Abstract

In this thesis I present some aspects of the coherent sets theory in Hilbert space and some applications in signal processing. The general theory will focus on three important types of coherent sets: Fourier sets, Weyl-Heisenberg sets and wavelet sets. In a square-integrable unitary representation of a locally compact group one chooses a generator (an admissible vector of the Hilbert space) and a discrete subset of the locally compact group. Then the coherent set (associated to the given generator and the discrete subset) is given by discretizing the continuous orbit passing through the generator, with respect to the discrete subset.

The analysis of Fourier sets is intimately connected with the theory of nonharmonic Fourier series and irregular sampling. Weyl-Heisenberg sets are obtained from a function (called window) by translations and modulations given by a discrete subset of the time-frequency plane. Wavelet sets are obtained starting again from a function (called wavelet) and then translating and dilating it with parameters taken from a discrete subset of the time-scale plane.

My analysis concentrates around three problems: stability, localization and density.

In chapter 2 a geometric theory of frames is presented, emphasizing certain equivalence relations. Within an equivalence class, a distance between equivalent frames is introduced.

In the next chapter two stability results are analyzed; one is an extension of Kadec'$\frac{1}{4}$-stability theorem for nonharmonic Fourier series from Riesz bases to frames.; the other result generalizes an observation by Daubechies and Tchamitchian that Meyer's wavelet basis is preserved under small perturbations of the translation parameter.

In chapter 4 the localization of the wavelet generator is studied. An uncertainty inequality of Battle type is proved, where the lower bound of $\frac{1}{2}$ (as in the case of the classical uncertainty principle) is replaced by $\frac{3}{2}$.

In the last chapter an application of the Weyl-Heisenberg Riesz bases for their span to a signal processing problem is presented. The problem is to find the best approximation of a stochastic signal by Weyl-Heisenberg expansions. Different sources of error (distortion) in an encoding-decoding scheme are further analyzed.

Acknowledgments

I am first of all grateful to Professor Ingrid Daubechies, my advisor and mentor during these years spent at Princeton. Her continuous help and care guided me from a complete outsider of wavelet theory to an analysis-friendly person. Her moral support made me pass over some strained moments of this period. I wholeheartedly thank her for everything she did for me.

During my last two years I came to know a couple of people who gave momentum in one way or another to my research. The week I spent at Texas A&M University in the fall of 1997 will always remain as a very concentrated and helpful time of discussions and exchange of ideas. There I met David Larson and his student Deguang Han working on similar problems as mine. Due to some time constraint I was not able to finish writing up this chapter on what I call "super sets" and whose main object (superframe) was the central topic of our discussions at Texas A&M. I am thankful to Professor Larson for this opportunity and his warm hospitality.

My first year at Princeton meant the closing of a previous chapter my research, namely the systems theory. Yet, it was here where I (re)discovered the beautiful theory of dynamical systems. For this I am grateful to Professor Phil Holmes.

Sometimes even a single word heard at the right time can worth more than tons of papers acquired at wrong moments. This so happened when I had a talk with Professor Charles Fefferman three years ago and he tossed the magic words of microlocal analysis. I am double indebted to him: once for that stimulating discussion, and second for his careful reading of my thesis and his valuable comments.

Moving from the abstract analysis theory toward the applications realm, I had a strong support from Vinay Vaishampayan at AT&T Research Labs. He introduced me to the multidescription compression problem, a goal and starting point at the same time for chapter 5. At AT&T I (re)met Zoran Cvetcković who showed me how to obtain better quantizers and what would be the price to pay when one deals with redundant samples.

Thinking of the people around me, I see my colleagues and friends with whom I had interesting discussions. Particularly I thank to Sinan Günturk for having patiently explained to me some details of the quantization and encoding theory. I also thank to my officemates Jonathan Mattingly and Ralf Wittenberg for some ideas, many hours of debates, English improving opportunities and (most important to me) for not disturbing me when I had to sleep overnight in the office during the last year. When I started working on this time-frequency analysis topic I had a lengthy discussion with Alexandru Ionescu, a pure mathematician colleague to whom I thank for some clever examples.

Last, but not least (how could that be), I thank my wife Andreia Maer for bearing with me during the busy periods (especially in the last six months) and for the happy moments we have been having together.

List of Notations

H	Separable complex Hilbert space
$\mathbb{I}$	Finite or countable index set
$l^2(\mathbb{I})$	the space of the complex-valued square summable sequences indexed by $\mathbb{I}$
$B(B_1, B_2)$	the space of bounded operator between two (Banach) Hilbert spaces B_1 and B_2
$\mathbf{1}_X$	the characteristic function of the set X
$\mathbf{1}$ or 1	the identity operator or the constant function
$\hat{f} = \mathcal{F}(f)$	the Fourier transform of f defined at ξ by $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx$
$f^v = \mathcal{F}^{-1}(f)$	the Fourier inverse transform of f
$\sum_m$	$\sum_{m\in\mathbb{Z}}$
$\int$	$\int_{-\infty}^{\infty}$

Contents

Chapter 1

Coherent Sets: The Group Representation Point of View

1.1 Square Integrable Representations of l.c.g.'s

Assume Γ is a *locally compact group* (l.c.g.) with the left invariant measure μ and U a strongly continuous unitary representation on the complex Hilbert space H, $U : \Gamma \to \mathcal{U}(H)$ ($\mathcal{U}(H)$ denotes the space of unitary operators on H). Then the representation U is called *irreducible* if the only closed subspaces of H invariant under the action of every unitary operator $U(\gamma)$, $\gamma \in \Gamma$ are $\{0\}$ and H itself. A vector $h \in H$ is called *cyclic* if the linear span of $\{U(\gamma)h, \gamma \in \Gamma\}$ is dense in H. A vector $h \in H$ is called *admissible* if the map $\gamma \to < h, U(\gamma)h >$ is in $L^2(\Gamma, d\mu)$, i.e. $\int_\Gamma | < h, U(\gamma)h > |^2 d\mu(\gamma) < \infty$.

DEFINITION 1.1 *The strongly continuous representation U is called* square integrable *if i) it has a cyclic vector and ii) there is an* admissible vector $h \in H$.

The central result that is of interest for us is the following theorem proved in [GMP85]:

THEOREM 1.2 *Suppose $U : \Gamma \to \mathcal{U}(H)$ is a square integrable representation on H. Then:*

1. The set of admissible vectors form a dense subset D of H;

2. There is a nonnegative selfadjoint operator A (unbounded, in general) on H with domain D such that for every $f_1, f_2 \in H$ and $g_1, g_2 \in D$ the following relation holds true:

$$\int_\Gamma < f_1, U(\gamma)g_1 >< U(\gamma)g_2, f_2 > d\mu(\gamma) =< g_2, Ag_1 >< f_1, f_2 > \tag{1.1}$$

3. If the group is unimodular (i.e. the left and right invariant measures coincide) then the selfadjoint operator is a multiple of the identity, $A = \lambda \cdot \mathbf{1}$, $\lambda \geq 0$ and any vector is admissible, i.e. $D = H$. $\square$

REMARK 1.3 *The relation (1.1) represents the weak form of a* continuous resolution of identity *that, for the square integrable representation U takes the following form:*

$$f = \frac{1}{< h, Ah >} \int_\Gamma < f, U(\gamma)h > U(\gamma) h d\mu(\gamma) \quad , \quad \forall f \in H, h \in D \tag{1.2}$$

with the vectorial integration converging strongly (i.e. in the sense of Bochner). The word "continuous" refers to the fact that we integrate over the entire group Γ.

REMARK 1.4 *From (1.1) by letting $f_1 = f_2 = g_1 = g_2 = g \in D$ we obtain:*

$$< g, Ag >= \frac{1}{\|g\|^2} \int_\Gamma | < g, U(\gamma)g > |^2 d\mu(\gamma) \tag{1.3}$$

Since the diagonal elements (i.e. of the form $< x, Ax >$) of a quadratic form are sufficient to uniquely define that form we draw the conclusion that (1.3) specifies completely the selfadjoint A.

The relation (1.1) (or 1.2) lies at the basis of the continuous transforms frequently used in signal processing. Suppose we fix an admissible $g \in D$. Then we can define an operator:

$$T_g : H \to L^2(\Gamma, d\mu) \quad , \quad T_g f(\gamma) =< f, U(\gamma)g > \tag{1.4}$$

Note that:

$$\|T_g f\|^2_{L^2(\Gamma, d\mu)} = \int_\Gamma | < f, U(\gamma)g > |^2 d\mu(\gamma) =< g, Ag > \|f\|^2 \tag{1.5}$$

Thus:

$$\|T_g\|_{B(H, L^2(\Gamma, d\mu))} =< g, Ag > \tag{1.6}$$

(by $B(H_1, H_2)$ we denote the space of bounded operators from H_1 to H_2 with the usual operator norm).

Relation (1.5) proves also the set $\mathcal{H}_g = Ran\, T_g = \{T_g f; f \in H\}$ is a closed subspace of $L^2(\Gamma, d\mu)$. Using now (1.2) we obtain that the vector f is "recovered" from the transformation T_g by:

$$f = \frac{1}{< g, Ag >} \int_\Gamma T_g f(\gamma) \cdot U(\gamma) g d\mu(\gamma) \tag{1.7}$$

We point out that the mapping $f \mapsto T_g f$ is one-to-one (because of (1.5)) but not onto (that is $\mathcal{H}_g \nsubseteq L^2(\Gamma, d\mu)$). In fact, for recovering f we might as well have used any function of the form $T_g f + T_z h$, with $h \in H$, $z \in D$ and $< z, Ag >= 0$. Indeed, for every $k \in H$:

$$\begin{aligned} &\frac{1}{< g, Ag >} \int_\Gamma (T_g f + T_z h)(\gamma) < U(\gamma) g, k > d\mu(\gamma) \\ &= \frac{1}{< g, Ag >} \int_{Ga} T_g f(\gamma) < U(\gamma) g, k > d\mu(\gamma) + \frac{1}{< g, Ag >} \int_\Gamma < h, U(\gamma) z >< U(\gamma), k > d\mu(\gamma) \\ &\qquad =< f, k > + < h, k > \frac{\overline{< Ag, z >}}{< g, Ag >} =< f, k > \end{aligned}$$

There exists thus many functions $\varphi \in L^2(\Gamma, d\mu)$ that allow reconstruction of f via the formula $\frac{1}{<g,Ag>} \int_\Gamma \varphi(\gamma) U(\gamma) g d\mu(\gamma)$; this points out that the continuous transform $T_g f$ contains redundant information on f. One way to cut down this redundancy is to discretize the continuous transform, i.e. to consider only a sequence of the form $\{< f, U(\gamma_i) g >\}_{i \in \mathbb{I}}$ for some particular discrete subset $\{\gamma_i\}_{i \in \mathbb{I}}$ of Γ. Sets of the form $\{U(\gamma_i) g\}_{i \in \mathbb{I}}$ are called *coherent subsets* and make the subject of the present thesis.

We end this section by recalling another property of the subspace $\mathcal{H}_g \subset L^2(\Gamma, d\mu)$. Recall a Hilbert space of functions $(V, <,>_V)$ is called a *reproducing kernel Hilbert space* if the mappings $f \in V \mapsto f(x)$ are bounded for every x, i.e. for every x there is a constant C_x such that $|f(x)| \leq C_x \|f\|_V$.

Take an arbitrarily $f \in H$ and the corresponding $T_g f \in \mathcal{H}_g$. Then, by using (1.4), (1.5) and the Cauchy-Schwartz inequality we get:

$$|T_g f(\gamma)| \leq \|f\| \cdot \|U(\gamma) g\| = \frac{\|g\|}{(< g, Ag >)^{1/2}} \|T_g f\|_{L^2(\Gamma, d\mu)} \tag{1.8}$$

which proves that $\mathcal{H}_g$ is a reproducing kernel Hilbert space.

1.2 Frames, Riesz Bases, s-Riesz Bases

Consider a Hilbert space H and an indexed set of vectors $\mathcal{F} = (f_i)_{i\in\mathbb{I}}$ in H with $\mathbb{I}$ a finite or countably infinite index set.

DEFINITION 1.5 *The set $\mathcal{F}$ is called a* frame *for H if there are two positive constants $0 < A, B < \infty$ such that for every $h \in H$:*

$$A\|h\|^2 \leq \sum_{i\in\mathbb{I}} |< h, f_i >|^2 \leq B\|h\|^2 \tag{1.9}$$

DEFINITION 1.6 *The set $\mathcal{F}$ is called a* Riesz basis *for H if it is a frame and a Schauder basis for H.*

DEFINITION 1.7 *The set $\mathcal{F}$ is called a* Riesz basis for its span *(or a* s-Riesz basis*) if there are two positive constants $0 < A, B < \infty$ such that for every finite sequence of complex numbers $(c_i)_{i\in\mathbb{I}}$ (i.e. $c_i \neq 0$ for only a finite number of i's):*

$$A\sum_{i\in\mathbb{I}} |c_i|^2 \leq \|\sum_{i\in\mathbb{I}} c_i f_i\|^2 \leq B\sum_{i\in\mathbb{I}} |c_i|^2 \tag{1.10}$$

DEFINITION 1.8 *The set $\mathcal{F}$ is called a* Bessel sequence *if there is a $B > 0$ such that for every $h \in H$:*

$$\sum_{i\in\mathbb{I}} |< h, f_i >|^2 \leq B\|h\|^2 \tag{1.11}$$

REMARK 1.9 *Any frame is a complete set, i.e. its closed linear span is all of H. Indeed, if $h \in H$ is such that $< h, f_i >= 0$ then (1.9) implies $h = 0$. Any frame, Riesz basis or s-Riesz basis is a Bessel sequence as well. However the converse is not true in general.*

The positive numbers A and B in (1.9) are called *frame bounds* and in (1.10) are called *Riesz basis bounds*. The number B in (1.11) is called a *Bessel sequence bound.* If in (1.9) we can choose $A = B$ then the frame is called *tight.*

Suppose $\mathcal{F}$ is a Bessel sequence. Then the following operators are well-defined and bounded:

$$T : H \to l^2(\mathbb{I}) \quad , \quad Th = (< h, f_i >)_{i\in\mathbb{I}} \tag{1.12}$$

$$T^* : l^2(\mathbb{I}) \to H \quad , \quad T^*c = \sum_{i\in\mathbb{I}} c_i f_i \tag{1.13}$$

$$S : H \to H \ , \quad S = T^*T \quad , \ Sh = \sum_{i\in\mathbb{I}} < h, f_i > f_i \tag{1.14}$$

$$G : l^2(\mathbb{I}) \to l^2(\mathbb{I}) \ , \quad G = TT^* \quad , \ Gc = (< \sum_{i\in\mathbb{I}} c_i f_i, f_j >)_{j\in\mathbb{I}} \tag{1.15}$$

where $l^2(\mathbb{I})$ is the Hilbert space of the complex-valued square summable sequences indexed by $\mathbb{I}$. Let $E = Ran\, T \subset l^2(\mathbb{I})$ and $\mathcal{E} = Ran\, T^* \subset H$. The operator T is called the *analysis operator*, T^* (the adjoint of T) is called the *synthesis operator*, S is called the *frame operator* and G is called the *grammian operator*. The space E is called the *coefficients range* and $\mathcal{E}$ represents the *linear span* of the set $\mathcal{F}$. Let us denote by δ_i the sequence $(\delta_i)_j = 1$ if $i = j$ and $(\delta_i)_j = 0$ if $i \neq j$. Then the set $\{\delta_i\}_{i\in\mathbb{I}}$ is an orthonormal basis in $l^2(\mathbb{I})$. The matrix of the grammian operator in the canonical basis $\{\delta_i\}_{i\in\mathbb{I}}$ of $l^2(\mathbb{I})$ is given by: $(G)_{ij} =< f_j, f_i >$, $i, j \in \mathbb{I}$.

When S is invertible we let $\tilde{f}_i$ and $f_i^{\#}$ be:

$$\tilde{f}_i = S^{-1} f_i \quad , \quad f_i^{\#} S^{-1/2} f_i \tag{1.16}$$

When G is invertible we denote by $\breve{f}_i$ and $f^{\natural}{}_i$ the following vectors:

$$\breve{f}_i = T^*(G)^{-1}\delta_i \quad , \quad f^{\natural}{}_i = T^*(G)^{-1/2}\delta_i \tag{1.17}$$

Then the following result is known in the literature (see [Daub90, HeWa89, Chris93] for proof):

PROPOSITION 1.10 *I. Consider $\mathcal{F} = (f_i)_{i\in\mathbb{I}}$ a Bessel sequence in the Hilbert space H. Then*

1. *The set $\mathcal{F}$ is a frame with frame bounds A, B if and only if $A\mathbf{1} \leq S \leq B\mathbf{1}$ (where $T \leq S$ stands for $< Tf, f >\leq< Sf, f >$ for all $f \in H$);*

2. *The set $\mathcal{F}$ is a s-Riesz basis with Riesz basis bounds A, B if $A\mathbf{1} \leq G \leq B\mathbf{1}$;*

II. Suppose that $\mathcal{F}$ is a frame for H with frame bounds A, B. Then

1. *$\mathcal{E} = H$ and E is a closed subspace of $l^2(\mathbb{I})$*

2. *The set $\tilde{\mathcal{F}} = \{\tilde{f}_i, i \in \mathbb{I}\}$ is a frame for H (called the* standard dual frame*) with frame bounds $\frac{1}{A}$, $\frac{1}{B}$ and having the same coefficients range;*

3. *The following* reconstruction formula *(or* discrete resolution of identity*) holds true for every $h \in H$:*

$$h = \sum_{i \in \mathbb{I}} < h, f_i > \tilde{f}_i = \sum_{i \in \mathbb{I}} < h, \tilde{f}_i > f_i \tag{1.18}$$

4. *The set $\mathcal{F}^{\#} = \{f_i^{\#}, i \in \mathbb{I}\}$ is a tight frame for H (called the* associated tight frame*) with frame bound 1 and having the same coefficients range E;*

5. *The orthogonal projection onto the coefficients range E is given by:*

$$P_E(c) = (< \sum_{j \in \mathbb{I}} c_j f_j, \tilde{f}_i >)_{i \in \mathbb{I}} = (< \sum_{j \in \mathbb{I}} c_j \tilde{f}_j, f_i >)_{i \in \mathbb{I}} = (< \sum_{j \in \mathbb{I}} c_j f_j^{\#}, f_i^{\#})_{i \in \mathbb{I}} \tag{1.19}$$

III. Suppose now that $\mathcal{F}$ is a s-Riesz basis for H with Riesz basis bounds A, B. Then

1. *$E = l^2(\mathbb{I})$ and $\mathcal{E}$ is a closed subspace of H;*

2. *The set $\breve{\mathcal{F}} = \{\breve{f}_i, i \in \mathbb{I}\}$ is a s-Riesz basis for H (called the* standard biorthogonal s-Riesz basis*) with Riesz basis bounds $\frac{1}{B}$, $\frac{1}{A}$ and having the same span $\mathcal{E}$; moreover, $< f_i, \breve{f}_j >= \delta_{ij}$ (the Kronecker symbol) and $\breve{\mathcal{F}}$ is the standard dual frame of $\mathcal{F}$ when the later is restricted to $\mathcal{E}$;*

3. *The following* reconstruction formula *holds for every $c \in l^2(\mathbb{I})$:*

$$c = (< \sum_{j \in \mathbb{I}} c_j f_j, \breve{f}_i >)_{i \in \mathbb{I}} = (< \sum_{j \in \mathbb{I}} c_j \breve{f}_j, f_i >)_{i \in \mathbb{I}} \tag{1.20}$$

4. *The set $\mathcal{F}^b = \{f^{\natural}{}_i, i \in \mathbb{I}\}$ is an orthonormal set in H (called the* associated orthonormal set*) and having the same span $\mathcal{E}$ (i.e. it is an orthonormal basis for E);*

5. *The orthogonal projection onto the span $\mathcal{E}$ is given by:*

$$P_E = \sum_{i \in \mathbb{I}} < \cdot, f_i > \breve{f}_i = \sum_{i \in \mathbb{I}} < \cdot, \breve{f}_i > f_i = \sum_{i \in \mathbb{I}} < \cdot, f^{\natural}{}_i > f^{\natural}{}_i \tag{1.21}$$

IV. The set $\mathcal{F}$ is a Riesz basis for H with Riesz basis bounds A, B if and only if it is complete and (1.10) holds for every finite sequence c (i.e. it is s-Riesz basis with Riesz basis bounds A, B). □

In figure 1.1 we pictured the action of various operators. We shall return in Chapter 2 to the geometry of frames. There, we shall analyze certain equivalency relations between frames and a distance between elements within the same class.

