (in Liapunov sense) of the equilibrium. This is implied by a Liapunov argument (i.e. positive Liapunov function) and thus we get some local boundedness of the trajectories.

2) In the second step we prove the attractivity of the equilibrium by means of Barbălat's Lemma. In fact, the LaSalle's Invariance Principle says that a special set (which is the maximal positive invariant set included in the kernel of the time derivative of the scalar function) has got a property of attractivity. But this set depends upon the function V (the scalar function). By changing the function we could obtain another set. Then the ω-limit set of our system is exactly the intersection of the whole these invariant sets and ... it is not very easy to be computed. This is the reason for which we try to throw away some subsets from a given invariant set. Our approach use a geometric technique and we need some regularity conditions for the invariant set. In the next section we study these conditions for a nonlinear system.

2 The Observability Decomposition for Nonlinear Systems

2.1 Integrability of Nonsingular Distributions

We consider a domain $D \subset \mathbf{R}^n$ and a set of vector fields $\{v_1, v_2, \ldots, v_r\}$ on D of class C^1 which are linear independent over $\mathbf{R}$ in every point $x \in D$.

We denote by $\{\omega_{r+1}, \ldots, \omega_n\}$ a set of $n - r$ 1-form linear independent which are ortoghonal to $\{v_1, \ldots, v_r\}$ (i.e. $\omega_i(v_j) \equiv 0$).

Definitions We say that the distribution spanned by $v_1, \ldots, v_r$ is *integrable* if through every point $p \in D$ passes a submanifold $N \hookrightarrow D$ of dimension r ($p \in \mathbf{N}$) such that for every point $x \in N$ the tangent space at x to N is given by $span_{\mathbf{R}}\{v_1|_x, \ldots, v_r|_x\}$.

Equivalent, the codistribution spanned by $\omega_{r+1}, \ldots, \omega_n$ is integrable if there are $n - r$ C^1 real-valued functions $h_{r+1}, \ldots, h_n : R \to \mathbf{R}$ such that $dh_{r+1}, \ldots, dh_n$ is a basis in the free module over $\mathcal{F}(D)$ (the ring of real-valued functions over D) spanned by $\omega_{r+1}, \ldots, \omega_n$.

We say that the distribution spanned by $v_1, \ldots, v_r$ is *involutive* if for any $1 \leq i, j \leq r$ the Lie brackets $[v_i, v_j]$ is a linear combination of $v_1, \ldots, v_r$.

We say a distribution is *nonsingular* if the vector fields that span the distribution are linear independent (as in our case).

A very known result about integrability of nonsingular distributions (or codistributions) is given by the Frobenius theorem which is stated below:

THEOREM 7 (Frobenius Theorem) *A nonsingular distribution is integrable if and only if it is involutive.* □

For the proof we refer the reader to [Isid89].

2.2 Observability and Detectability of Nonlinear Systems

Let us consider the following nonlinear system without input:

$$(S)\begin{cases} \dot{x} &= f(x) \\ y &= h(x) \end{cases} \quad , \quad x \in D \subset \mathbf{R}^n \tag{5}$$

where f is a vector field and h a vector-valued function both of class C^s with s large enough ($f : D \to TD$, $h : D \to \mathbf{R}^p$). We suppose $0 \in D$ and $f(0) = 0$, $h(0) = 0$. Now we can define the observability in terms of the above system:

Definition We say that the pair (h, f) is *observable* if from $y(t) \equiv 0$, $\forall t \geq 0$ we obtain that $x(t) \equiv 0$, $\forall t \geq 0$ (this means that the unobserved trajectory is only the trivial solution $x(t) \equiv 0$).

Let us define $K = \{x \in D | h(x) = 0\}$ and denote by N the maximal f-invariant set included in K. Then, to say that pair (h, f) is observable is equivalent to say that $N = \{0\}$. In the case when N contains more elements than 0, it is interesting to consider the restriction of the system (5) on this set. We set $\tilde{f} = f|_N$ and the system (5) becomes:

$$(S')\begin{cases} \dot{x} &= \tilde{f}(x) \\ y &= 0 \end{cases} \quad , \; x \in N \tag{6}$$

We should rather regard this system as a dynamics given by the flow generated by f on D and then restricted to N; we point out that N may not be a manifold. Regarding the system (6) we define:

Definition The pair (h, f) is said to be *zero-state detectable* if the system S' has $\bar{x} = 0$ an asymptotical equilibrium point.

In the case of zero-state detectability we obtain that from:

$$y(t) \equiv 0 \;,\; \forall t \geq 0 \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = 0$$

i.e. every unobserved trajectory goes to zero as t tends at infinity.

We note that the zero-state detectability requires more than the attractivity condition stated above (it requires also the stability for the restricted dynamics).

A criterion of sufficiency for zero-state detectability is given below:

THEOREM 8 *If there exists p vector fields on D: $k_1, \ldots, k_p$ such that the vector field $f + h_1k_1 + \cdots + h_pk_p$ has $\bar{x} = 0$ an asymptotical stable equilibrium, then the pair (h, f) is* zero-state detectable.

Proof *For the system:*

$$\dot{z} = f(z) + h_1(z)k_1(z) + \ldots + h_p(z)k_p(z) \quad , \quad z \in D$$

the set N is also $f + hk$-invariant and the restricted dynamics on N is exactly that given by (6). □

2.3 The Observability Decomposition of Nonlinear Systems

The set N introduces in the previous subsection has a very nice property: it is invariant and the restriction of the system to this set produces a null output. To use this property from a geometric point of view we need N to be a submanifold. We shall characterize N in terms of an involutive codistribution and then the condition of integrability reduces to the condition of regularity of this object.

We start by building up this object. From the conditions involved by N we get:

$$\begin{array}{l} h(x) \equiv 0 \\ \frac{dh}{dt} = L_f h(x) \equiv 0 \\ \frac{d^2h}{dt^2} = L_f^2 h(x) \equiv 0 \quad , \quad \forall x \in N \\ \cdots \\ \frac{d^qh}{dt^q} = L_f^q h(x) \equiv 0 \\ \cdots \end{array} \tag{7}$$

Then N is given by the intersection computed as follows:

$$N = \{x \in D | h(x) = L_f h(x) = L_f^2 h(x) = \cdots = L_f^q h(x) = \cdots = 0\}$$

Let us consider the codistribution:

$$\Omega = span_{\mathcal{F}(D)}\{dh, dL_f h, dL_f^2 h, \ldots, dL_f^q h, \ldots\} \tag{8}$$

Since D has dimension n, if Ω is a regular codistribution (i.e. has constant rank) then there exists q functions with $q \leq n$: φ_0, $\varphi_1, \cdots, \varphi_{q-1}$ such that:

$$dL_f^q h = \varphi_0 dh + \varphi_1 dL_f h + \cdots + \varphi_{q-1} dL_f^{q-1} h$$

Then the codistribution Ω is spanned only by:

$$\Omega = \{dh_i, dL_f h_i, \ldots, dL_f^{q-1} h_i | 1 \leq i \leq p\}$$

From these $q.p$ 1-forms we pick up only linear independent ones and, with them, we construct a basis whose entries are labeled as:

$$\{dg_{s+1}, dg_{s+2}, \ldots, dg_n\} \overset{b}{\subset} \Omega \tag{9}$$

with $g_i : D \rightarrow \mathbf{R}$, $s+1 \leq i \leq n$ and $s = n - rank\,\Omega$. From the condition of constant rank we obtain that:

$$N = \{x \in D | g_{s+1}(x) = g_{s+2}(x) = \cdots = g_n(x) = 0\} \tag{10}$$

Now we choose another s functions from $\mathcal{F}(D)$ $(g_1, \ldots, g_s)$ which are functional independent together with g_i's and we make the coordinate transformation:

$$z_k = g_k(x) \quad , \quad 1 \leq k \leq n$$

In this new coordinate frame, the manifold N is given by:

$$N = \{z \in g(D) | z_{s+1} = z_{s+2} = \cdots = z_n = 0\}$$

and, because of its f-invariance the system (5) becomes:

$$\left\{ \begin{array}{rcl} \dot{z}^1 & = & \tilde{f}_1(z^1, z^2) \\ \dot{z}^2 & = & \tilde{f}_2(z^2) \\ y & = & \tilde{h}(z^2) \end{array} \right. \tag{11}$$

where $z^1 = (z_1, z_2, \ldots, z_s)$, $z^2 = (z_{s+1}, \ldots, z_n)$ is a partition of coordinates in this new frame. In fact we have proved the following theorem called the observability decomposition of nonlinear systems theorem:

THEOREM 9 (The Observability Decomposition Theorem) *Suppose for system (5) the codistributions Ω has constant rank on D. Then there exists a coordinate system $(z_1, \ldots, z_n)$ in which the nonlinear system takes the form (11).* $\Box$

3 The Extension of Barbashin-Krasovski-LaSalle Theorem

3.1 Statement of the Main Results

Let us consider the following nonlinear system:

$$\dot{x} = f(x)$$

$x \in D \subset \mathbf{R}^n$, D a positive invariant bounded domain and $0 \in D$ an equilibrium: $f(0) = 0$. We suppose f to be of a class large enough.