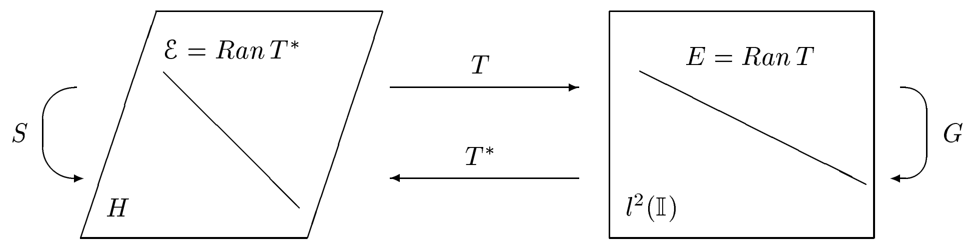

Figure 1.1: Operators and Subspaces associated to Frames and s-Riesz Bases

1.3 Three Examples: Fourier, Weyl-Heisenberg and Wavelet Sets

In section 1.1 we called *coherent set* a set of vectors obtained by discretizing a continuous orbit of a l.c.g. unitary representation. In this section we present three examples that preview the analysis of the forthcoming chapters.

Fourier Sets

The group is the additive group $\Gamma = (\mathbb{R}, +)$ with the usual Lebesgue measure as Haar measure and the Hilbert space is the Paley-Wiener space of band-limited functions:

$$H = B^2_\sigma := \{f \in L^2(\mathbb{R}) \mid supp\,\hat{f} \subset [-\sigma, \sigma]\} \tag{1.22}$$

with the usual scalar product inherited from $L^2(\mathbb{R})$. Then the unitary representation is given by translations:

$$U : \Gamma \to \mathcal{U}(H) \quad , \quad U(a)f(x) = f(x - a) \tag{1.23}$$

In this thesis we consider the Fourier transform with the following normalization:

$$f \in L^1(\mathbb{R}) \mapsto \hat{f}(\xi) = \mathcal{F}(f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx \tag{1.24}$$

defined on L^1 and next extended to $L^2(\mathbb{R})$ via the Plancherel theorem. The inverse Fourier transform will be denoted by $\mathcal{F}^{-1}(f)$ or $\check{f}$.

It can be easily seen that U given by (1.23) is highly reducible; however, it has cyclic vectors. For instance $g = \check{\mathbf{1}}_{[-\sigma,\sigma]}$ is such a cyclic vector. Next let us analyze the admissibility condition. Take a $g \in B^2_\sigma$. Then:

$$\begin{aligned} \int_\Gamma |<g, U(a)g>|^2 da &= \int_{-\infty}^{\infty} da \Big| \int_{-\infty}^{\infty} e^{-i\xi a} |\hat{g}(\xi)|^2 d\xi \Big|^2 \\ &= 2\pi \int_{-\sigma}^{\sigma} |\hat{g}(\xi)|^4 d\xi = 2\pi \|\hat{g}\|_4^4 \end{aligned} \tag{1.25}$$

Thus any function $g \in B^2_\sigma$ whose Fourier transform is in L^4 is an admissible function. In particular $g = \mathbf{1}^v_{[-\sigma,\sigma]}$ is such a function. Hence the representation (1.23) is a square integrable representation. The domain D of the selfadjoint operator A in theorem 1.2 is given by

$$D = B^2_\sigma \cap \mathcal{F}^{-1}(L^4[-\sigma,\sigma]) \tag{1.26}$$

Fix an admissible vector $d \in D$. We discretize the continuous orbit passing through g according to the set $\mathcal{L} = \{-\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$. Thus we obtain the set of vectors:

$$\mathcal{G} = \{g_n = g(\cdot + \lambda_n) \ , \ n \in \mathbb{Z}\} \tag{1.27}$$

indexed by $\mathbb{I} = \mathbb{Z}$. Note that $\hat{g}_n(\xi) = e^{i\lambda_n\xi}\hat{g}(\xi)$. Let us denote by

$$\hat{\mathcal{G}} = \{\hat{g}_n \ , \ n \in \mathbb{Z}\} \ ; \tag{1.28}$$

then obviously $\mathcal{G}$ is a frame, a Riesz basis or a s-Riesz basis for B^2_σ if and only if $\hat{\mathcal{G}}$ is a frame, a Riesz basis or a s-Riesz basis for $L^2[-\sigma,\sigma]$. Thus the analysis of $\mathcal{G}$ reduces to the analysis of $\hat{\mathcal{G}}$ for $L^2[-\sigma,\sigma]$.

A very popular choice for g is $\hat{g} = \frac{1}{\sqrt{2\pi}}\mathbf{1}_{[-\sigma,\sigma]}$. In this case, for $\lambda_n = \frac{\pi}{\sigma}n$ the set $\mathcal{G}$ is an orthonormal basis for B^2_σ. Since $g_n(x) = \sqrt{\frac{\sigma}{\pi}} sinc(\sigma x - n\pi)$ we obtain, for every $f \in B^2_\sigma$:

$$f(x) = \sum_{n\in\mathbb{Z}} <f, g_n> g_n = \sum_{n\in\mathbb{Z}} f(n\frac{\pi}{\sigma}) \cdot sinc(\sigma x - n\pi) \tag{1.29}$$

which is the classical Shannon sampling theorem.

When $\mathcal{G}$ is a frame, a Riesz basis or a s-Riesz basis, it is called a *Fourier frame, Fourier Riesz basis* or a *Fourier s-Riesz basis.* Likewise the sequence $(\lambda_n)_{n\in\mathbb{Z}}$ is called a *frame sequence, Riesz basis sequence* or a *s-Riesz basis sequence* if $\mathcal{G}$ is such, for $g = \check{\mathbf{1}}_{[-\sigma,\sigma]}$.

REMARK 1.11 *The frame sequence definition can be extended to the complex plane as well. In this case we are interested whether for $\lambda_n = \sigma_n + i\rho_n \in \mathbb{C}$ the set $\{f_n(x) = e^{i\lambda_n x}, n \in \mathbb{Z}\}$ is a frame, a Riesz basis or a s-Riesz basis in $L^2[-\sigma,\sigma]$. A basic principle due to Duffin and Schaeffer in [DuSc52] proves that if $(\sigma_n)_{n\in\mathbb{Z}}$ is a frame sequence and $|\rho_n| \le M$ for some $M < \infty$, then $(\lambda_n)_{n\in\mathbb{Z}}$ is a frame sequence as well. Thus, in general, the complex frame sequence problem reduces to a real sequence frame problem plus uniform bound of the imaginary part.*

REMARK 1.12 *Sets of the form $\{g(\cdot - \lambda_n), n \in \mathbb{Z}\}$ have been also studied in $L^2(\mathbb{R})$ instead of the smaller space B^2_σ. In this case, one is interested in studying conditions under which $\{g(\cdot - \lambda_n), n \in \mathbb{Z}\}$ is a frame for its span or a s-Riesz basis. It turned out for the case $\lambda_n = na$, $a > 0$ that the function $\sum_m |\hat{g}(\xi - \frac{2\pi m}{a})|^2$ determines the behaviour of the set $\{g(\cdot - na), n \in \mathbb{Z}\}$ (see [BeLi95]).*

Weyl-Heisenberg Sets

We set Γ to be the Weyl-Heisenberg group:

$$\Gamma = H^1 := (T^1 \times \mathbb{R} \times \mathbb{R}, \star) \tag{1.30}$$

$$(z_1, p_1, q_1) \star (z_2, p_2, q_2) = (z_1 z_2 e^{i(p_1 q_2 - q_1 p_2)}, p_1 + p_2, q_1 + q_2)$$

where $T^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the complex 1-torus, and $H = L^2(\mathbb{R})$. Both left and right invariant measures are given by $d\mu(z,p,q) = \frac{1}{z} dz\, dp\, dq$. Therefore the group is unimodular. We consider the following representation of H^1 on $L^2(\mathbb{R})$:

$$U(z,p,q)f(x) = ze^{ipx} f(x-q) \tag{1.31}$$

Choose a $g \in L^2(\mathbb{R})$. Then, using the Plancherel identity again, the 2-norm of the coefficient map turns into:

$$\int_\Gamma | < g, U(z,p,q)g > |^2 d\mu(z,p,q) = 2\pi \|g\|_2^4$$

Thus, from (1.3) we get:

$$< g, Ag >= 2\pi\|g\|_2^2 \;\Rightarrow\; A = 2\pi \cdot 1 \tag{1.32}$$

This shows that $D = L^2(\mathbb{R})$, i.e. every vector is admissible. Take now an arbitrary function $g \in L^2(\mathbb{R})$ and consider a discrete subset of H^1, $\mathcal{L} = \{(z_i, p_i, q_i), i \in \mathbb{I}\}$. Then we call $\{U(z_i, p_i, q_i)g, i \in \mathbb{I}\}$ a *Weyl-Heisenberg set* (or, simply, a *WH set*) and g a *window*. A particular but very important case is when we choose $\mathcal{L}$ to be the lattice $\mathcal{L} = \{(1, 2\pi m\alpha, n\beta); m, n \in \mathbb{Z}\}$ for some $\alpha, \beta > 0$. Then we denote $U(1, 2\pi m\alpha, n\beta)g$ by $g_{mn;\alpha\beta}$ or, when there is no danger of confusion, by g_{mn}. Likewise, we denote the Weyl-Heisenberg set $\{g_{mn;\alpha\beta}; m, n \in \mathbb{Z}\}$ by $\mathcal{WH}_{g;\alpha,\beta}$. Also, in the coherent case, we denote by $T_{g;\alpha\beta}$ the analysis operator associated to $\mathcal{WH}_{g;\alpha,\beta}$.

The following theorem summarizes the well-known results in literature (see [Daub90, HeWa89, DaLaLa95, Jans95]):

THEOREM 1.13 *I. a) If the window g belongs to the Wiener amalgam space $W(L^\infty, l^1)$ defined as follows:*

$$W(L^\infty, l^1) := \{f : \mathbb{R} \to \mathbb{C} \mid \|f\|_{W(L^\infty,l^1)} := \sum_{n\in\mathbb{Z}} ess \sup_{x\in[n,n+1]} |f(x)| < \infty\}\ , \tag{1.33}$$

then the set $\mathcal{WH}_{g;\alpha,\beta}$ is a WH Bessel sequence for every $\alpha, \beta > 0$;

b) If $\mathcal{WH}_{g;\alpha,\beta}$ is a WH Bessel sequence with bound B then:

$$\frac{1}{\alpha}\sum_n |g(x - n\beta)|^2 \le B \qquad \frac{2\pi}{\beta}\sum_m |\hat{g}(\xi - 2\pi m\alpha)|^2 \le B \tag{1.34}$$

for every $x, \xi \in \mathbb{R}$.

II. Suppose $\mathcal{WH}_{g;\alpha,\beta}$ is a frame with bounds A, B. Set $\tilde{g} = S^{-1}g$ and $\widetilde{g_{mn}} = S^{-1}g_{mn}$, where S is the frame operator.

a) $\widetilde{g_{mn}}(x) = e^{2\pi i m\alpha x}\tilde{g}(x - n\beta)$, for every $m, n \in \mathbb{Z}$;

b) $\alpha \cdot \beta \le 1$;

c) If $\alpha\beta = 1$ the $\mathcal{WH}_{g;\alpha,\beta}$ is a Riesz basis (for $L^2(\mathbb{R})$);

d) The associated tight frame is given by $\mathcal{WH}_{g^\#;\alpha,\beta}$ where $g^\# = S^{-1/2}g$;

e) For every $x, \xi \in \mathbb{R}$*:*

$$A \leq \frac{1}{\alpha} \sum_n |g(x - n\beta)|^2 \leq B \quad , \quad A \leq \frac{2\pi}{\beta} \sum_m |\hat{g}(\xi - 2\pi m\alpha)|^2 \leq B \tag{1.35}$$

III.

1. *If* $T_{g;\alpha,\beta}$ *is bounded, so is* $T_{g;\frac{1}{\beta},\frac{1}{\alpha}}$*;*

2. *Let* $f, h \in L^2(\mathbb{R})$ *such that* $T_{g;\alpha,\beta}$*,* $T_{f;\alpha,\beta}$*,* $T_{h;\frac{1}{\beta},\frac{1}{\alpha}}$ *are all bounded. Then:*

$$T^*_{f;\alpha,\beta} T_{g;\alpha,\beta} h = \frac{1}{\alpha\beta} T^*_{h;\frac{1}{\beta},\frac{1}{\alpha}} T_{g;\frac{1}{\beta},\frac{1}{\alpha}} f \tag{1.36}$$

(*the* Wexler-Raz identity*).*

IV. $\mathcal{WH}_{g;\alpha\beta}$ *is a frame if and only if* $\mathcal{WH}_{g;\frac{1}{\beta},\frac{1}{\alpha}}$ *is a Riesz basis for its span;*

V. Suppose $\mathcal{WH}_{g;\alpha\beta}$ *is a frame and* $\mathcal{WH}_{g;\frac{1}{\beta},\frac{1}{\alpha}}$ *a Riesz basis for its span.*

1. $\mathcal{WH}_{g';\alpha,\beta}$ *is a dual of* $\mathcal{WH}_{g;\alpha\beta}$ *if and only if* $\mathcal{WH}_{\frac{1}{\alpha\beta}g';\frac{1}{\beta},\frac{1}{\alpha}}$ *is a pseudodual of* $\mathcal{WH}_{g;\frac{1}{\beta},\frac{1}{\alpha}}$*;*

2. $\mathcal{WH}_{\tilde{g};\alpha,\beta}$ *is the standard dual of* $\mathcal{WH}_{g;\alpha\beta}$ *if and only if* $\mathcal{WH}_{\frac{1}{\alpha\beta}\tilde{g};\frac{1}{\beta},\frac{1}{\alpha}}$ *is the standard dual of* $\mathcal{WH}_{g;\frac{1}{\beta},\frac{1}{\alpha}}$*;*

3. $\mathcal{WH}_{g^{\#};\alpha,\beta}$ *is the associated tight frame of* $\mathcal{WH}_{g;\alpha\beta}$ *if and only if* $\mathcal{WH}_{\frac{1}{\sqrt{\alpha\beta}}g^{\#};\frac{1}{\beta},\frac{1}{\alpha}}$ *is the associated orthonormal set of* $\mathcal{WH}_{g;\frac{1}{\beta},\frac{1}{\alpha}}$*;* $\square$

As this theorem suggests, the product $\alpha \cdot \beta$ plays a major role in the behavior of the set $\mathcal{WH}_{g;\alpha,\beta}$. We shall call $r = \frac{1}{\alpha\beta}$ the *deficit* or *redundancy* of the set $\mathcal{WH}_{g;\alpha,\beta}$ depending on whether $\alpha\beta \geq 1$ or $\alpha\beta \leq 1$. Hence a WH s-Riesz basis has a deficit $r = \frac{1}{\alpha\beta} \leq 1$ and a WH frame has a redundancy $r = \frac{1}{\alpha\beta} \geq 1$.

Wavelet Sets

Take for Γ the $ax + b$ group (or the translations and dilations group) defined by:

$$ax + b = (\mathbb{R}^* \times \mathbb{R}, o) \qquad (a_1, b_1)o(a_2, b_2) = (a_1 a_2, b_1 + a_1 b_2) \tag{1.37}$$

Set $H = L^2(\mathbb{R})$ and consider the unitary representation of $ax + b$ on $L^2(\mathbb{R})$ given by:

$$U : ax + b \to \mathcal{U}(L^2(\mathbb{R})) \quad , \quad U(a, b)h(x) = \frac{1}{\sqrt{|a|}} h(\frac{x - b}{a}) \tag{1.38}$$

Note that the $ax+b$ group is not unimodular; the left and right invariant measures are different. The measure of interest to us is the left invariant measure $\frac{da\,db}{a^2}$. After some computations, the 2-norm of the coefficient map $(a,b) \mapsto < h, U(a,b)h >$ turns into:

$$\int_\Gamma | < h, U(a,b)h > |^2 \frac{da\,db}{a^2} = \|h\|^2 \int_{-\infty}^{\infty} \frac{1}{|\xi|} |\hat{h}(\xi)|^2 d\xi \tag{1.39}$$

Thus, if we take $h \in D_{ax+b} = \{h \in L^2(\mathbb{R}) | \int |\xi|^{-1} |\hat{h}(\xi)|^2 d\xi < \infty\}$ then (1.39) is finite. This shows that U admits an admissible vector. On the other hand, although (1.38) is not an irreducible representation (the Hardy space $H_2 = \{f \in L^2(\mathbb{R}) | supp\,\hat{f} \subset [0,\infty)\}$ is invariant under the action of all unitary translation and dilation operators) it has cyclic vectors: for instance take $h = \check{\mathbf{1}}_{[-2\pi,-\pi]\cup[\pi,2\pi]}$; then, for each fixed a, $span\{U(a,b)h, b \in \mathbb{R}\}$ is dense in $\mathcal{F}^{-1}(L^2([-\frac{2\pi}{a_0}, -\frac{\pi}{a_0}] \cup [\frac{\pi}{a_0}, \frac{2\pi}{a}]))$; thus, taking union over a we get $\mathcal{F}^{-1}(L^2(\mathbb{R})) = L^2(\mathbb{R})$. Hence (1.3) is given by: $\hat{A}h(\xi) = \frac{1}{|\xi|}\widehat{h}(\xi)$.