THEOREM 10 *Let $V : D \to \mathbf{R}$ be a positive semidefinite function of class $\mathcal{C}^{p+1}$, $V \geq 0$, $V(0) = 0$ and consider the following sequence of functions:*

$$\begin{array}{rcl} I_1 & \stackrel{\text{def}}{=} & \dot{V} = \nabla V \cdot f \\ I_2 & \stackrel{\text{def}}{=} & \dot{I}_1 = \nabla I_1 \cdot f \\ & & \cdots \\ I_p & \stackrel{\text{def}}{=} & \dot{I}_{p-1} = \nabla I_{p-1} \cdot f \\ I_{p+1} & \stackrel{\text{def}}{=} & \dot{I}_p = \nabla I_p \cdot f \end{array} \tag{12}$$

If the following conditions are fulfilled:

1. $I_1 \leq 0$ and the dynamics restricted to the maximal invariant set S included in $K = \{ x \in D \mid \dot{I}_1(x) = 0 \}$ is asymptotically stable.

2. $I_1, \ldots, I_p$ are functional independent on D (i.e. $span_{\mathbf{R}}\{ dI_1, \ldots, dI_p \}$ keeps constant rank on D)

3. I_{p+1} is functional dependent of class $\mathcal{C}^1$ on $I_1, \ldots, I_p$ (or, equivalent, there exist p continuous functions $\phi_1, \ldots, \phi_p$ such that $dI_{p+1} = \phi_1 dI_1 + \cdots + \phi_p dI_p$)

then the equilibrium point $x_0 = 0$ is asymptotically stable. $\Box$

In terms of systems theory the above result could be restated as it follows. Let us consider a nonlinear dynamical system of the form:

$$(S) \left\{ \begin{array}{rcll} \dot{x} & = & f(x) & , \ f(0) = 0 \\ y & = & h(x) & , \ h(0) = 0 \end{array} \right. \tag{13}$$

where f and h are of class large enough and $x \in D$, a positive invariant domain.

We consider the following geometric condition:

G. There exists $p \in \mathbf{N}$, $p \leq n$ such that:

1. $\{ dh, dL_f h, \ldots, dL_f^{p-1} h \}$ are linear independent on D.

2. There exist p continuous functions $\phi_1, \ldots, \phi_p$ such that:

$$dL_f^p h = \phi_1 dh + \phi_2 dL_f h + \cdots + \phi_p dL_f^{p-1} h \quad \text{on } D$$

Now we are able to state the "systemic" version of Theorem 10:

THEOREM 11 *Consider the nonlinear dynamical system* (13). *If:*

1. The pair (h, f) *is zero-state detectable*

2. There exists a $\mathcal{C}^{p+1}$ *positive semidefinite function* V*, solution on a neighborhood of* x_0 *of the equation:*

$$\nabla V \cdot f + \| h \|^q = 0$$

for some q $(V \geq 0,\ V(0) = 0)$ *such that* $V^{-1}([0, \delta])$ *is bounded for some* $\delta > 0$

3. The geometric condition **G** *is fulfilled*

then $x_0 = 0$ *is an asymptotic stable equilibrium.*

3.2 Proof of Theorems

We shall prove Theorem 10 using Theorem 11 and then we shall give the complete proof of Theorem 11. But first we need a lemma which gives a partial answer to our question:

LEMMA 12 (see [Hahn67], Theorem 34.2) *Consider the nonlinear system:*

$$\begin{cases} \dot{x} &= f(x), f(0) = 0 \\ y &= h(x), h(0) = 0 \end{cases}$$

and suppose :