Wavelet sets are coherent sets obtained by discretizing a continuous orbit of the $ax+b$ group according to the discrete subset $\mathcal{L} = \{(a_0^m, a_0^m n b_0); m,n \in \mathbb{Z}\}$ for some fixed $a_0 > 1, b_0 > 0$. Thus the wavelet set associated to the *wavelet* $\Psi \in D_{ax+b}$ and parameters $a_0 > 1, b_0 > 0$ is:

$$\mathcal{W}_{\Psi;a_0 b_0} = \{\Psi_{mn;a_0 b_0}(x) = a_0^{-m/2}\Psi(a_0^{-m}x - nb_0) \quad , \quad m,n \in \mathbb{Z}\} \tag{1.40}$$

If (1.40) is a frame, a Riesz basis or a s-Riesz basis, it is called a *wavelet frame*, a *wavelet Riesz basis* or a *wavelet s-Riesz basis*, accordingly.

The following theorem summarizes the relevant known results in literature:

THEOREM 1.14 *I. a) If the wavelet Ψ satisfies the following decay condition for some $\gamma > 1+\alpha > 1$ and $C > 0$:*

$$|\hat{\Psi}(\xi)| \leq \frac{C|\xi|^\alpha}{(1+|\xi|)^\gamma} \tag{1.41}$$

then the set $\mathcal{W}_{\Psi;ab}$ is a wavelet Bessel sequence for every $a > 1$, $b > 0$. Moreover, the upper bound is estimated by the following:

$$B \leq \frac{2\pi}{b}\{ess \sup_{|\xi|\in[1,a]} \sum_{m\in\mathbb{Z}} |\hat{\Psi}(a^m\xi)|^2 + 2\sum_{k=1}^{\infty}[\beta(\frac{2\pi k}{b})\beta(-\frac{2\pi k}{b})]^{1/2}\} \tag{1.42}$$

where $\beta(s) = ess\sup_{|\xi|\in[1,a]} \sum_{m\in\mathbb{Z}} |\hat{\Psi}(a^m\xi)| \cdot |\hat{\Psi}(a^m\xi + s)|$;

b) If $\mathcal{W}_{\Psi;ab}$ is a wavelet Bessel sequence with bound B, then for every $\xi \in \mathbb{R}$:

$$\frac{2\pi}{b}\sum_{m}|\hat{\Psi}(a^m\xi)|^2 \leq B \tag{1.43}$$

II. a) If Ψ satisfies (1.41) then for any $a > 1$ such that $\sum_m |\hat{\Psi}(a^m\xi)|^2 \geq const > 0$ there is a $b^C > 0$ such that for every $0 < b < b^C$ the set $\mathcal{W}_{\Psi;ab}$ is a wavelet frame. Moreover, the upper bound is given by (1.42), whereas for the lower bound we have the following estimate:

$$A \geq \frac{2\pi}{b}\{ess \inf_{|\xi|\in[1,a]}\sum_{m\in\mathbb{Z}}|\hat{\Psi}(a^m\xi)|^2 - \sum_{k=1}^{\infty}[\beta(\frac{2\pi k}{b})\beta(-\frac{2\pi k}{b})]^{1/2}\} \tag{1.44}$$

b) If $\mathcal{W}_{\Psi;ab}$ is a wavelet frame with bounds A, B, then for every $\xi \in \mathbb{R}$:

$$A \leq \frac{2\pi}{b}\sum_{m}|\hat{\Psi}(a^m\xi)|^2 \leq B \tag{1.45}$$

c) In general, the standard dual frame of $\mathcal{W}_{\Psi;ab}$ (when this is a frame) is not a wavelet set. However, the following is true: $\widetilde{\Psi_{mn}} = U(a^m, 0)\widetilde{\Psi}_n$.

1.4 General Problems: Stability, Localization, Density

In this section we introduce three problems related to coherent sets. As we have seen before, a coherent set is defined by two pieces of data: a generator (usually an admissible vector) and a discrete subset of the l.c.g.

Stability. The stability problem refers to the coherent set behaviour when either the generator or the l.c.g.'s subset is modified. If the generator is perturbed, we can introduce a new norm to measure how large the perturbation can be, to preserve the coherent set's property (either frame, Riesz basis or s-Riesz basis). If the discrete subset $\mathcal{L}$ is deformed we distinguish between *structural perturbation*, when only a norm condition characterizes the perturbation (as, for instance,in the Fourier frame case when we perturb $\lambda_n \mapsto \lambda_n + \delta + n$, with $|\delta_n| < \delta$), or *parametric perturbation* when the initial subset $\mathcal{L}$ and the perturbed one $\mathcal{L}'$ have the same parametric structure, but with different values of the parameters (for instance in the wavelet Riesz basis case when we perturb only the translation parameter b into $b + \delta$).

Under such perturbation, our task is to determine whether the coherent set keeps its original property (frame, Riesz basis, s-Riesz basis) and to estimate the new bounds.

Localization. The localization problem refers to the generator of the coherent set. Basically we study how well localized in time-frequency domain it is. The quantity we are interested in is the uncertainty product defined as:

$$\sigma_g(Q)\sigma_g(P) = (\int_{-\infty}^{\infty} (x-\bar{x})^2 |g(x)|^2 dx)^{1/2} (\int_{-\infty}^{\infty} (\xi - \bar{\xi})^2 |\hat{g}(\xi)|^2 d\xi)^{1/2} \tag{1.46}$$

where $\bar{x} = \int_{-\infty}^{\infty} x|g(x)|^2 dx$, $\bar{\xi} = \int_{-\infty}^{\infty} \xi |\hat{g}(\xi)|^2 d\xi$ and assuming the generator has been previously normed, $\|g\| = 1$. The classical Fourier inequality (called also the Heisenberg uncertainty principle for its quantum physics interpretation) states that $\sigma_g(Q)\sigma_g(P) \geq \frac{1}{2}$.

On the other hand, if $\mathcal{WH}_{g;\alpha,\beta}$ is a Weyl-Heisenberg Riesz basis, then by the Balian-Low theorem, $\sigma_g(Q)\sigma_g(P) = \infty$. Thus, between $\frac{1}{2}$ (the lowest nontrivial possible bound) and ∞ we may have a lot of room for the other coherent set generators. We shall analyze the localization problem for both the WH and wavelet sets.

Density. The density problem refers to the discrete subset $\mathcal{L}$ of the l.c.g. Γ. The problem is to find necessary conditions (in terms of the subset $\mathcal{L}$) such that the coherent set to be a frame, a Riesz basis or a s-Riesz basis. It is clearly connected with the stability problem since this density condition should be invariant under structural or parametric perturbations; for instance, it is known that if $\mathcal{WH}_{g;\alpha,\beta}$ is a frame then $\alpha\beta < 1$; thus any parametric stability result should preserve this condition. It turns out that the density concept is very well suited for the Weyl-Heisenberg sets, but not for wavelet sets.

Chapter 2

Geometry of Frames

2.1 Equivalency Relations

Suppose H is an infinite dimensional separable Hilbert space. A theorem due to Paley-Wiener [PaWi34] states the following: let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of H and let $\{f_i\}_{i\in\mathbb{N}}$ be a family of vectors in H. If there exists a constant $\lambda \in [0,1)$ such that

$$\| \sum_{i=1}^{n} c_i(e_i - f_i) \| \leq \lambda \| \sum_{i=1}^{n} c_i e_i \| = \lambda (\sum_{i=1}^{n} |c_i|^2)^{1/2} \tag{2.1}$$

for all $n, c_1, c_2, \dots, c_n$, then $\{f_i\}_{i\in\mathbb{N}}$ is a Riesz basis in H with Riesz basis bounds $(1-\lambda)^2$, $(1+\lambda)^2$. An extension of this theorem was given by Christensen in [Chris95] to Hilbert frames and by Christensen and Heil in [ChHe96] to Banach frames.

Duffin and Eachus ([DuSc52]) proposed a converse of the above result by proving that every Riesz basis, after a proper scaling, is close to an orthonormal basis in the sense of (2.1). We are going to extend this result to Hilbert frames and to prove some results about quadratic closeness and distance between two frames.

In this chapter we shall discuss mainly the relations between two frames. Let $\mathcal{F} = (f_i)_{i\in\mathbb{I}}$ and $\mathcal{G} = (g_i)_{i\in\mathbb{I}}$ be two frames in H. We define the following notions:

- If Q is an invertible bounded operator $Q : H \to H$ with bounded inverse, and if $g_i = Qf_i$, then we say that $\mathcal{F}$ and $\mathcal{G}$ are *Q-equivalent*.

- We say they are *unitarily equivalent* if they are Q-equivalent for a unitary operator Q.

- If Q is a bounded operator $Q : H \to H$ (not necessarily invertible) and $g_i = Qf_i$, then we say $\mathcal{F}$ is *Q-partial-equivalent* with $\mathcal{G}$.

- We say $\mathcal{F}$ is *partial-isometric-equivalent* with $\mathcal{G}$ if there exists a partial isometry $J : H \to H$ such that $g_i = Jf_i$ (then J should satisfy $JJ^* = 1$ since $g_i \in Ran\ J$ and $\mathcal{G}$ is a complete set in H).

The last two relations (Q-partial-equivalent and partial- isometric-equivalent) are not equivalence relations, because they are not symmetric.

We say that a frame $\mathcal{G} = (g_i)_{i\in\mathbb{I}}$ is *(quadratically) close* to a frame $\mathcal{F} = (f_i)_{i\in\mathbb{I}}$ if there exists a positive number $\lambda \geq 0$ such that:

$$\| \sum_{i\in\mathbb{I}} c_i(g_i - f_i) \| \leq \lambda \| \sum_{i\in\mathbb{I}} c_i f_i \| \tag{2.2}$$

for any $c = (c_i)_{i\in\mathbb{I}} \in l^2(\mathbb{I})$ (see [Youn80]). The infimum of such λ's for which (2.2) holds for any $c \in l^2(\mathbb{I})$ will be called the *closeness bound* of the frame $\mathcal{G}$ to the frame $\mathcal{F}$ and denoted by $c(\mathcal{G}, \mathcal{F})$.

The closeness relation is not an equivalence relation (it is transitive, but not symmetric in general). However, if $\mathcal{G}$ is quadratically close to $\mathcal{F}$ with a closeness bound less than 1, then $\mathcal{F}$ is also quadratically close to $\mathcal{G}$ but the closeness bound is different, in general. Indeed, from (2.2) it follows that:

$$\| \sum_{i\in\mathbb{I}} c_i(g_i - f_i) \| \leq \frac{\lambda}{1-\lambda} \| \sum_{i\in\mathbb{I}} c_i g_i \|$$

The closeness bound can be related to a relative operator bound used in perturbation theory (see [Kato76]). More specifically, if T^g, T^f denote the analysis operators associated respectively to the frames $\mathcal{G}$ and $\mathcal{F}$, then $c(\mathcal{G}, \mathcal{F})$ is the $(T^f)^*$-bound of $(T^g)^* - (T^f)^*$ (in the terminology of Kato).

The next step is to correct the nonsymmetry of the closeness relation. We say that two frames $\mathcal{F} = (f_i)_{i\in\mathbb{I}}$ and $\mathcal{G} = (g_i)_{i\in\mathbb{I}}$ are *near* if $\mathcal{F}$ is close to $\mathcal{G}$ and $\mathcal{G}$ is close to $\mathcal{F}$. Nearness is now an equivalence relation. We define the *predistance* $d^0(\mathcal{F}, \mathcal{G})$ between $\mathcal{F}$ and $\mathcal{G}$, two frames that are near to each other, as the maximum between the two closeness bounds:

$$d^0(\mathcal{F}, \mathcal{G}) = \max(c(\mathcal{F}, \mathcal{G}), c(\mathcal{G}, \mathcal{F})) \tag{2.3}$$

It is easy to prove that d^0 is positive and symmetric, but does not satisfy the triangle inequality. This inconvenience can be removed if we define the *(quadratic) distance* between $\mathcal{F}$ and $\mathcal{G}$ by:

$$d(\mathcal{F},\mathcal{G}) = \log(d^0(\mathcal{F},\mathcal{G})+1) \tag{2.4}$$

Then, as we shall see later (Theorem 2.7), this defines a metric on the set of frames which are near to one another.

Since the nearness relation is an equivalence relation, we can partition the set of all frames on H, denoted $\mathcal{F}(H)$, into disjoint equivalent classes, indexed by an index set $\mathbf{A}$:

$$\mathcal{F}(H) = \bigcup_{\alpha\in\mathbf{A}} \mathcal{E}_\alpha \tag{2.5}$$

with the following properties:

$$\mathcal{E}_\alpha \cap \mathcal{E}_\beta = \emptyset \ , \ \text{for } \alpha \neq \beta$$

$$\forall \mathcal{F},\mathcal{G} \in \mathcal{E}_\alpha, d(\mathcal{F},\mathcal{G}) < \infty, \quad \text{and } \forall \mathcal{F} \in \mathcal{E}_\alpha, \mathcal{G} \in \mathcal{E}_\beta \text{ with } \alpha \neq \beta \ , \ d(\mathcal{F},\mathcal{G}) = \infty.$$

Let π denote the index projection: $\pi : \mathcal{F}(H) \to \mathbf{A}$ with $\mathcal{F} \mapsto \pi(\mathcal{F}) = \alpha$ if $\mathcal{F} \in \mathcal{E}_\alpha$. We shall prove that the partition (2.5) corresponds to the nondisjoint partition of $l^2(\mathbb{I})$ into closed infinite dimensional subspaces. Moreover, the two equivalence relations introduced before are identical (i.e. two frames are near if and only if they are Q-equivalent) as we shall prove later.

We shall be interested in finding the nearest tight frame to a given frame. For a frame $\mathcal{G}$ we denote by $\mathcal{T}^1_\mathcal{G}$ the set of tight frames which are quadratically close to $\mathcal{G}$ and by $\mathcal{T}^2_\mathcal{G}$ the set of tight frames such that $\mathcal{G}$ is close to them:

$$\mathcal{T}^1_\mathcal{G} = \{\mathcal{F} = (f_i)_{i\in\mathbb{I}} \mid \mathcal{F} \text{ is a tight frame and } c(\mathcal{G},\mathcal{F}) < +\infty\} \tag{2.6}$$

$$\mathcal{T}^2_\mathcal{G} = \{\mathcal{F} = (f_i)_{i\in\mathbb{I}} | \mathcal{F} \text{ is a tight frame and } c(\mathcal{F},\mathcal{G}) < +\infty\} \tag{2.7}$$

When no confusion can arise, we shall drop the subscript $\mathcal{G}$. Let $d^1 : \mathcal{T}^1 \to \mathbb{R}_+$, $d^2 : \mathcal{T}^2 \to \mathbb{R}_+$ denote the map from each $\mathcal{F}$ to the associated closeness bound, i.e. $d^1(\mathcal{F}) = c(\mathcal{G},\mathcal{F})$ and $d^2(\mathcal{F}) = c(\mathcal{F},\mathcal{G})$. If $\mathcal{G}$ is a tight frame itself then $\mathcal{G} \in \mathcal{T}^1 \cap \mathcal{T}^2$ and $\min d^1 = \min d^2 = 0$.

Consider now the intersection between these two sets:

$$\mathcal{T}_{\mathcal{G}} = \mathcal{T}^1 \cap \mathcal{T}^2 = \{\mathcal{F} = (f_i)_{i \in \mathbb{I}} \mid \mathcal{F} \text{ is a tight frame and } d(\mathcal{F}, \mathcal{G}) < +\infty\} \subset \mathcal{E}_{\pi(\mathcal{G})} \tag{2.8}$$

In section 2.3 we will be looking for the minima of the functions d^1, d^2 and $d|_{\mathcal{T}}$.

2.2 Geometry of Equivalent Frames

In this section we are mainly concerned with the relations introduced before. We shall prove that Q-equivalence is the same as nearness (in other words, two frames are Q-equivalent if and only if they are near). The following lemmas are fundamental for all constructions and results in this chapter:

LEMMA 2.1 *Consider $\mathcal{F}_1 = \{f_i^1\}_{i \in \mathbb{I}}$ and $\mathcal{F}_2 = \{f_i^2\}_{i \in \mathbb{I}}$ two tight frames in H with frame bounds 1. Denote by T_1 and T_2 respectively their analysis operators. Then:*

1) $Ran\, T_2 \subset Ran\, T_1$ if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$ are partial isometric equivalent; moreover, if J is the corresponding partial isometry, then $Ker\, J \simeq Ran\, T_1 / Ran\, T_2$, more specifically: $Ker\, J = T_1^*(Ran\, T_1 \cap (Ran\, T_2)^{\perp})$;

2) $Ran\, T_1 = Ran\, T_2$ if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$ are unitarily equivalent.

Proof

1. Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are partial isometric equivalent. Then $f_i^2 = J f_i^2$ and $T_2 = T_1 J^*$ for some partial isometry J. Obviously, $Ran\, T_2 \subset Ran\, T_1$. Now, recall that T_1 and T_2 are isometries from H onto their ranges (since $\mathcal{F}_1$ and $\mathcal{F}_2$ are tight frames with bound 1). Therefore they preserve the scalar product and linear independency. Thus:

$$Ran\, T_1 = T_1(Ran\, J^* \oplus Ker\, J) = T_1 J^*(H) \oplus T_1(Ker\, J) = Ran\, T_2 \oplus T_1(Ker\, J)$$

and $T_1(Ker\, J)$ is the orthogonal complement of $Ran\, T_2$ into $Ran\, T_1$. On the other hand $T_1^*|_{Ran\, T_1}$ is the inverse of $T_1 : H \to Ran\, T_1$ and thus $Ker\, J = T_1^*(Ran\, T_1 \cap (Ran\, T_2)^{\perp})$ fixing canonically the isometric isomorphism $Ker\, J \simeq Ran\, T_1 / Ran\, T_2$.

Conversely, suppose $Ran\, T_2 \subset Ran\, T_1$. Then, the two projectors are $P_1 = T_1 T_1^*$ onto $Ran\, T_1$ and $P_2 = T_2 T_2^*$ onto $Ran\, T_2$ and we have $P_1 T_2 = T_2$. Now, consider $J : H \to H$, $J = T_2^* T_1$ which acts in the following way:

$$J(x) = \sum_{i\in\mathbb{I}} < x, f_i^1 > f_i^2$$

We have:

$$JJ^* = T_2^* T_1 T_1^* T_2 = T_2^* P_1 T_2 = T_2^* T_2 = 1$$

We want to prove now that $f_j^2 = Jf_j^1$ for all j. We have, for fixed j,

$$Jf_j^1 - f_j^2 = \sum_{i\in\mathbb{I}} (< f_j^1, f_i^1 > - < f_j^2, f_i^2 >) f_i^2 = T_2^* c$$

where $c = \{c_i\}_{i\in\mathbb{I}}$, $c_i =< f_j^1, f_i^1 > - < f_j^2, f_i^2 >$. On the other hand:

$$0 = f_j^1 - \sum_{i\in\mathbb{I}} < f_j^1, f_i^1 > f_i^1 = \sum_{i\in\mathbb{I}} (\delta_{ij} - < f_j^1, f_i^1 >) f_i^1 = T_1^* a^j$$

where $a^j = \{a_i^j\}_{i\in\mathbb{I}}$, $a_i^j = \delta_{ij} - < f_j^1, f_i^1 >$ and δ_{ij} is the Kronecker symbol. Similarly $0 = T_2^* b^j$ with $b^j = \{b_i^j\}_{i\in\mathbb{I}}$, $b_i^j = \delta_{ij} - < f_j^2, f_i^2 >$. Thus $a^j \in Ker\, T_1^*$ and $b^j \in Ker\, T_2^*$. But $Ker\, T_1^* = (Ran\, T_1)^\perp \subset (Ran\, T_2)^\perp = Ker\, T_2^*$. Therefore $a^j \in Ker\, T_2^*$ and then $c^j = a^j - b^j \in Ker\, T_2^*$ which means $T_2^* c^j = 0$ or $f_j^2 = Jf_j^1$. Moreover, $T_2 = T_1 J^*$ and, as we have proved before, $Ker\, J = T_1^*(Ran\, T_1 \cap (Ran\, T_2)^\perp)$.

2. The conclusion comes from point 1: the partial isometry will have a zero kernel ($Ker\, J = \{0\}$) and therefore it is a unitary operator (recall that the range of J should be H).

This ends the proof of the lemma. □

LEMMA 2.2 *Consider $\mathcal{F}_1 = \{f_i^1\}_{i\in\mathbb{I}}$ and $\mathcal{F}_2 = \{f_i^2\}_{i\in\mathbb{I}}$ two frames in H. Let us denote their analysis operators by T_1 and T_2, respectively. Then:*

1) $Ran\, T_2 \subset Ran\, T_1$ if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$ are Q-partial equivalent for some bounded operator Q; furthermore, $Ker\, Q = T_1^*(Ran\, T_1 \cap (Ran\, T_2)^\perp)$.

2) $Ran\, T_1 = Ran\, T_2$ if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$ are Q-equivalent, for some invertible operator Q with bounded inverse.

Proof

Let us denote the frame operators by $S_1 = T_1^* T_1$, $S_2 = T_2^* T_2$.

1. Suppose $Ran\, T_2 \subset Ran\, T_1$. As before we define $(f_i^1)^\# = S_1^{-1/2} f_i^1$. Then $\mathcal{F}_1{}^\#$ is $S_1^{1/2}$-equivalent with $\mathcal{F}_1$. By Lemma 2.1, $\mathcal{F}_1{}^\#$ is J-partial equivalent with $\mathcal{F}_2{}^\#$, where $J = (T_2^\#)^* T_1^\#$ is

a partial isometry and $\mathcal{F}_2{}^{\#}$ defined by $f_i^2 = S_2^{1/2}(f_i^2)^{\#}$ is $S_2^{1/2}$-equivalent with $\mathcal{F}_2$. By composing , we get that $\mathcal{F}_1$ is Q-partial equivalent with $\mathcal{F}_2$ with $Q = S_2^{1/2} J S_1^{-1/2}$. Furthermore, since S_1 and S_2 are invertible, $Ker\,Q = S_1^{1/2} Ker\,J = T_1^*(Ran\,T_1 \cap (Ran\,T_2)^{\perp})$.

Conversely, if $\mathcal{F}_1$ is Q-partial equivalent with $\mathcal{F}_2$ and Q is the bounded operator relating $\mathcal{F}_1$ to $\mathcal{F}_2$, then $T_2 = T_1 Q^*$ and obvious $Ran\,T_2 \subset Ran\,T_1$. On the other hand, since $T_1^* T_1 = S_1$ is invertible, $Q = T_2^* T_1 S_1^{-1}$ and then $\mathcal{F}_1{}^{\#}$ is J-partial equivalent with $\mathcal{F}_2{}^{\#}$ with $J = S_2^{-1/2} Q S_1^{1/2}$. We have:

$$JJ^* = S_2^{-1/2} Q S_1^{1/2} S_1^{1/2} Q^* S_2^{-1/2} = S_2^{-1/2} T_2^* P_1 T_2 S_2^{-1/2}$$

where $P_1 = T_1 S_1^{-1} T_1^*$ is the orthogonal projection onto $Ran\,T_1$. But $Ran\,T_2 \subset Ran\,T_1$, hence $P_1 T_2 = T_2$. Thus: $JJ^* = S_2^{-1/2} T_2^* T_2 S_2^{-1/2} = 1$, proving that J is a partial isometry. Now we apply the conclusion of Lemma 2.1 and obtain that $Ker\,J = (T_1^{\#})^*(Ran\,T_1 \cap (Ran\,T_2)^{\perp})$. Substituting this into $Ker\,Q = S_1^{1/2} Ker\,J$ we obtain the result.

2. The statement is obtained from 1), by observing that $Ker\,Q = \{0\}$; since we also know that $Ran\,Q = H$, Q is therefore invertible with bounded inverse. □

We now present the connection between the closeness relation and partial equivalence.

LEMMA 2.3 *Consider $\mathcal{F}_1 = \{f_i^1\}_{i\in\mathbb{I}}$ and $\mathcal{F}_2 = \{f_i^2\}_{i\in\mathbb{I}}$ two frames in H. Let us denote their analysis operators by T_1 and T_2, respectively. Then $\mathcal{F}_1$ is close to $\mathcal{F}_2$ (i.e. $c(\mathcal{F}_1, \mathcal{F}_2) < \infty$) if and only if $\mathcal{F}_2$ is Q-partial equivalent with $\mathcal{F}_1$ for some bounded operator Q and therefore $Ran\,T_2 \subset Ran\,T_1$. Moreover $c(\mathcal{F}_1, \mathcal{F}_2) = \| Q - 1 \|$.*

Proof

$\Rightarrow$

Suppose $\mathcal{F}_1$ is close to $\mathcal{F}_2$. Then $\| \sum_{i\in\mathbb{I}} c_i(f_i^1 - f_i^2) \| \leq \lambda \| \sum_{i\in\mathbb{I}} c_i f_i^2 \|$ for $\lambda = c(\mathcal{F}_1, \mathcal{F}_2)$. If $c = \{c_i\}_{i\in\mathbb{I}} \in Ker\,T_2^*$, then necessarily $c \in Ker\,T_1^*$. Therefore $Ker\,T_2^* \subset Ker\,T_1^*$ or $Ran\,T_1 = (Ker\,T_1^*)^{\perp} \subset (Ker\,T_2^*)^{\perp} = Ran\,T_2$. Now, applying Lemma 2.2 we get that $\mathcal{F}_2$ is Q-partial equivalent with $\mathcal{F}_1$. Then $f_i^1 = Q f_i^2$. For any $v \in H$ we can find $(c_i)_{i\in\mathbb{I}} \in l^2(\mathbb{I})$ such that $v = \sum_{i\in\mathbb{I}} c_i f_i^2$; we then have: $\sum_{i\in\mathbb{I}} c_i(f_i^1 - f_i^2) = (Q-1)v$, so that

$$\inf_{T_2^* c \neq 0} \frac{\|(T_1^* - T_2^*)c\|}{\|T_2^* c\|} = \inf_{v\neq 0} \frac{\|(Q-1)v\|}{\|v\|} = \|Q - 1\|$$

Therefore $c(\mathcal{F}_1, \mathcal{F}_2) = \| Q - 1 \|$.

$\Leftarrow$

Suppose $\mathcal{F}_2$ is Q-partial equivalent with $\mathcal{F}_1$. Then, it is easy to check that $c(\mathcal{F}_1, \mathcal{F}_2) = \| Q - 1 \|$ and then $\mathcal{F}_1$ is close to $\mathcal{F}_2$. $\Box$

As a consequence of this lemma, we obtain the following result:

THEOREM 2.4 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two frames. Then they are near if and only if they are Q-equivalent for some invertible operator Q. Moreover, $d^0(\mathcal{F}_1, \mathcal{F}_2) = \max(\| Q - 1 \|, \| 1 - Q^{-1} \|)$.* $\diamond$

Applying this theorem to the set $\mathcal{T}$ defined in (2.8) we obtain the following corollary:

COROLLARY 2.5 *Consider a frame $\mathcal{G} = (g_i)_{i \in \mathbb{I}}$ in H and consider also the set $\mathcal{T}_{\mathcal{G}}$ defined by* (2.8)*. Then $\mathcal{T}$ is parametrized in the following way:*

$$\mathcal{T}_{\mathcal{G}} = \{\mathcal{F} = (f_i)_{i \in \mathbb{I}} \mid f_i = \alpha U g_i^{\#} \text{ where } \alpha > 0 \text{ and } U \text{ is unitary}\}$$

Proof

Indeed, let $\alpha > 0$ and U unitary. Then, by computing its frame operator one can easily check that $\mathcal{F} = (f_i)_{i \in \mathbb{I}}$, $f_i = \alpha U g_i^{\#}$ is a tight frame with bound α^2.

Conversely, suppose $\mathcal{F} = (f_i)_{i \in \mathbb{I}} \in \mathcal{T}$. Then, from Theorem 2.4 we obtain $f_i = Q g_i^{\#}$ for some invertible Q. We compute its frame operator:

$$S^{\mathcal{F}} = \sum_{i \in \mathbb{I}} < \cdot, f_i > f_i = Q(\sum_{i \in \mathbb{I}} < \cdot, g_i^{\#} > g_i^{\#}) Q^* = Q Q^*$$

Therefore $QQ^* = A \cdot 1$ which means that $\frac{1}{\sqrt{A}} Q$ is unitary. Thus $Q = \sqrt{A} U$ for some unitary U. $\Box$

The following result makes a connection between the extension of the Paley and Wiener theorem given by Christensen in [Chris95] and the relations introduced so far:

THEOREM 2.6 *Let $\mathcal{F} = (f_i)_{i \in \mathbb{I}}$ be a frame in H and $\mathcal{G} = (g_i)_{i \in \mathbb{I}}$ be a set of vectors in H. Suppose there exists $\lambda \in [0, 1)$ such that*

$$\| \sum_{i \in \mathbb{I}} c_i (g_i - f_i) \| \leq \lambda \| \sum_{i \in \mathbb{I}} c_i f_i \|$$

for any $n \in \mathbb{N}$ *and* $c_1, c_2, \dots$ *in* $\mathbb{C}$. *Then* $\mathcal{G}$ *is a frame in* H *and:*

1) $\mathcal{G}$ *is Q-equivalent with* $\mathcal{F}$*;*

2) If T^f *and* T^g *are the analysis operators associated respectively to* $\mathcal{F}$ *and* $\mathcal{G}$*, then* $Ran\, T^f = Ran\, T^g$*;*

3) $c(\mathcal{G}, \mathcal{F}) \leq \lambda < 1$ *and* $d^0(\mathcal{G}, \mathcal{F}) < \infty$.

Proof

The conclusion that $\mathcal{G}$ is a frame follows from a stability result proved by Christensen in [Chris95]. As we have checked before, from $c(\mathcal{G}, \mathcal{F}) < 1$ we get $c(\mathcal{F}, \mathcal{G}) \leq \frac{\lambda}{1-\lambda} < \infty$. Therefore $\mathcal{F}$ and $\mathcal{G}$ are near and we can apply Theorem 2.4 and complete the proof. $\square$

Theorem 2.4 allows us to partition the set of all frames on H, denoted $\mathcal{F}(H)$, into equivalent classes, as follows:

$$\mathcal{F}(H) = \bigcup_{\alpha \in A} \mathcal{E}_\alpha$$

where $\mathcal{E}_\alpha \subset \mathcal{F}(H)$ is a set of frames such that any $\mathcal{F}, \mathcal{G} \in \mathcal{E}_\alpha$, $\mathcal{F}$ is Q-equivalent with $\mathcal{G}$ or, equivalent, $\mathcal{F}$ is near to $\mathcal{G}$. Therefore, for each index $\alpha \in A$, the function $d^0 : \mathcal{E}_\alpha \times \mathcal{E}_\alpha \to \mathbb{R}_+$ is well-defined and finite. We want to prove now that the function:

$$d : \mathcal{E}_\alpha \times \mathcal{E}_\alpha \to \mathbb{R}_+ \quad , \quad d(\mathcal{F}, \mathcal{G}) = \log(1 + d^0(\mathcal{F}, \mathcal{G}))$$

is a distance on each class $\mathcal{E}_\alpha$.

THEOREM 2.7 *The function* d *defined above is a distance on* $\mathcal{E}_\alpha$. *Moreover, for any* $\mathcal{F} \in \mathcal{E}_\alpha$ *and* $\mathcal{G} \in \mathcal{F}(H)$*, if* $d(\mathcal{F}, \mathcal{G}) < \infty$ *then* $\mathcal{G} \in \mathcal{E}_\alpha$.

Proof

The second part of the statement is immediate: if $d(\mathcal{F}, \mathcal{G})$ is finite so is $d^0(\mathcal{F}, \mathcal{G})$; hence $\mathcal{F}$ is close to $\mathcal{G}$ and therefore they belong to the same class. To prove that d is a distance we need to check only the triangle inequality. Let $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathcal{E}_\alpha$. Then there exist Q and R invertible bounded operators on H such that $g_i = Qf_i$, $h_i = Rg_i$ and therefore $h_i = RQf_i$. We have:

$$d(\mathcal{F}, \mathcal{G}) = \log(1 + \max(\| Q - 1 \|, \| Q^{-1} - 1 \|))$$

$$d(\mathcal{G},\mathcal{H}) = \log(1+\max(\| R-1 \|, \| R^{-1}-1 \|))$$

$$d(\mathcal{F},\mathcal{H}) = \log(1+\max(\| RQ-1 \|, \| Q^{-1}R^{-1}-1 \|))$$

and:

$$\begin{aligned} \| RQ-1 \| = \| (R-1)(Q-1)+R+Q-2 \| &\leq \| R-1 \| \cdot \| Q-1 \| + \| R-1 \| + \| Q-1 \| \\ &= (\| R-1 \| +1)(\| Q-1 \| +1)-1 \end{aligned}$$

Hence:

$$\log(\| RQ-1 \| +1) \leq \log(\| R-1 \| +1) + \log(\| Q-1 \| +1)$$

Similarly for $\| Q^{-1}R^{-1}-1 \|$ and therefore $d(\mathcal{F},\mathcal{H}) \leq d(\mathcal{F},\mathcal{G}) + d(\mathcal{G},\mathcal{H})$. $\square$

The next step is to relate the partition (2.5) with the set of infinite dimensional closed subspaces of $l^2(\mathbb{I})$. We suppose H is infinite dimensional and $\mathbb{I}$ is countably infinite. Otherwise the following result still holds providing we replace "infinite dimensional closed subspaces" by "subspaces of dimension equal with the dimension of H".

Let us denote by $\mathcal{S}(l^2(\mathbb{I}))$ the set of all infinite dimensional closed subspaces of $l^2(\mathbb{I})$. Then Lemma 2.2 and Theorem 2.4 assert that $\mathcal{F}(H)$ is mapped into $\mathcal{S}(l^2(\mathbb{I}))$ by:

$$i : \mathcal{F}(H) \rightarrow \mathcal{S}(l^2(\mathbb{I})) \quad , \quad i(\mathcal{E}_\alpha) = Ran\, T \tag{2.9}$$

where T is the analysis operator associated to any frame $\mathcal{F} \in \mathcal{E}_\alpha$. The natural question that can be asked is whether i is surjective, i.e. if for any closed infinite dimensional subspace of $l^2(\mathbb{I})$ we can find a corresponding frame in $\mathcal{F}(H)$. The answer is yes as the following theorem proves (see Christensen in [Chris93], Aldroubi in [Ald94] or Holub in [Hol94] for this type of argument):

THEOREM 2.8 *For any infinite dimensional closed subspace E of $l^2(\mathbb{I})$ there exists a frame $\mathcal{F} \in \mathcal{F}(H)$ (and therefore a class $\mathcal{E}_\alpha$) such that $i(\mathcal{F}) = E$ (in other words, $Ran\, T = E$ with T the analysis operator associated to $\mathcal{F}$). Therefore i is a bijective mapping from the set of classes $\mathcal{E}_\alpha$ into $\mathcal{S}(l^2(\mathbb{I}))$.*

Proof

Let $E \subset l^2(\mathbb{I})$ be an infinite dimensional closed subspace. Choose an orthonormal basis $\{d_i\}_{i\in\mathbb{I}}$ in E and a basis $\{e_i\}_{i\in\mathbb{I}}$ in H (recall H is infinite dimensional and separable and $\mathbb{I}$ countably infinite). Let $p_i : l^2(\mathbb{I}) \to \mathbb{C}$ be the canonical projections, $p_i(c) = c_i$, where $c = \{c_j\}_{j\in\mathbb{I}}$, $i \in \mathbb{I}$. Let $P : l^2(\mathbb{I}) \to \mathbb{C}$ be the orthogonal projection onto E. Let us denote by $\{\delta_i\}_{i\in\mathbb{I}}$ the canonical basis in $l^2(\mathbb{I})$, i.e. $\delta_i = \{\delta_{ij}\}_{j\in\mathbb{I}}$. Then it is known (see [Hol94]) that $\{P\delta_i\}_{i\in\mathbb{I}}$ is a tight frame with bound 1 in E (and any tight frame indexed by I with bound 1 in E is of this form, i.e. the orthogonal projection of some orthonormal basis of $l^2(\mathbb{I})$) since:

$$\sum_{i\in\mathbb{I}} < c, P\delta_i > P\delta_i = P \sum_{i\in\mathbb{I}} < Pc, \delta_i > \delta_i = Pc = c \quad , \quad \forall c \in E$$

We define a tight frame with bound 1 in H in the following way:

$$f_i = \sum_{j\in\mathbb{I}} < P\delta_i, d_j > e_j = \sum_{j\in\mathbb{I}} < \delta_i, d_j > e_j = \sum_{j_i} p_i(d_j)e_j$$