1) There exist a semipositive solution $(V \geq 0)$ *of the Liapunov equation:*

$$\frac{dV}{dt} + \| h \|^q = 0$$

2) The pair (h, f) *is detectable.*

3) The origin is a stable equilibrium for f*.*

Then the origin is an asymptotically stable equilibrium for f*.*

Proof

Let us denote by S *the largest* f*-invariant set included in* $N = \{ x \in \mathbf{R}^n | \dot{V}(x) = 0 \}$*. Let* $\tilde{x}(t)$ *be a trajectory. Since it is bounded then there exists* $(t_n)_n \to \infty$ *such that* $(x(t_n))_n$ *is a convergent sequence and let* $x^* = \lim_{n \to \infty} x(t_n)$*. Now, if we apply Barbălat's Lemma (lemma 5) we get that* $V : [0, \infty) \to \mathbf{R}$, $V(t) = V(x_0) - \int_0^t \| h \|^q (x(t)) dt$ *is an uniformly continuous function and then* $\lim_{t \to \infty} \dot{V}(x(t)) = 0$ *or* $\lim_{n \to \infty} \dot{V}(x_n) = 0$*, which means* $x^* \in S$*. We have to prove that* $x^* = 0$*. Let us suppose* $x^* \neq 0$*. We know that* $x(t, x^*) \stackrel{t \to \infty}{\longrightarrow} 0$ *(where* $x(t, .)$ *denotes the associated flow to* f*) because it is a trajectory included in* S*. Let* $\varepsilon = \| x^* \| / 2$*. Then* $\exists \delta_\varepsilon > 0$ *such that for any* $x_0 \in \mathbf{R}^n$ *with* $\| x_0 \| < \delta_\varepsilon \Rightarrow \| x(t, x_0) \| < \varepsilon$*,* $\forall t > 0$*. For* $t_1 > 0$ *such that*

$\| x(t,x^*) \| < \delta_\varepsilon/2$ we have $\lim_{n\to\infty} \tilde{x}(t_n+t_1) = x(t_1,x^*)$. We choose N such that $\| \tilde{x}(t_N) \| < \delta_\varepsilon$. Then for $n > N$ $\tilde{x}(t_n) = x(t_n - t_N, \tilde{x}(t_N))$ and from stability: $\| \tilde{x}(t_n) \| < \varepsilon$, $\forall t_n > t_N$. This is a contradiction with $\lim_{n\to\infty} \| \tilde{x}(t_n) \| = 2\varepsilon$. Then $x^* = 0$ and the ω-limit set of each trajectory is $\Omega(x) = \{0\}$, $x \in D$. Now, by Birkoff's Theorem we get the attractivity of the origin. □

Proof of Theorem 10. Since $I_1 \leq 0$ it is sufficient to consider $q = 1$ and $h = -I_1$. Then $I_1 = \nabla V \cdot f$ becomes:

$$\nabla V \cdot f + h = 0$$

and the condition of zero-state detectability is given by the first condition of Theorem 10. The geometric conditions G are given by the next two conditions of Theorem 10. Then, using Theorem 11 we conclude that $x_0 = 0$ is an asymptotically stable equilibrium.

Proof of Theorem 11. We change the coordinate system such that our dynamics is brought into a special form. We see that under condition **G** we can apply the Observability Decomposition Theorem (Theorem 9) and then in this new coordinates the dynamics is described by:

$$\begin{cases} \dot{\xi} &= \varphi(\xi) \\ \dot{\eta} &= \psi(\xi,\eta) \\ y &= \xi_1 \end{cases}$$

and $\varphi^T(\xi) = [\xi_2, \xi_3, \ldots, \xi_{n-s}, F(\xi)]$, $\varphi(0) = 0$, $\psi(0,0) = 0$. Now, let us consider the decoupled system:

$$(DS) \begin{cases} \dot{\xi} &= \varphi(\xi) \\ y &= \xi_1 \end{cases}$$

Since $\frac{dV}{dt} = \nabla V \cdot f = - \| h \|^2 = -|\xi_1|^2$ and $\xi = \xi(t,\xi_0)$ we have that $V(x) = V(\xi)$ (it depends only on the first p coordinates). On the other hand, if $y(t) \equiv 0 \Rightarrow \xi_1 \equiv 0 \Rightarrow \xi_2 = \dot{\xi}_1 \equiv 0$ and so on. Then $\xi \equiv 0$. That means that (ξ_1, φ) is zero-state observable (see [ByMa94]). Now, applying Theorem 3.1 from [ByMa94] for V to (DS) we conclude that $\xi_0 = 0$ is an asymptotically stable equilibrium for φ.

From zero-state detectability we claim that $\psi(0,\eta)$ has at $\eta_0 = 0$ an asymptotically stable equilibrium.

In that it follows we shall prove the stability of the origin for f. (This proof is borrowed from [Vidy80]). Since:

$$\begin{aligned} \dot{\xi} &= \varphi(\xi) \\ \dot{\eta} &= \Psi(0,\eta) \end{aligned}$$

are asymptotic stable systems, there exist $V_1 : D \to \mathbf{R}$ and $V_2 : D \to \mathbf{R}$ two strict positive Liapunov functions of class $\mathcal{C}^\infty$ (i.e. $V_{1,2} > 0$ and $\dot{V}_{1,2} < 0$ for $x \neq 0$) for which there are $F_1, F_2, F_3, G_1, G_2, G_3$ of class K