It is easy to prove that f_i's are well defined, since $\| f_i\|^2 = \sum_{j\in\mathbb{I}} | < P\delta_i, d_j > |^2 = \| P\delta_i\|^2 < \infty$. Let T be the analysis operator associated to $\{f_i\}_{i\in\mathbb{I}}$ and $x \in H$ be arbitrarly. Then:

$$< x, f_i >= \sum_{j\in\mathbb{I}} p_i(d_j) < x, e_j >= p_i(\sum_{j\in\mathbb{I}} < x, e_j > d_j) \quad , \quad \forall i \in \mathbb{I}$$

Thus: $T(x) = \{< x, f_i > \}_{i\in\mathbb{I}} = \sum_{j\in\mathbb{I}} < x, e_j > d_j$ and obviously $Ran\, T = E$. It is simple to check that $Tf_i = P\delta_i$ and therefore $\{f_i\}_{i\in\mathbb{I}}$ is a tight frame with bound 1. □

2.3 The Closest Tight Frames

We are concerned here with close frames and with the distance functions d^1, d^2 and $d|_{\mathcal{T}}$ introduced earlier; we would like to characterize the minima of these functions. Here is the main result:

THEOREM 2.9 *Consider $\mathcal{G} = (g_i)_{i\in\mathbb{I}}$ a frame in H with optimal frame bounds A, B and consider the sets $\mathcal{T}^1$, $\mathcal{T}^2$ and $\mathcal{T}$ introduced in* (2.6), (2.7) *and* (2.8). *Let us denote by $\theta = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}$ and $\rho = \frac{1}{4}(\log B - \log A)$. Then the following conclusions hold:*

1. *The values of the minima of d^1, d^2 and $d|_{\mathcal{T}}$ are given by:*

$$\min d^1 = \min d^2 = \theta \qquad \min d|_{\mathcal{T}} = \rho$$

2. *These values are achieved by the following scalings of the associated tight frames of $\mathcal{G}$:*

$$\mathcal{F}^1 = \{f_i^1\}_{i\in\mathbb{I}} \quad , \quad f_i^1 = \frac{\sqrt{A} + \sqrt{B}}{2} g_i^{\#} \tag{2.10}$$

$$\mathcal{F}^2 = \{f_i^2\}_{i\in\mathbb{I}} \quad , \quad f_i^2 = \frac{2\sqrt{AB}}{\sqrt{A}+\sqrt{B}} g_i^{\#} \tag{2.11}$$

$$\mathcal{F}^0 = \{f_i^0\}_{i\in\mathbb{I}} \quad , \quad f_i^0 = \sqrt[4]{AB} g_i^{\#} \tag{2.12}$$

Hence $d^1(\mathcal{F}^1) = d^2(\mathcal{F}^2) = \theta$ *and* $d(\mathcal{F}^0) = \rho$

3. *Any tight frame that achieves the minimum of one of the three functions* d^1, d^2 *or* d *is unitarily equivalent with the corresponding solution* (2.10), (2.11) *or* (2.12) *in the following way:*

$$(d^1)^{-1}(\theta) = \{\mathcal{K} = \{k_i\}_{i\in\mathbb{I}} | k_i = U f_i^1,\ U \text{ unitary and } \| U - \frac{2}{\sqrt{A}+\sqrt{B}} S^{1/2} \| = \theta\} \tag{2.13}$$

$$\begin{aligned}(d^2)^{-1}(\theta) \quad &= \quad \{\mathcal{K} = \{k_i\}_{i\in\mathbb{I}} | k_i = U f_i^2,\ U \text{ unitary} \\ &\quad \text{and } \| U - \frac{2\sqrt{AB}}{\sqrt{A}+\sqrt{B}} S^{-1/2} \| = \theta\}\end{aligned} \tag{2.14}$$

$$\begin{aligned}d^{-1}(\rho) \quad &= \quad \{\mathcal{K} = \{k_i\}_{i\in\mathbb{I}} | k_i = U f_i^0,\ U \text{ unitary} \\ &\quad \text{and } \| U - \sqrt[4]{AB} S^{-1/2} \| = \| U - \frac{1}{\sqrt[4]{AB}} S^{1/2} \| = \rho\}\end{aligned} \tag{2.15}$$

where S *is the frame operator associated to* $\mathcal{G}$. *Moreover, any unitary operator that parametrizes* $(d^1)^{-1}(\theta)$, $(d^2)^{-1}(\theta)$ *or* $d^{-1}(\rho)$ *as above, has the value* 1 *in its spectrum.*

Proof

If $\mathcal{G}$ is a tight frame then $\mathcal{F}^1 = \mathcal{F}^2 = \mathcal{F}^0 = \mathcal{G}$ and $\theta = \rho = 0$ and the problem is solved. Therefore we may suppose that $A < B$.

The proof proceeds in three steps. In the first step we check that $d^1(\mathcal{F}^1) = d^2(\mathcal{F}^2) = \theta$ and $d(\mathcal{F}^0) = \rho$. Then, since $\theta < 1$, it follows that the infimum of d^1 and d^2 are less than 1. Now, using Corollary 2.5 and Theorem 2.4 we can reduce our problem to an infimum of an operator norm. In the third step we will prove two lemmas, one to be applied to d^1 and d^2, and the other to d, and this will end the proof.

i) Let us check that (2.10), (2.11), (2.12) achieve the desired values for d^1, d^2 and d, respectively. For $f_i^1 = Q g_i$ with $Q = \frac{\sqrt{A}+\sqrt{B}}{2} S^{-1/2}$ we have $d^1(\mathcal{F}^1) = c(\mathcal{G}, \mathcal{F}^1) = \| 1 - Q^{-1} \|$. Now, $\sqrt{A} \leq S^{1/2} \leq \sqrt{B}$ and these bounds are optimal. Therefore:

$$-\frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}} \leq 1 - Q^{-1} \leq \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}$$

which means $\| 1 - Q^{-1} \| = \theta$. Similar, for $f_i^2 = Lg_i$ with $L = \frac{2\sqrt{AB}}{\sqrt{A}+\sqrt{B}} S^{-1/2}$ we have $d^2(\mathcal{F}^2) = c(\mathcal{F}^2, \mathcal{G}) = \| L - 1 \|$ and a similar calculus shows that $d^2(\mathcal{F}^2) = \theta$.

For $\mathcal{F}^0$ we have $f_i^0 = Rg_i$ with $R = \sqrt[4]{AB} S^{-1/2}$; an easily calculation shows that:

$$\| R - 1 \| = \| 1 - R^{-1} \| = \max(\sqrt[4]{\frac{B}{A}} - 1, 1 - \sqrt[4]{\frac{A}{B}}) = \sqrt[4]{\frac{B}{A}} - 1.$$

Therefore:

$$d(\mathcal{F}^0) = \log(1 + \max(\| R - 1 \|, \| 1 - R^{-1} \|)) = \log \sqrt[4]{\frac{B}{A}} = \rho$$

ii) Since we are looking for the infimum of the functions d^1, d^2 and since $\theta < 1$ we may restrict our attention to the tight frames $\mathcal{F} \in \mathcal{T}^1$ (or in $\mathcal{T}^2$) such that $d^1(\mathcal{F}) < 1$ (respectively $d^2(\mathcal{F}) < 1$). But this implies also that $d^2(\mathcal{F}) < \infty$ (respectively $d^1(\mathcal{F}) < \infty$). Therefore we may restrict our attention to tight frames in $\mathcal{T}^1 \cap \mathcal{T}^2 = \mathcal{T}$.

Corrolary 2.5 tells us that these frames must have the form: $\mathcal{F} = (f_i)_{i \in \mathbb{I}}$ and $f_i = \sqrt{C} U g_i^{\#} = \sqrt{C} U S^{-1/2} g_i$ for some $C > 0$ and U unitary. Hence:

$$d^1(\mathcal{F}) = \| 1 - \frac{1}{\sqrt{C}} S^{1/2} U^{-1} \| = \| \frac{1}{\sqrt{C}} S^{1/2} - U \| \tag{2.16}$$

$$d^2(\mathcal{F}) = \| \sqrt{C} U S^{-1/2} - 1 \| = \| \sqrt{C} S^{1/2} - U \| \tag{2.17}$$

$$d^0(\mathcal{F}) = \max(\| \frac{1}{\sqrt{C}} S^{1/2} - U \|, \| \sqrt{C} S^{-1/2} - U \|) \tag{2.18}$$

To minimize d is equivalent to minimize d^0; since d^0 has a simpler expression, we prefer to work with d^0 from now on.

Thus, our problem is reduced to find minima of the operator norms (2.16), (2.17), (2.18) subject to $C > 0$ and U unitary.

iii) The next step is to solve these norm problems. For d^1 and d^2 we apply the following lemma to be proved later:

LEMMA 2.10 *Consider R a selfadjoint operator on H with $a = \| R^{-1}\|^{-1}$ and $b = \| R \|$. Then, the solution of the following inf-problem:*

$$\mu = \inf_{\substack{\alpha > 0 \\ U \text{ unitary}}} \| \alpha R - U \| \tag{2.19}$$

is given by $\mu = \frac{b-a}{b+a}$ and $\alpha = \frac{2}{a+b}$. This infimum is achieved by the identity operator; any other unitary U that achieves the infimum must have 1 in its spectrum.

If we apply this lemma with $R = S^{1/2}$, $\alpha = \frac{1}{\sqrt{C}}$ and $a = \sqrt{A}$, $b = \sqrt{B}$, then we get $\mu = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}} \equiv \theta$ and $\alpha = \frac{2}{\sqrt{A}+\sqrt{B}}$, hence the parametrization (2.13) of the solutions. This proves (2.16). For (2.17) we apply the lemma with $R = S^{-1/2}$, $\alpha = \sqrt{C}$ and $a = \frac{1}{\sqrt{B}}$, $b = \frac{1}{\sqrt{A}}$. We get $\mu = \theta$ and $\alpha = \frac{2\sqrt{AB}}{\sqrt{A}+\sqrt{B}}$, hence the parametrization (2.14) of the solutions.

For d we need a similar lemma, but this time for another optimization problem:

LEMMA 2.11 *Consider R a bounded invertible selfadjoint operator on H with $a = \| R^{-1}\|^{-1}$ and $b = \| R \|$. Then, the solution of the following optimization problem:*

$$\mu = \inf_{\substack{\alpha > 0 \\ U \text{ unitary}}} \max(\| \alpha R - U \|, \| \frac{1}{\alpha} R^{-1} - U \|) \tag{2.20}$$

is given by $\mu = \sqrt{\frac{b}{a}} - 1$, $\alpha = \frac{1}{\sqrt{ab}}$ and U in the set:

$$\{U : H \to H \,|\, U \text{ unitary and } \| \frac{1}{\sqrt{ab}} R - U \| = \| \sqrt{ab} R^{-1} - U \| = \sqrt{\frac{b}{a}} - 1\} \tag{2.21}$$

Moreover, the set (2.24) contains the identity and therefore is not empty, and the spectrum of any U contains 1.

The solution for d^0 is now straightforward: we apply this lemma to (2.18) with $R = S^{1/2}$, $\alpha = \frac{1}{\sqrt{C}}$ and $a = \sqrt{A}$, $b = \sqrt{B}$. We get $\mu = \min d^0 = \sqrt[4]{\frac{B}{A}} - 1$ and $\alpha = \frac{1}{\sqrt[4]{AB}}$, hence the parametrization (2.15) of the solution and the proof of theorem is complete. $\Box$

It still remains to prove the two lemmas:

Proof of Lemma 2.10

We denote by $\sigma(X)$ the spectrum of the operator X. Thus $a, b \in \sigma(R)$. Now, by Weyl's criterion (see for instance [ReSi80]), there are two sequences of normed vectors in H, $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ such that $\| v_n \| = \| w_n \| = 1$ and $\lim_n \| (R-a)v_n \| = 0$, $\lim_n \| (R-b)w_n \| = 0$.

Let $\delta = \alpha - \frac{2}{a+b}$. Suppose $\delta > 0$. Let $\varepsilon = \frac{\delta}{2}b$. Then there exists an index N such that for any $n > N$, $\| Rw_n - bw_n \| \leq \frac{\varepsilon}{\alpha}$. We get $\| \alpha R w_n \| \geq \alpha b - \varepsilon > 1$ and:

$$\| (\alpha R - U)w_n \| \geq | \, \| \alpha R w_n \| - \| U w_n \| \, | = \| \alpha R w_n \| - 1 \geq \alpha b - \varepsilon - 1 = \frac{b-a}{b+a} + \varepsilon$$

Therefore:

$$\| \alpha R - U \| \geq \frac{b-a}{b+a} + \varepsilon > \frac{b-a}{b+a} = \mu \tag{2.22}$$

Suppose now $\delta < 0$. Let $\varepsilon = -\frac{\delta}{2}a > 0$. Then, there exists an N such that for any $n > N$, $\| Rv_n - av_n \| \leq \frac{\varepsilon}{\alpha}$. We get $\| \alpha R v_n \| \leq \alpha a + \varepsilon < 1$ and:

$$\| (\alpha R - U)v_n \| \geq | \, \| \alpha R v_n \| - \| U v_n \| \, | = 1 - \| \alpha R v_n \| \geq 1 - \alpha a - \varepsilon = \frac{b-a}{b+a} + \varepsilon$$

Therefore:

$$\| \alpha R - U \| \geq \frac{b-a}{b+a} + \varepsilon > \frac{b-a}{b+a} = \mu \tag{2.23}$$

From (2.22) and (2.23) we observe that the infimum of $\| \alpha R - U \|$ has the value $\frac{b-a}{b+a}$ and may be achieved only if $\delta = 0$, i.e. $\alpha = \frac{2}{a+b}$. Thus, the first part of the lemma has been proved.

The set of all unitary U that achieve the infimum is then given by:

$$\{U : H \to H \,|\, U \text{ unitary and } \| \frac{2}{a+b}R - U \| = \frac{b-a}{b+a}\} \tag{2.24}$$

We still have to prove that the set (2.24) contains the identity and 1 is in spectrum of any unitary operator from this set.

From $a \leq R \leq b$ we get $-\frac{b-a}{b+a} \leq \frac{2}{a+b}R - 1 \leq \frac{b-a}{b+a}$. Therefore $\| \frac{2}{a+b}R - 1 \| \leq \frac{b-a}{b+a}$. But, as we have proved, $\frac{b-a}{b+a}$ is the minimum that can be achieved. Therefore $\| \frac{2}{a+b}R - 1 \| = \frac{b-a}{b+a} = \mu$ and thus 1 is in the set (2.24).

Now recall the sequence $(v_n)_n$ and the inequality (2.22) which is realized on $(v_n)_n$. For U in the set (2.24) we have: $\| (\frac{2}{a+b}R - U)v_n \| \to \mu$. But:

$$\| (\frac{2}{a+b}R - U)v_n\|^2 = \frac{4}{(a+b)^2} < v_n, R^2 v_n > - \frac{2}{a+b} < v_n, (RU + U^*R)v_n > +1$$

From $(R-a)v_n \to 0$ we get $< v_n, R^2 v_n > \to a^2$. Therefore:

$$\lim_n < v_n, (RU + U^*R)v_n > = \frac{a+b}{2}\left(\frac{4a^2}{(a+b)^2} + 1 - \sigma^2\right) = 2a$$

Now:

$$RU + U^*R = (R-a)U + U^*(R-a) + a(U + U^*)$$

and the previous limit gives $\lim_n < v_n, (U + U^*)v_n > = 2$.

Therefore:

$$\| (U-1)v_n\|^2 = < v_n, (2 - (U + U^*))v_n > \to 0$$

or $\lim_n \| (U-1)v_n \| = 0$ which proves $1 \in \sigma(U)$. $\square$

Proof of Lemma 2.11

First, let us solve the following scalar problem:

$$\bar{\mu} = \inf_{\alpha > 0} \max\left(\max_{a \le x \le b} |\alpha x - 1|, \max_{a \le x \le b} \left|\frac{1}{\alpha x} - 1\right|\right) \tag{2.25}$$

Because of monotonicity:

$$\max_{a \le x \le b} |\alpha x - 1| = \max\left(|\alpha a - 1|, |\alpha b - 1|\right)$$

$$\max_{a \le x \le b} \left|\frac{1}{\alpha x} - 1\right| = \max\left(\left|\frac{1}{\alpha a} - 1\right|, \left|\frac{1}{\alpha b} - 1\right|\right)$$

Therefore $\bar{\mu} = \inf_{\alpha > 0} f(\alpha)$ where $f(\alpha) = \max\left(|\alpha a - 1|, |\alpha b - 1|, \left|\frac{1}{\alpha a} - 1\right|, \left|\frac{1}{\alpha b} - 1\right|\right)$

It is now simple to check that where the infimum is achieved at least two moduli are equal. This condition is fulfilled at the following points:

$$\alpha_1 = \frac{2}{a+b}; \quad \alpha_2 = \frac{1}{a}; \quad \alpha_3 = \frac{1}{a} \pm \frac{1}{a}\sqrt{1 - \frac{a}{b}}; \quad \alpha_4 = \frac{1}{\sqrt{ab}}; \quad \alpha_5 = \frac{1}{b}; \quad \alpha_6 = \frac{a+b}{2ab}$$

We evaluate $f(\alpha)$ at these points and we get:

$$f(\alpha_1) = \frac{b-a}{2a}; \quad f(\alpha_2) = \frac{b-a}{a}; \quad f(\alpha_3) = \frac{\sqrt{b-a}}{a}(\sqrt{b} - \sqrt{b-a})$$

$$f(\alpha_4) = \sqrt{\frac{b}{a}} - 1; \quad f(\alpha_5) = \frac{b-a}{a}; \quad f(\alpha_6) = \frac{b-a}{2a}$$

It is obvious now that: $f(\alpha_4) \le f(\alpha_1) = f(\alpha_6) \le f(\alpha_2) = f(\alpha_5) \le f(\alpha_3)$ and therefore $\bar{\mu} = f(\alpha_4) = \sqrt{\frac{b}{a}} - 1$ and $\alpha_{optim} = \alpha_4 = \frac{1}{\sqrt{ab}}$. Observe also that for $\alpha = \alpha_4$ we have:

$$\max_{a \le x \le b} |\alpha_4 x - 1| = \max_{a \le x \le b} |\frac{1}{\alpha_4 x} - 1|$$

Let us now return to the norm problem (2.20). We are going to prove now that $\mu = \bar{\mu} = \sqrt{\frac{b}{a} - 1}$ is the optimum and $\alpha = \alpha_4 = \frac{1}{\sqrt{ab}}$. As in the previous lemma, consider $(v_n)_{n\ge1}$ and $(w_n)_{n\ge1}$ two sequences of normed vectors in H ($\| v_n \| = \| w_n \| = 1$) such that $\lim_n \| (R-a)v_n \| = 0$, $\lim_n \| (R-b)w_n \| = 0$. It is simple to check that $\lim_n \| (R^{-1} - \frac{1}{a})v_n \| = 0$ and $\lim_n \| (R^{-1} - \frac{1}{b})w_n \| = 0$ hold too. Now, consider some $\alpha > 0$, $\alpha \ne \alpha_4 = \frac{1}{\sqrt{ab}}$. Then, as in the scalar problem above, we have:

$$\text{either} \quad \max_{a \le x \le b} |\alpha x - 1| > \bar{\mu} \quad \text{or} \quad \max_{a \le x \le b} |\frac{1}{\alpha x} - 1| > \bar{\mu} \tag{2.26}$$

Suppose the first inequality holds. Now, either $|\alpha a - 1| > \bar{\mu}$ or $|\alpha b - 1| > \bar{\mu}$. In the former case we use the sequence $(v_n)_n$ as follows: Let $\varepsilon = \frac{1}{2}(|\alpha a - 1| - \bar{\mu}) > 0$ and let N_ε be such that $\| (R-a)v_n \| \le \frac{\varepsilon}{\alpha}$ for any $n \ge N_\varepsilon$. Then:

$$\| (\alpha R - U)v_n \| \ge | \| \alpha R v_n \| - \| U v_n \| | = |\alpha \| a v_n + (R-a)v_n \| - 1| \ge$$

$$\ge |\alpha a - 1| - \alpha \| (R-a)v_n \| > \bar{\mu} + \varepsilon$$

which implies $\| \alpha R - U \| > \bar{\mu} + \varepsilon$.

Similarly, in the later case ($|\alpha b - 1| > \bar{\mu}$) we take $\varepsilon = \frac{1}{2}(|\alpha b - 1| - \bar{\mu}) > 0$ and N_ε such that $\| (R-b)w_n \| \le \frac{\varepsilon}{\alpha}$ for any $n \ge N_\varepsilon$. Therefore:

$$\| (\alpha R - U)w_n \| \ge | \| \alpha R w_n \| - \| U w_n \| | = |\alpha \| b w_n + (R-b)w_n \| - 1| \ge$$

$$\ge |\alpha b - 1| - \alpha \| (R-b)w_n \| > \bar{\mu} + \varepsilon$$

Thus, in both cases we obtain $\| \alpha R - U \| > \bar{\mu}$. If the second inequality in (2.26) holds, a similar argument can be used to prove that, for $\alpha \neq \alpha_4$ we have $\| \frac{1}{\alpha} R^{-1} - U \| > \bar{\mu}$. Therefore the optimum in (2.20) is achieved for $\alpha = \frac{1}{\sqrt{ab}}$ and the value of it is $\mu = \sqrt{\frac{b}{a}} - 1$. It is obvious now that the set of unitary operators that achieve the optimum is given by (2.21) and also that the identity operator is in that set. The only problem that still remains to be proved is that all these unitary operators have 1 in their spectrum.

The previous argument proves the following conclusion: fix $\delta_0 > 0$ small enough and let U be in the set (2.21). Then, since $\bar{\mu} < \|(\alpha R - U)w_n\|$ for all $\alpha \neq \frac{1}{\sqrt{ab}}$ we can substitute $\alpha = \frac{1}{\sqrt{ab}} + \delta$, where $0 < \delta \leq \delta_0$, and obtain:

$$\bar{\mu} \leq \| (\delta R + \frac{1}{\sqrt{ab}} R - U) w_n \|$$

for $n \geq N_\delta$ where N_δ is an integer depending on δ. Then $\bar{\mu} \leq \| (\delta R + \frac{1}{\sqrt{ab}} R - U) w_n \| < \delta \| R \| + \bar{\mu}$ for $n \geq N_\delta$ (use (2.21); it follows that $| \|(\frac{1}{\sqrt{ab}} R - U) w_n\| - \bar{\mu}| \leq 2\delta \|R\|$ can be made arbitrarily small by choosing n sufficiently large, so that $\| (\frac{1}{\sqrt{ab}} R - U) w_n \| \to \bar{\mu}$ when $n \to \infty$. Now, by repeating the argument given in the proof of lemma 2.10 we obtain $\lim_n \| (U-1) w_n \| = 0$ which proves $1 \in \sigma(U)$ and the lemma is proved. □

REMARK 2.12 *We point out that the entire theory can be carried out on the set of Hilbert frames over different Hilbert spaces, but indexed by the same index set. All the results are similar, the changes being straightforward.*

REMARK 2.13 *As a final remark we acknowledge that the two Lemmas 2.1 and 2.2 have also been independently obtained by D.Han and D.R.Larson in a recent paper ([HaLa97]).*

Chapter 3

Stability of Coherent Frames

3.1 General Stability Results

All the stability results for frames known in the literature are based on the perturbation of the identity principle (which says that if A is a bounded linear operator with $\|A\| \leq 1$ then $I + A$ is invertible and $\|(I+A)^{-1}\| \leq (1 - \|A\|)^{-1}$ or small variations of it. These various stability results can be summarized in the following theorem due mainly (in this form) to O.Christensen:

THEOREM 3.1 (Stability Theorem for Hilbert Frames) *Suppose H a separable complex Hilbert space, $\mathbb{I}$ a countable index set and $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ a frame in H with bounds A, B. Consider $\mathcal{G} = (g_i)_{i\in\mathbb{I}}$ a family of vectors in H. If one of the following two conditions is fulfilled $\forall x \in H$:*

$$\text{(Type 1)} \quad (\sum_{i\in\mathbb{I}} |<x, f_i - g_i>|^2)^{1/2} \leq \lambda(\sum_{i\in\mathbb{I}} |<x, f_i>|^2)^{1/2} + \mu\|x\| \tag{3.1}$$

or $\forall n \geq 0$, $c_i \in \mathbb{C}$:

$$\text{(Type 2)} \quad \|\sum_{n\in I_N} c_i(f_i - g_i)\| \leq \lambda\|\sum_{n\in I_N} c_i f_i\| + \mu(\sum_{n\in I_N} |c_i|^2)^{1/2} \tag{3.2}$$

where $(I_n)_{n\geq 0}$ is an increasing sequence of finite subsets of $\mathbb{I}$: $I_0 \subset I_1 \subset I_2 \subset \ldots \subset \mathbb{I}$ such that $\cup_{n\in\mathbb{N}} I_n = \mathbb{I}$, and $\lambda + \frac{\mu}{\sqrt{A}} < 1$; then $(g_i)_{i\in\mathbb{I}}$ is also a frame in H with bounds $A(1 - \lambda - \frac{\mu}{\sqrt{A}})^2$, $B(1 + \lambda + \frac{\mu}{\sqrt{B}})^2$. Moreover, if $\mathcal{F}$ is a Riesz basis then $\mathcal{G}$ is also a Riesz basis. $\square$

This result was first stated by Paley and Wiener in their celebrated paper [PaWi34]. They considered only the stability of Riesz basis and the type 2 condition. Later on, in a different context, Kato ([Kato76]) proved a perturbation theorem which basically incorporates the above theorem. Recently, Christensen and Heil ([Chris95], [ChHe96]) established the link between Kato's perturbation theorem and frames in both Hilbert and Banach contexts.

In this chapter we are going to prove three stability results. The first one refers to general coherent frames and claims that the frame generators set is open in the set of Bessel sequence generators with respect to some topology. The second result extends a long sequence of results in nonharmonic analysis. Using a Kadec-type estimate we give a stractural stability bound for Fourier frames (or more specific, frame sequences). The last theorem extends to general wavelet bases a surprising result due to Daubechies and Tchamitchian, but proved by them for the Meyer's orthogonal wavelet basis only. Basically, this last result shows that the time-frequency density is not a well-defined quantity for wavelet sets.

3.2 Stability of Coherent Frames

Recall from chapter 1 that a *coherent set* is characterized by a *generator* g and a collection of unitary operators $\mathcal{C} = \{U_i\}_{i\in\mathbb{I}}$ obtained by discretizing a continuous unitary representation of a l.c.g. Thus we obtain the following coherent set:

$$\mathcal{S}(\mathcal{C}, g) := \{Ug \ , \ U \in \mathcal{C}\}$$

Let us denote by $\mathbf{B}^{\mathcal{C}}$ the set of Bessel sequence generators and by $\mathbf{F}^{\mathcal{C}}$ the set of frame generators with respect to the collection $\mathcal{C}$. Obviously $\mathbf{F}^{\mathcal{C}} \subset \mathbf{B}^{\mathcal{C}}$.

We introduce now an unbounded operator associated to the collection $\mathcal{C}$ by:

$$T^{\mathcal{C}} : H \to B(H, l^2(\mathbb{I})) \ \ , \ \ (T^{\mathcal{C}} g)(x) = \{< x, U_i g >\}_{i\in\mathbb{I}} \tag{3.3}$$

with domain $D(T^{\mathcal{C}}) = \mathbf{B}^{\mathcal{C}}$. For any $g \in \mathbf{B}^{\mathcal{C}}$ we denote $|||g||| = \|T^{\mathcal{C}} g\|_{B(H,l^2(\mathbb{I}))}$. Is is straightforward to prove that $|||\cdot||| : \mathbf{B}^{\mathcal{C}} \to \mathbb{R}_+$ is a norm and thus $(\mathbf{B}^{\mathcal{C}}, |||\cdot|||)$ is a normed space.

The main result is the following:

THEOREM 3.2 *$T^{\mathcal{C}}$ is closed and $(\mathbf{B}^{\mathcal{C}}, |||\cdot|||)$ is a Banach space.*

Proof

a) $T^{\mathcal{C}}$ is closed: Let $(x_n, T^{\mathcal{C}}x_n)_n$ be a Cauchy sequence on the graph of $T^{\mathcal{C}}$ in $H \times B(H, l^2(\mathbb{I}))$. Then $x_n \in \mathbf{B}^{\mathcal{C}}$, $x_n \to x$ and $T^{\mathcal{C}}x_n \to T$ in $B(H, l^2(\mathbb{I}))$, for some $x \in H$ and $T \in B(H, l^2(\mathbb{I}))$. Since $T \in B(H, l^2(\mathbb{I}))$ we obtain, by applying Riesz lemma, that $T(y) = \{< y, z_i >\}_{i\in\mathbb{I}}$ for some $z_i \in H$. Thus $\{z_i\}_{i\in\mathbb{I}}$ is a Bessel sequence in H. On the other hand, $\|T^{\mathcal{C}}x_n - T\| = k_n$ and $\lim_{n\to\infty} k_n = 0$ which turns into:

$$\sum_{i\in\mathbb{I}} | < y, U_i x_n - z_i > |^2 \leq k_n^2 \|y\|^2 \quad , \quad \forall y \in H$$

It also implies, for all i, $| < y, U_i x_n - z_i > | \leq k_n\|y\|$ and thus $\|U_i x_n - z_i\| \leq k_n$. Therefore $\lim_{n\to\infty} U_i x_n = z_i$ in H. But $\lim_{n\to\infty} x_n = x$ in H and thus $U_i x = z_i$. We obtain that $T^{\mathcal{C}}x = T$ and therefore $x \in \mathbf{B}^{\mathcal{C}}$ and $T^{\mathcal{C}}$ is closed.

b) Since $T^{\mathcal{C}}$ is closed, it follows that $(\mathbf{B}^{\mathcal{C}}, |\!|\!|\cdot|\!|\!|)$ is a Banach space with the norm $\|x\|_1 = \|x\| + |\!|\!|x|\!|\!|$. But:

$$|\!|\!|x|\!|\!| = \|T^{\mathcal{C}}x\|_{B(H,l^2(\mathbb{I}))} = \|(T^{\mathcal{C}}x)^*\|_{B(H,l^2(\mathbb{I}))} \text{ and } (T^{\mathcal{C}}x)^*(c) = \sum_{i\in\mathbb{I}} c_i U_i x$$

Thus:

$$\|(T^{\mathcal{C}}x)^*\|_{B(H,l^2(\mathbb{I}))} \geq \|U_i x\| = \|x\|$$

Therefore $|\!|\!|x|\!|\!| \leq \|x\|_1 \leq 2|\!|\!|x|\!|\!|$ which means that $\|\cdot\|_1$ and $|\!|\!|\cdot|\!|\!|$ are equivalent. □

We have also the following stability result:

THEOREM 3.3 $\mathbf{F}^{\mathcal{C}}$ *is open in* $(\mathbf{B}^{\mathcal{C}}, |\!|\!|\cdot|\!|\!|)$. *More specific, for any* $g \in \mathbf{F}^{\mathcal{C}}$ *if* A *denotes the lower bound of the coherent frame* $\mathcal{S}(\mathcal{C}, g)$ *then the ball of center* g *and radius* $\sqrt{A}$ *is included in* $\mathbf{F}^{\mathcal{C}}$, *i.e.* $B_{\sqrt{A}}(g) \subset \mathbf{F}^{\mathcal{C}}$.

Proof

The second statement implies the first one, therefore we shall prove only the inclusion of that ball. Let $y \in B_{\sqrt{A}}(g)$. Then $\mu = |\!|\!|g - y|\!|\!| < \sqrt{A}$ and:

$$\Big(\sum_{i\in\mathbb{I}} | < z, U_i g - U_i y > |^2\Big)^{1/2} \leq \mu(\|z\|^2)^{1/2} \quad , \quad \forall z \in H$$

Now, by the stability theorem of Hilbert frames 3.1, Type 1 (3.1) with $\lambda = 0$ and μ as above, we obtain that $y \in \mathbf{F}^{\mathcal{C}}$ also. This concludes the proof. □

3.3 Structural Stability of Fourier Frames

Consider $\gamma > 0$ and $L^2[-\gamma,\gamma]$ with the usual scalar product inherited from L^2. Consider a sequence $(\lambda_n)_{n\in\mathbb{Z}}$ of complex numbers and construct the sequence of functions $\mathcal{F} = \{f_n\}_{n\in\mathbb{Z}}$ by $f_n : [-\gamma,\gamma] \to \mathbb{C}$, $f_n(x) = \frac{1}{\sqrt{2\gamma}}e^{i\lambda_n x}$. Recall that we called $(\lambda_n)_{n\in\mathbb{Z}}$ a *frame sequence* if $\mathcal{F}$ is a frame for $L^2[-\gamma,\gamma]$, in which case $\mathcal{F}$ is called a *Fourier frame*. Likewise, $\mathcal{F}^v$ is then a Fourier frame for B^2_γ.

Our problem is the following: given a frame sequence of real numbers $(\lambda_n)_{n\in\mathbb{Z}}$ with bounds A, B, find a positive constant L such that any sequence of real numbers $(\mu_n)_{n\in\mathbb{Z}}$ with $|\mu_n - \lambda_n| \leq \delta < L$ is also a frame sequence. An extension of this problem will take into account the complex case.

In the context of an orthonormal Fourier basis ($\lambda_n = n$, $\gamma = \pi$) this problem was first considered by Paley and Wiener. By using their stability result, they obtained a first value for L, $L_1 = \frac{1}{\pi^2}$. Later on, Duffin and Eachus in [DuEa42] improved this constant to $L_2 = \frac{\ln 2}{\pi} = 0.22...$. Finally, Kadec in [Kadec64] proved that the optimal value of this constant (called the Paley-Wiener constant) is $L_K = \frac{1}{4}$ (earlier, Levinson in [Levin40] proved that for $\delta = \frac{1}{4}$ one can perturb the orthonormal Fourier basis to a noncomplete set).

The stability question of Fourier frames was considered by Duffin and Schaeffer in their seminal paper [DuSc52]. They used a type (3.1) inequality with $\mu = 0$ and they obtained $L_{DS} = \frac{1}{\gamma}\ln[1+\sqrt{\frac{A}{B}}]$ (see proof of Theorem 13, §4.8 in [Youn80]). This value has been used recently by [CvVet95] in a quantization error analysis of Weyl-Heisenberg frame expansions. For $\gamma = \pi$ and $A = B$ one can obtain $L_{DS} = \frac{\ln 2}{\pi}$ which is less than Kadec' estimate. A better estimate for L is given in Theorem 3.4:

THEOREM 3.4 *Suppose $(\lambda_n)_{n\in\mathbb{Z}}$ a frame sequence of real numbers for $L^2[-\gamma,\gamma]$ with bounds A, B. Set:*

$$L(\gamma) = \frac{\pi}{4\gamma} - \frac{1}{\gamma}arcsin(\frac{1}{\sqrt{2}}(1-\sqrt{\frac{A}{B}})) \tag{3.4}$$

Consider the sequence $\{\rho_n\}_{n\in\mathbb{Z}}$ of complex numbers $\rho_n = \mu_n + i\sigma_n$ such that $\sup_n|\mu_n - \lambda_n| = \delta < L(\gamma)$ and $\sup_n|\sigma_n| = M < \infty$. Then the following two conclusions hold true:

1) The sequence $\{\rho_n\}_{n\in\mathbb{Z}}$ is a frame sequence for $L^2[-\gamma,\gamma]$;

2) The real sequence $(\mu_n)_{n\in\mathbb{Z}}$ is a frame sequence with bounds:

$$A(1-\sqrt{\frac{A}{B}}(1-\cos\gamma\delta+\sin\gamma\delta))^2\ ,\ B(2-\cos\gamma\delta+\sin\gamma\delta)^2 \tag{3.5}$$

□

Proof of Theorem 3.4

By Theorem II from [DuSc52] (see also Theorem 14, §4.8 in [Youn80]) we need to prove Theorem 3.4 only for real sequences $\rho_n=\mu_n$. On the other hand, if we scale the sequence we can reduce the problem to the case $\gamma=\pi$. Indeed, if $(\lambda_n)_{n\in\mathbb{Z}}$ is a frame sequence for $L^2[-\gamma,\gamma]$ then $\{\lambda'_n=\frac{\gamma}{\pi}\lambda_n\}_{n\in\mathbb{Z}}$ is a frame sequence for $L^2[-\pi,\pi]$ with the same bounds (in the former case $f_n(x)=\frac{1}{\sqrt{2\gamma}}e^{i\lambda_n x}$, in the latter $f'_n(x)=\frac{1}{\sqrt{2\pi}}e^{i\lambda'_n x}$). Thus $L(\gamma)=\frac{\pi}{\gamma}L(\pi)$ and we have to prove:

$$L(\pi)=\frac{1}{4}-\frac{1}{\pi}arcsin(\frac{1}{\sqrt{2}}(1-\sqrt{\frac{A}{B}})) \tag{3.6}$$

Observe that this is consistent also with the frame bounds since $\gamma\delta=\pi\delta'$.