$$K = \{\varphi : [0,\infty) \to \mathbf{R} | \, \varphi(0) = 0 \, , \, \varphi \textit{ continuous} \, , \, \varphi \textit{ strict increasing} \}$$

such that:

$$F_1(\| \xi \|) \leq V_1(\xi) \leq F_2(\| \xi \|)$$

$$\frac{dV_1}{dt} = \nabla V_1(\xi) \cdot \varphi(\xi) \leq -F_3(\| \xi \|)$$

$$G_1(\| \eta \|) \leq V_2(\eta) \leq G_2(\| \eta \|)$$
$$\frac{dV_2}{dt} = \nabla V_2(\eta) \cdot \Psi(0,\eta) \leq -G_3(\| \eta \|)$$
Moreover, we suppose Ψ to be at least of class $\mathcal{C}^1$. Then:

$$\sup_{\|\xi\| \leq c} \| \nabla_\xi \Psi(\xi,\eta) \| = L_1 < \infty$$

We set:

$$M = \sup_{\|x_2\| \leq c} \| \nabla_{x_2} V_2(x_2) \|$$

where c is a constant such that $B(0,c) \times B(0,c) \subset D$. Let $\varepsilon > 0$; we shall find $\delta_1 > 0, \delta_2 > 0$ such that for any $\| \xi_0 \| \leq \delta_1$, $\| \eta_0 \| \leq \delta_2$ we obtain $\| \xi(t) \| < \varepsilon$ and $\| \eta(t) \| < \varepsilon$. We take δ_2 to be such that $G_2(\delta_2) \leq G_1(\varepsilon)$. Let $\varepsilon_1 < min(\varepsilon, \frac{G_3(\delta_3)}{ML_1})$ and δ_1 such that $F_2(\delta_1) < F_1(\varepsilon_1)$. Then for $\| \xi \| \leq \delta_1 \Rightarrow \| \xi(t) \| \leq \varepsilon_1 \leq \varepsilon$. We claim that $\frac{dV_2}{dt}|_s = \nabla V_2(\eta) \cdot \Psi(\xi,\eta) \leq 0$ for $\| \xi(t) \| \leq \varepsilon_1$ and $\| \eta \| \geq \delta_2$. We have:

$$\nabla V_2(\eta) \cdot \Psi(\xi,\eta) = \nabla V_2(\eta) \cdot \Psi(0,\eta) + \nabla V_2(\eta) \cdot (\Psi(\xi,\eta) - \Psi(0,\eta)) \leq -G_3(\| \eta \|) + ML_2 \| \xi \| \leq$$

$$\leq -G_3(\| \eta \|) + ML_2 \varepsilon_1$$

Now, since G_3 is increasing and $\varepsilon_1 \leq \frac{G_3(\delta_3)}{ML_1}$ the above claiming follows.

Then, for any $\| \eta_0 \| \leq \delta_2$ we have $V_2(\eta(t)) \leq G_2(\delta_2) \leq G_1(\varepsilon)$. Then $\| \eta \| \leq \varepsilon$ and this proves the stability.

The asymptotic stability comes from the Lemma 12.

This ends the proof. □

3.3 An Example

Let us consider the following nonlinear system:

$$\begin{aligned} \dot{x} &= -x^3 \\ \dot{y} &= -y^3 - yx^2 \end{aligned}$$

We see that it has an unique equilibrium, namely $(x_0, y_0) = (0,0)$. For the following Liapunov candidate function:

$$V = \frac{x^2}{2}$$

we obtain:

$$\dot{V} = -x^4 = -h^q \leq 0$$

and we choose $q = 4$, $h(x) = x$. Then $L_f h = -x^3 = -h^3(x)$. Then:

1) The pair (h, f) is zero-state detectable because the maximal invariant set included in $Ker\, h$ is

$$S = \{ (0,y) \mid y \in \mathbf{R} \}$$

and the dynamics restricted to S is given by $y = -y^3$ which is asymptotically stable.

2) V is solution of $\nabla V \cdot f + h^4 = 0$

3) For $p = 1$, $dh = dx \neq 0$ and $L_f h = -h^3$.

Then according to Theorem 11 we assert that the equilibrium point $(x_0, y_0) = (0, 0)$ is asymptotically stable.

On the other hand, if we consider:

$$\tilde{V} = \frac{x^2}{2} + \frac{y^2}{2}$$

as a Liapunov candidate we have:

$$\frac{d\tilde{V}}{dt} = -x^4 - y^4 - x^2y^2 \leq 0$$

Thus, by Liapunov's theorem on asymptotic stability we obtain the same conclusion: $(0, 0)$ is an asymptotically stable equilibrium.

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