To prove (3.6), we shall use Kadec' estimations from his theorem and then the Type 2 form of the Stability Theorem with $\lambda=0$. Let $N\in\mathbb{N}$ and $c_n\in\mathbb{C}$, $n\in I_N$ be arbitrary. Set $\delta_n=\mu_n-\lambda_n$. We obtain:

$$U=\|\sum_{n\in I_N}c_n(\frac{1}{\sqrt{2\pi}}e^{i\lambda_n x}-\frac{1}{\sqrt{2\pi}}e^{i\mu_n x})\| \ =\ \frac{1}{\sqrt{2\pi}}\|\sum_{n\in I_N}c_ne^{i\lambda_n x}(1-e^{i\delta_n x})\| \tag{3.7}$$

By expanding $1-e^{i\delta_n x}$ into a Fourier series relative to the orthogonal system $\{1,\cos\nu x,\sin(\nu-\frac{1}{2})x\}$, $\nu=1,2,\ldots$ we obtain:

$$\begin{aligned}1-e^{i\delta_n x} &= (1-\frac{\sin\pi\delta_n}{\pi\delta_n})+\sum_{\nu=1}^{\infty}\frac{(-1)^\nu 2\delta_n\sin\pi\delta_n}{\pi(\nu^2-{\delta_n}^2)}\cos(\nu x)\\ &\quad +i\sum_{\nu=1}^{\infty}\frac{(-1)^\nu 2\delta_n\cos\pi\delta_n}{\pi((\nu-\frac{1}{2})^2-{\delta_n}^2)}\sin((\nu-\frac{1}{2})x)\end{aligned} \tag{3.8}$$

We plug (3.8) into (3.7), we change the order of summation, we use the triangle inequality and then we use the bounds $\|\cos(\nu x)\varphi(x)\|\ \le\ \|\varphi\|$ and $\|\sin((\nu-\frac{1}{2})x)\varphi(x)\|\ \le\ \|\varphi\|$. We obtain:

$$\begin{aligned}U &\le \|\sum_{n\in I_N}(1-\frac{\sin\pi\delta_n}{\pi\delta_n})c_ne^{i\lambda_n x}\|+\sum_{\nu=1}^{\infty}(\|\sum_{n\in I_N}\frac{2\delta_n\sin\pi\delta_n}{\pi(\nu^2-{\delta_n}^2)}c_ne^{i\lambda_n x}\|\\ &\quad +\|\sum_{n\in I_N}\frac{2\delta_n\cos\pi\delta_n}{\pi((\nu-\frac{1}{2})^2-{\delta_n}^2)}c_ne^{i\lambda_n x}\|)\end{aligned}$$

Now we use that $(\lambda_n)_{n\in\mathbb{Z}}$ is a frame sequence with upper bound B. Therefore each norm can be bounded as:

$$\| \sum_{n\in I_N} a_n c_n e^{i\lambda_n x}\| \le \sqrt{B}\|\{a_n c_n\}\| \le \sqrt{B}\sup_n |a_n|\,\|\{c_n\}\|;$$

since we have:

$$|1-\frac{\sin\pi\delta_n}{\pi\delta_n}| \le 1-\frac{\sin\pi\delta}{\pi\delta}$$

$$|\frac{2\delta_n\sin\pi\delta_n}{\pi(\nu^2-{\delta_n}^2)}| \le \frac{2\delta\sin\pi\delta}{\pi(\nu^2-\delta^2)}$$

$$|\frac{2\delta_n\cos\pi\delta_n}{\pi((\nu-\frac{1}{2})^2-{\delta_n}^2)}| \le \frac{2\delta\cos\pi\delta}{\pi((\nu-\frac{1}{2})^2-\delta^2)}$$

(the last inequality holds because $\delta<\frac{1}{4}$); it follows that:

$$U \le \sqrt{B}(Re(1-e^{i\pi\delta})-Im(1-e^{i\pi\delta}))\,(\sum_{n\in I_N}|c_n|^2)^{1/2}$$

or:

$$U \le \sqrt{B}(1-\cos\pi\delta+\sin\pi\delta)\,(\sum_{n\in I_N}|c_n|^2)^{1/2}$$

Now we can apply the Stability Theorem (Type 2) with $\lambda=0$ and $\mu=\sqrt{B}(1-\cos\pi\delta+\sin\pi\delta)$. The condition of that theorem turns into $\mu<\sqrt{A}$ or $1-\cos\pi\delta+\sin\pi\delta<\sqrt{\frac{A}{B}}$ and then, by a little trigonometry we get:

$$\delta < L = \frac{1}{4}-\frac{1}{\pi}arcsin(\frac{1}{\sqrt{2}}(1-\sqrt{\frac{A}{B}}))$$

The frame bounds for $(\mu_n)_{n\in\mathbb{Z}}$ come from $A(1-\frac{\mu}{\sqrt{A}})^2$ and $B(1+\frac{\mu}{\sqrt{B}})^2$. This ends the proof. $\diamond$

3.4 Parametric Stability of Wavelet Riesz Bases

Consider two positive numbers $a_0>1$, $b_0>0$ and a function $\Psi\in L^2(\mathbb{R})$. Recall that a wavelet set $\mathcal{W}_{\Psi;a_0,b_0}$ is defined by $\mathcal{W}_{\Psi;a_0,b_0}=\{\Psi_{mn;a_0b_0}|(m,n)\in\mathbb{Z}^2\}$ where $\Psi_{mn;a_0b_0}(x)=a_0^{-m/2}\Psi(a_0^{-m}x-nb_0)$. If the set $\mathcal{W}_{\Psi;a_0b_0}$ is a frame (respectively, a Riesz basis or a s-Riesz basis) in $L^2(\mathbb{R})$ we call it a *wavelet frame* (a *wavelet Riesz basis* or a *wavelet s-Riesz basis*).

Our problem concerns the behavior of the set $\mathcal{W}_{\Psi;ab}$ when $a=a_0$ and b runs through a neighborhood of b_0. This problem was first considered by Daubechies and Tchamitchian in 1990 for the

Meyer orthogonal wavelet basis (see [Daub90]) where $a_0 = 2$, $b_0 = 1$. They proved that for all b in some nontrivial interval $[1-\varepsilon, 1+\varepsilon]$, the corresponding $\mathcal{F}_{\Psi;2b}$ constituted a Riesz basis; their proof exploited the very particular structure of the Meyer basis. We are going to extend this stability result to a more general class of wavelet Riesz basis, using a different argument. The precise statement is given in Theorem 3.5:

THEOREM 3.5 *Suppose that the function $\Psi \in L^2(\mathbb{R})$ generates a wavelet Riesz basis with bounds A, B for some $a_0 > 1, b_0 > 0$ (i.e. $\mathcal{W}_{\Psi;a_0b_0}$ is a Riesz basis with bounds A, B). Furthermore, let $\hat{\Psi}$, the Fourier transform of Ψ, satisfy the following requirement: $\hat{\Psi}$ is of class $\mathcal{C}^1$ on $\mathbb{R}$ and both $\hat{\Psi}$ and $\hat{\Psi}'$ are bounded by:*

$$|\hat{\Psi}(\xi)|, |\hat{\Psi}'(\xi)| \leq C\frac{|\xi|^\alpha}{(1+|\xi|)^\gamma} \quad , \quad \forall \xi \in \mathbb{R} \tag{3.9}$$

for some $C > 0$ and $\gamma > 1+\alpha > 1$. Then there exists an $\varepsilon > 0$ such that for any b with $|b - b_0| < \varepsilon$, the set $\mathcal{W}_{\Psi;a_0b}$ is a Riesz basis. $\square$

Proof of Theorem 3.5

To prove this theorem, we shall use the Type 1 criterion of stability together with an upper bound estimation given in theorem 1.14 in section 1.3.

Consider Ψ and $a_0 > 1$, $b_0 > 0$ and $b > 0$ as in the hypothesis and denote by $U_b : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the unitary operator $(U_bf)(x) = \sqrt{\frac{b}{b_0}}f(\frac{b}{b_0}x)$. We define $\Phi = U_b\Psi$, or more specifically $\Phi(x) = \sqrt{\frac{b}{b_0}}\Psi(\frac{b}{b_o}x)$. One can easily check that $U_b\Psi_{mn;a_0b} = \Phi_{mn;a_0b_0}$, therefore U_b maps $\mathcal{F}_{\Psi;a_0b}$ into $\mathcal{F}_{\Phi;a_0b_0}$ unitarily. Thus $\mathcal{W}_{\Psi;a_0b}$ is a Riesz basis (respectively, frame) if and only if $\mathcal{W}_{\Phi;a_0b_0}$ is a Riesz basis (frame). Moreover, they have the same bounds. In order to prove that $\mathcal{W}_{\Psi;a_0b}$ is a Riesz basis, we show that $\mathcal{W}_{\Phi;a_0b_0}$ is a Riesz basis by comparing it with $\mathcal{W}_{\Psi;a_0b_0}$. We note that:

$$\Psi_{mn;a_0b_0} - \Phi_{mn;a_0b_0} = (\Psi - \Phi)_{mn;a_0b_0}$$

Therefore the condition (3.1) with $\lambda = 0$ is equivalent with the condition that $\mathcal{F}_{\Psi-\Phi;a_0b_0}$ be a Bessel set with upper bound less than A, the lower frame bound of the Riesz basis $\mathcal{W}_{\Psi;a_0b_0}$.

Let us denote by $B_{\alpha,\gamma}$ the constant B given by (1.42) for $\hat{f}(\xi) = \frac{|\xi|^\alpha}{(1+|\xi|)^\gamma}$. It is simple to check that $|\hat{\Psi}(\xi) - \hat{\Phi}(\xi)| \leq C_b\frac{|\xi|^\alpha}{(1+|\xi|)^\gamma}$. Therefore an upper bound for the Bessel set $\mathcal{F}_{\Psi-\Phi;a_0b_0}$ is given by

$C_b B_{\alpha,\gamma}$. On the other hand, using the Ascoli-Arzelá lemma and the hypotheses on $\hat{\Psi}(\xi)$ and $\hat{\Psi}'(\xi)$ we obtain that $g_b(\xi) = \frac{(1+|\xi|)^\gamma}{|\xi|^\alpha}\hat{\Phi}(\xi)$ converges uniformly to $g_{b_0}(\xi) = \frac{(1+|\xi|)^\gamma}{|\xi|^\alpha}\hat{\Psi}(\xi)$ as $b \to b_0$. Thus we may choose C_b to depend continuously on b around b_0 and $C_{b_0} = 0$. Then, for some neighborhood of b_0 for which $C_b B_{\alpha\gamma} < A$ we may set $\mu = \sqrt{C_b B_{\alpha\gamma}}$ in (3.1) and we obtain that $\mathcal{W}_{\Phi;a_0 b_0}$ is a Riesz basis. Now the proof is complete. $\diamond$

Chapter 4

An Uncertainty Principle for Wavelet Sets

4.1 Introduction

In this chapter we look for lower bounds of various uncertainty quantities. On $L^2(\mathbb{R})$ we introduce the following unbounded selfadjoint operators:

$$Q\,f(x) = x\,f(x)\ ,\ D(Q) = \{f \in L^2(\mathbb{R}) | \int x^2 |f(x)|^2 dx < \infty\} \tag{4.1}$$

$$P\,f(x) = -i\frac{df}{dx}\ ,\ D(P) = \{f \in L^2(\mathbb{R}) |\ f' \in L^2(\mathbb{R})\} \tag{4.2}$$

Now take a $f \in D(P) \cap D(Q)$ with $\|f\| = 1$ and let us denote by:

$$\bar{x} = < f, Qf > \quad , \quad \bar{\xi} = < f, Pf > \tag{4.3}$$

the first moments of $|f|^2$, respectively $|\hat{f}|^2$. Then, by *uncertainty quantity* we mean one of the following products:

$$\Delta_1(f) = \|Pf\| \cdot \|Qf\| \tag{4.4}$$

$$\Delta_2(f) = \|(P - \bar{\xi})f\| \cdot \|(Q - \bar{x})f\| \tag{4.5}$$

$$\Delta_3(f) = \|Pf\| \cdot \|(Q - \bar{x})f\| \tag{4.6}$$

We point out that $\Delta_2(f)$ is the product of the two variances associated to $|f|^2$ and, respectively, $|\hat{f}|^2$. Notice also that $\Delta_1(f) \geq \Delta_3(f) \geq \Delta_2(f)$.

The classical Fourier inequality (or the uncertainty principle) states that for every $f \in L^2(\mathbb{R})$ with $\|f\| = 1$, $\Delta_2(f) \geq \frac{1}{2}$. Thus $\frac{1}{2}$ is an absolute lower bound achieved only by (possibly modulated or translated) gaussian functions.

On the other hand, for Weyl-Heisenberg Riesz bases $\mathcal{WH}_{g;\alpha,\beta}$ (with $\alpha\beta = 1$) the Balian-Low theorem states that $\Delta_1(g) = \infty$ - see [Batt88] for a nice proof.

Thus, one may naturally ask whether this nonlocalization is due to the Riesz basis property. The answer is negative and, in fact, it has been proved by J.Bourgain in [Bourg88] that for every $\varepsilon > 0$ there is an orthonormal basis $\{h_n\}_{n\geq 0}$ of $L^2(\mathbb{R})$ such that $\Delta_2(h_n) < \frac{1}{2} + \varepsilon$. Unfortunatelly his construction does not yield a coherent set. Thus the next question could be whether the coherence is the obstacle for localization. Again the answer is (at least partially) negative because Y.Meyer constructed in [Mey86] an orthonormal wavelet basis that is localized in time-frequency domain (i.e. $\Delta_2(\Psi) < \infty$) - note that the much older orthonormal wavelet basis given by the Haar wavelet basis has uncertainty infinite because of its discontinuity. Since then a lot of other wavelet basis (orthogonal or biorthogonal) have been constructed (see [Daub88]), many of them with good time-frequency localization.

Given these results, we ask whether the lower bound $\frac{1}{2}$ is still optimal for wavelet sets. As we shall see later, the answer is negative and the bound $\frac{1}{2}$ should be replaced by $\frac{3}{2}$. This type of result has been first proved by G.Battle in [Batt97]. He assumed either $\mathcal{W}_{\Psi;ab}$ is an orthonormal set, or (4.8) holds true. We show below that (4.8) is always satisfied for localized wavelet Bessel sequence generators, so that Battle's lower bound $\frac{3}{2}$ will hold for all wavelet Bessel sequence generators. We also give a shorter proof of Battle's result.

4.2 An Uncertainty Inequality for Wavelet Bessel Sequences

Suppose the wavelet set $\mathcal{W}_{\Psi;ab}$ is a Bessel sequence (see chapter 1, section 1.3 for definitions and properties). Then a necessary condition on the wavelet Ψ is the inequality (1.43). If we divide it by ξ and integrate from 1 to a we get:

$$\sum_m \int_1^a \frac{|\hat{\Psi}(a^m\xi)|^2}{\xi} d\xi \leq \frac{b \ln a}{2\pi} B$$

or, equivalently:

$$\int_0^\infty \frac{|\hat{\Psi}(\xi)|^2}{\xi} d\xi \leq \frac{b \ln a}{2\pi} B \tag{4.7}$$

LEMMA 4.1 *Suppose $\mathcal{W}_{\Psi;ab}$ is a wavelet Bessel sequence and $\Psi \in D(Q)$. Then:*

$$\int_{-\infty}^{\infty} \Psi(x) dx = 0 \tag{4.8}$$

Proof

Suppose we proved that Ψ is in $L^1(\mathbb{R})$. Then $\hat{\Psi}$ is continuous and for (4.7) to hold it is necessary that $\hat{\Psi}(0) = 0$. This implies (4.8).

Thus it remains to prove that $\Psi \in L^1(\mathbb{R})$. We know that $Q\Psi \in L^2(\mathbb{R})$. Then:

$$\int |\Psi(x)| dx \leq (\int \frac{1}{(1+|x|)^2} dx)^{1/2} (\int (1+|x|)^2 |\Psi(x)|^2 dx)^{1/2} < \infty$$

Thus $\Psi \in L^1(\mathbb{R})$ and the proof is done. $\square$

Let us introduce two linear spaces and a norm that next will play a very important role :

$$V_0 = \{f \in D(P) \cap D(Q) \mid \int f(x) dx = 0\} \tag{4.9}$$

$$S_0 = \{\varphi \in \mathcal{S} \mid \int \varphi(x) dx = 0\} \tag{4.10}$$

where $\mathcal{S}$ is the Schwartz class of rapidly decreasing functions, and:

$$\|f\|_{(1,1)} = \|f\| + \|P f\| + \|Q f\| \tag{4.11}$$

for which norm the space V_0 is closed.

LEMMA 4.2 *S_0 is dense in V_0 with respect to the norm $\|\cdot\|_{(1,1)}$.*

Proof

Take a $f \in V_0$ and a sequence $\varphi_n \in \mathcal{S}$ such that $\|f - \varphi_n\|_{(1,1)} \to 0$, as $n \to \infty$ (this is possible because $\mathcal{S}$ is dense in $D(p) \cap D(q)$ w.r.t. the norm (4.11)). Choose $G \in \mathcal{S}$ such that $\int G(x) dx = 1$

and set $c_n = \int \varphi_n(x)dx$. Then:

$$|c_n| = |\int(\varphi_n(x) - f(x))dx| \leq \int |\varphi_n(x) - f(x)|dx$$
$$\leq (\int \frac{1}{(1+|x|)^2}dx)^{1/2}(\int (1+|x|)^2|\varphi_n(x) - f(x)|^2dx)^{1/2} \stackrel{n\to\infty}{\longrightarrow} 0$$

Hence $c_n \to 0$ as $n \to \infty$. Set $\varphi_n^0 = \varphi_n - c_n G \in S_0$. We obtain:

$$\|\varphi_n^0 - f\|_{(1,1)} \leq \|\varphi_n - f\|_{(1,1)} + |c_n|\|G\|_{(1,1)} \stackrel{n\to\infty}{\longrightarrow} 0$$

Thus S_0 is dense in V_0. Q.E.D. □

Now we are ready for the main result:

THEOREM 4.3 *If $\mathcal{W}_{\Psi;ab}$ is a wavelet Bessel sequence with $\|\Psi\| = 1$ then for every $c \in \mathbb{R}$:*

$$\|(Q-c)\Psi\| \cdot \|P\Psi\| \geq \frac{3}{2} \tag{4.12}$$

Proof

If $\Psi \notin D(P) \cap D(Q)$ then at least one of the two norms in (4.12) is infinite, the product is infinite as well and therefore (4.12) is trivially checked.

Suppose now $\Psi \in D(P) \cap D(Q)$. By Lemma 4.1 we know $\int \Psi(x)dx = 0$. Thus $\Psi \in V_0$. Let us take a sequence $(\Psi_n)_{n\geq 1}$ in S_0 converging to Ψ in $\|\cdot\|_{(1,1)}$ norm. It follows that $\|\Psi_n\| \to 1$; we can assume, without loss of generality that $\|\Psi_n\| = 1$. Then clearly $\|(Q-c)\Psi_n\| \cdot \|P\Psi_n\| \to \|(Q-c)\Psi\| \cdot \|P\Psi\|$. Thus it is enough to prove (4.12) for Ψ_n. The following argument is taken from Battle's paper ([Batt97]). Let us introduce the following unbounded selfadjoint operator:

$$Z = \frac{1}{2}(P\,Q + Q\,P) \quad , \quad D(Z) = \{f \in D(P) \cap D(Q) | P\,Q\,f \in L^2(\mathbb{R})\} \tag{4.13}$$

Since $P\,Q - Q\,P = -i$ we get $Z = P\,Q + \frac{i}{2} = Q\,P - \frac{i}{2}$. Note also:

$$P\,Z = Z\,P - i\,P \tag{4.14}$$

Since $\Psi_n \in S_0$, it follows that $x \mapsto \varphi_n(x) = i\int_{-\infty}^x \Psi_n(t)dt$ is a map in $L^2(\mathbb{R})$ and $\Psi_n = P\varphi_n$. Then:

$$\|(Q-c)\Psi_n\| = \|(Q\,P - c\,P)\varphi_n\| = \|(Z - c\,P + \frac{i}{2})\varphi_n\| = \|(Z - c\,P - \frac{i}{2})\varphi_n\|$$

where for the second equality we used (4.14) and the last equality is due to the fact that $Z - cP$ and $\frac{1}{2}\mathbf{1}$ are commuting selfadjoint operators. Next:

$$\|(Q-c)\Psi_n\| \cdot \|P\Psi_n\| \geq |< P\Psi_n, (Z - cP - \frac{i}{2})\varphi_n >| = |< \Psi_n, P(Z - cP - \frac{i}{2})\varphi_n >|$$
$$= |< \Psi_n, (Z - cP - \frac{3}{2}i)P\varphi_n >| = |< \Psi_n, (Z - cP)\Psi_n > + \frac{3}{2}i| \geq \frac{3}{2}$$

because $< \Psi_n, (Z - cP)\Psi_n >$ is a real number. Thus (4.12) holds for Ψ_n and this proves that it holds for Ψ as well. Q.E.D. □

From this theorem we draw immediately the following corollary:

COROLLARY 4.4 *If $\mathcal{W}_{\Psi;ab}$ is a wavelet Bessel sequence with $\|\Psi\| = 1$ then:*

1. $\Delta_1 \geq \frac{3}{2}$, $\Delta_3 \geq \frac{3}{2}$;
2. *If Ψ is a real-valued function then $\Delta_2 \geq \frac{3}{2}$.*

Proof

1. is straightforward;

2. If Ψ is real-valued, then $|\hat{\Psi}|$ is even and $\bar{\xi} = 0$. Thus $\Delta_2(\Psi) = \Delta_3(\Psi)$ and we are done. □

4.3 Uncertainty Inequalities for Higher-Order Vanishing Moment Wavelets

In [Batt97], G.Battle was interested to find lower bounds for quantities of the form $\|P^n\Psi\| \cdot \|Q^n\Psi\|$ when Ψ is a n^{th} order vanishing moment wavelet. On the other hand, our interest lays in finding lower bounds for the uncertainty quantities introduced in (4.4)–(4.6), when Ψ is a n^{th} order vanishing moment wavelet. Here I shall present a result proving the previous estimates are optimal for higher order vanishing moment wavelets too.

A n^{th} order vanishing moment wavelet means a function $\Psi \in L^2(\mathbb{R})$ such that the following integrals are well-defined and vanish:

$$\int \Psi(x)dx = 0 \ , \quad \int x\Psi(x)dx = 0 \ , \ \ldots \ , \quad \int x^n\Psi(x)dx = 0 \tag{4.15}$$

Let us introduce the following space:

$$S_n = \{\varphi \in \mathcal{S} | \varphi \text{ is a } n^{th} \text{ order vanishing moment wavelet}\} \tag{4.16}$$

Note that S_0 agrees with the definition (4.10) and $S_n \subset S_{n-1} \subset S_0 \subset V_0$ are linear spaces. Our problem concerns the infimum of the uncertainty quantities (4.4)–(4.6) for $f \in S_n$, $\|f\| = 1$. The next lemma tells us an important property regarding S_n:

LEMMA 4.5 *For every $n \geq 0$, S_n is dense in V_0 with respect to the topology induced by the norm $\|\cdot\|_{(1,1)}$.*

Proof

The proof follows in two steps. In the first step we shall construct a nice system of vectors in S_n. In the second step we shall use this system to approximate arbitrary elements in V_0 by functions from S_n.

Consider $\varphi^0, \varphi^1, \ldots, \varphi^n$ a set of n functions in $\mathcal{S}$ which are biorthogonal to $1, x, x^2, \ldots, x^n$ in the following sense:

$$\int_{-\infty}^{\infty} x^j \varphi^l(x) dx = \delta_{jl} \quad , \quad j = 0, 1, 2, \ldots, n \tag{4.17}$$

Such functions exist and are easy to construct; for instance if we denote $h^j(x) = e^{-x^2/2} \cdot x^j \in \mathcal{S}$, we can denote by $\{\tilde{h}^1, \ldots, \tilde{h}^n\}$ the standard biorthogonal s-Riesz basis of the s-Riesz basis $\{h^1, \ldots, h^n\}$ in $L^2(\mathbb{R})$ (the $\tilde{h}^l$'s can be expressed in terms of Hermite polynomials); then we can take $\varphi^l(x) = e^{-x^2/2}\tilde{h}^l(x) \in \mathcal{S}$. Next let us denote:

$$\varphi^l_\varepsilon(x) = \varepsilon^{l+1}\varphi^l(\varepsilon x) \tag{4.18}$$

a special scaling of φ^l's. One can check that (4.17) is invariant under this normalization, i.e. $\int x^j \varphi^l_\varepsilon(x) dx = \delta_{jl}$. The Fourier transform of $\hat{\varphi^l_\varepsilon}(\xi) = \varepsilon^l \hat{\varphi}^l(\frac{\xi}{\varepsilon})$. Note also the following relations that will be useful later:

$$\begin{aligned} \|\varphi^l_\varepsilon\| &= \varepsilon^{l+\frac{1}{2}}\|\varphi^l\| \\ \|Q\varphi^l_\varepsilon\| &= \varepsilon^{l-\frac{1}{2}}\|Q\varphi^l\| \\ \|P\varphi^l_\varepsilon\| &= \varepsilon^{l+\frac{3}{2}}\|P\varphi^l\| \end{aligned}$$

We prove now that S_n is dense in S_0. Since S_0 is dense in V_0 (Lemma 4.2), the conclusion will then follow.

Choose an arbitrary $f \in S_0$. Fix $\varepsilon > 0$. Set $a_j = \int x^j f(x)dx$, $j = 0, 1, \ldots, n$. Note that $a_0 = 0$ by the definition of S_0. Let us denote by $g_\delta = f - \sum_{j=1}^n a_j \varphi_\delta^j$. We have $g_\delta \in S_n$ and:

$$\|f - g_\delta\|_{(1,1)} \leq \sum_{j=1}^{n} |a_j| \cdot \|\varphi_\delta^j\|_{(1,1)} \leq \sum_{j=1}^{n} \delta^{j-\frac{1}{2}} |a_j| \cdot \|\varphi^j\|_{(1,1)} \leq \delta^{\frac{1}{2}} \sum_{j=1}^{n} |a_j| \cdot \|\varphi^j\|_{(1,1)}$$

Thus, by choosing $\delta < \varepsilon^2 / (\sum_{j=1}^n |a_j| \cdot \|\varphi^j\|_{(1,1)})^2$ we obtain $\|f - g_\delta\| < \varepsilon$ and the proof is done. □

From this lemma we get immediately the following result.

THEOREM 4.6 *For every $\varepsilon > 0$ and $n \geq 0$ there is a n^{th} order vanishing moment wavelet $\Psi \in S_n$ such that $\Delta_j(\Psi) \leq \frac{3}{2} + \varepsilon$, $j = 1, 2, 3$.*

Proof

We know the Hermite function $H_1(x) = (\pi)^{-1/2} x e^{-x^2/2}$ achieves the lower bound $\frac{3}{2}$. However it does not belong to S_n in general (except for the case $n = 0$). Yet, since S_n is dense in S_0, we can approximate H_1 by a sequence $\Psi_\varepsilon \in S_n$, $\Psi_\varepsilon \to H_1$, in $\|\cdot\|_{(1,1)}$-norm as $\varepsilon \to 0$. The convergence in $\|\cdot\|_{(1,1)}$-norm implies the convergence of $\Delta_k(\Psi_\varepsilon)$ to $\Delta_j(H_1) = \frac{3}{2}$, $j = 1, 2, 3$. □

REMARK 4.7 *This result shows that we cannot say more about the uncertainty products based solely on the number of vanishing moments. To obtain larger lower bounds we need to know more about the wavelet.*

Chapter 5

Approximation of Stochastic Signals by Weyl-Heisenberg Pairs

5.1 Weyl-Heisenberg Pairs and Signal Models

By a (deterministic) signal we mean a function f belonging to some Banach space X that will be specified by the context. The Banach space will be either $L^2(\mathbb{R})$ if the signal has finite energy, or a weighted L^2_w space, for a suitable weight w, if the signal has infinite energy but finite power. When dealing with stochastic signals, it is assumed that a probability space (Ω, Σ, μ) is given and the stochastic model will represent a (measurable) map from Ω into X satisfying some additional conditions.

In this section we are interested in describing certain ways to approximate a signal, whether deterministic or stochastic, by coherent Weyl-Heisenberg pairs.

In 1992, P.J. Munch analyzed the dependency of colored noise optimal reduction on the redundancy for a particular class of WH frames (see [Munch92]). Our goal is to analyze how the approximation error depends on the deficit when using a WH s-Riesz basis.

Consider now two functions $g^1, g^2 \in L^2(\mathbb{R})$ and two positive numbers $\alpha > 0, \beta > 0$.

Definition The four-tuple $(g^1, g^2; \alpha, \beta)$ is called a *Weyl-Heisenberg pair* (or, shortly, a *WH pair*) if $\mathcal{WH}_{g^1;\alpha,\beta}$ and $\mathcal{WH}_{g^2;\alpha,\beta}$ are both WH Bessel sequences.

The pair $(g^1, g^2; \alpha, \beta)$ is called a *standard biorthogonal WH pair* if $\mathcal{WH}_{g^1;\alpha,\beta}$ and $\mathcal{WH}_{g^2;\alpha,\beta}$ are

WH s-Riesz bases and the latter is the standard biorthogonal WH s-Riesz basis of the former.

The pair $(g^1, g^2; \alpha, \beta)$ is called a *standard dual WH pair* if $\mathcal{WH}_{g^1;\alpha,\beta}$ and $\mathcal{WH}_{g^2;\alpha,\beta}$ are both WH frames and the latter is the standard dual frame of the former.

Suppose $(g^1, g^2; \alpha, \beta)$ is a WH pair. Then the following frame-like operator is bounded and well-defined:

$$S_{g^1,g^2;\alpha,\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \quad , \quad S_{g^1,g^2;\alpha,\beta}(f) = \sum_{m,n\in\mathbb{Z}} < f, g^1_{mn} > g^2_{mn} \tag{5.1}$$

Similarly the grammian-like operator:

$$G_{g^1,g^2;\alpha,\beta} : l^2(\mathbb{Z}^2) \to l^2(\mathbb{Z}^2) \quad , \quad G_{g^1,g^2;\alpha,\beta}(c) = \sum_{m',n'\in\mathbb{Z}} < g^1_{m'n'}, g^2_{mn} > c_{m'n'} \tag{5.2}$$

is bounded and well-defined (see [RnShn96]). We shall call $S_{g^1,g^2;\alpha,\beta}$ the *frame operator* and $G_{g^1,g^2;\alpha,\beta}$ the *grammian operator* of the pair $(g^1, g^2; \alpha, \beta)$.

We restrict our attention to WH pairs satisfying two additional assumptions:

$$A1. \text{ The pair has a deficit} \frac{1}{\alpha\beta} < 1 \tag{5.3}$$

$$A2. \ \mathcal{WH}_{g^1;\alpha,\beta} \text{ and } \mathcal{WH}_{g^2;\alpha,\beta} \text{ are WH s-Riesz bases.} \tag{5.4}$$

Under these assumptions, $Ran\, S_{g^1,g^2;\alpha,\beta}$ and $Ran\, {S_{g^1,g^2;\alpha,\beta}}^*$ are both proper closed subspaces of $L^2(\mathbb{R})$ and $S_{g^1,g^2;\alpha,\beta}$ is thus not invertible for any choice of g^1, g^2 (see [Rief81] and [RaSt95]).

For a signal (or a signal model) f in a Banach space X our goal is to analyze how close its coherent approximation $S_{g^1,g^2;\alpha,\beta}f$, given by a WH pair $(g^1, g^2; \alpha, \beta)$, is to the original signal (i.e. $\|f - S_{g^1,g^2;\alpha,\beta}f\|_X$). To do this we need to introduce certain signal models.

5.1.1 Deterministic Model

Suppose $f \in L^2(\mathbb{R})$ is an unknown deterministic signal and $(g^1, g^2; \alpha, \beta)$ is a WH pair. Then the approximation error is: $\mathcal{E}(f) = f - S_{g^1,g^2;\alpha,\beta}f \in L^2(\mathbb{R})$ and the error measure is $\|\mathcal{E}(f)\|$. Since f is unknown and we did not make any à priori assumption about the signal, we should consider the *worst-case*, namely:

$$\sup_{\substack{f\in L^2(\mathbb{R})\\ \|f\|=1}} \|\mathcal{E}(f)\|$$

Thus a measure of the approximation error given by a WH pair $(g^1, g^2; \alpha, \beta)$ is given by the operator norm:

$$J(g^1, g^2; \alpha, \beta) = \|1 - S_{g^1,g^2;\alpha,\beta}\|_{B(L^2(\mathbb{R}))} \tag{5.5}$$

where $B(L^2(\mathbb{R}))$ stands for the space of bounded operators on $L^2(\mathbb{R})$.

In this model, our optimization problem is to find g^1, g^2 that minimizes (5.5) for a given set of parameters α, β and subject to the assumption $A2$ made before:

$$arg \min_{\substack{g^1,g^2\\ A2 \text{ holds}}} \|1 - S_{g^1,g^2;\alpha,\beta}\| \tag{5.6}$$

Despite its rather complicated statement, the optimization problem (5.6) has a very simple solution with a nice geometric interpretation. As we shall prove further, the optimum in (5.6) is 1 if $\alpha\beta > 1$ and 0 if $\alpha\beta \leq 1$, and is achieved for a large class of optimizers (g^1, g^2).

To find the optimum in (5.6) notice that for $f \in Ker\, S_{g^1,g^2;\alpha,\beta}$, $\|f\| = 1$ we obtain $\|f - S_{g^1,g^2;\alpha,\beta} f\| = \|f\| = 1$. Therefore $\|1 - S_{g^1,g^2;\alpha,\beta}\| \geq 1$ for every WH pair satisfying $A1$ and $A2$. Let us show now that the bound 1 is actually achieved. We claim that $J(g^1, g^2; \alpha, \beta) = 1$ for any standard biorthogonal WH pair. Indeed, suppose $(g^1, g^2; \alpha, \beta)$ is a standard biorthogonal WH pair, then $S_{g^1,g^2;\alpha,\beta}$ is the orthogonal projection onto the span $\mathcal{E}$ of $\mathcal{WH}_{g^1;\alpha,\beta}$ and therefore $1 - S_{g^1,g^2;\alpha,\beta}$ is the orthogonal projection $P_{\mathcal{E}^\perp}$ onto the orthogonal complement $\mathcal{E}^\perp$ of $\mathcal{E}$. Hence:

$$J(g^1, g^2; \alpha, \beta) = \|1 - S_{g^1,g^2;\alpha,\beta}\| = \|P_{\mathcal{E}^\perp}\| = 1$$

Conversely, the following fact holds: Suppose $(g^1, g^2; \alpha, \beta)$ is a minimizer of (5.6), i.e. a WH pair satisfying $A1$ and $A2$, and $J(g^1, g^2; \alpha, \beta) = 1$. Let us denote by $\tilde{g}^2$ the generator of the standard biorthogonal WH s-Riesz basis $\mathcal{WH}_{\tilde{g}^2;\alpha,\beta}$ of $\mathcal{WH}_{g^2;\alpha,\beta}$. Choose an arbitrary $f \in L^2(\mathbb{R})$. Then f has a unique orthogonal decomposition:

$$f = f^{\perp} + \sum_{m,n\in\mathbb{Z}} < f, \tilde{g}^2_{mn} > g^2_{mn} \quad , \quad f^{\perp} \perp g^2_{mn} \ , \ \forall m,n$$

Then:

$$\|(1 - S_{g^1,g^2;\alpha,\beta})f\|^2 = \|f^{\perp} + \sum_{m,n} < f, (\tilde{g}^2 - g^1)_{mn} > g^2_{mn}\|^2 =$$

$$= \|f^{\perp}\|^2 + \|\sum_{m,n} < f, (\tilde{g}^2 - g^1)_{mn} > g^2_{mn}\|^2 \geq \|f^{\perp}\|^2 = \|(1 - S_{\tilde{g}^2,g^2;\alpha,\beta})f\|^2$$

Thus, for any minimizer $(g^1, g^2; \alpha, \beta)$, the biorthogonal WH pair $(\tilde{g}^2, g^2; \alpha, \beta)$ has an approximation error that is pointwise smaller than that of the minimizer (in the strong operator sense). We have thus proved the following result:

THEOREM 5.1 *For any fixed α, β, the optimal value of the optimization problem* $\min_{g^1,g^2} \|1 - S_{g^1,g^2;\alpha,\beta}\|$ *is given by:*

$$J^*(\alpha, \beta) = \begin{cases} 1 & , \quad \alpha \cdot \beta > 1 \\ 0 & , \quad \alpha \cdot \beta \leq 1 \end{cases} \tag{5.7}$$

For $\alpha\beta > 1$, any biorthogonal WH pair $(g, \tilde{g}; \alpha, \beta)$ is a minimizer of (5.6)*. Conversely, if $(g^1, g^2; \alpha, \beta)$ is a minimizer of* (5.6) *then, for every $f \in L^2(\mathbb{R})$:*

$$\|(1 - S_{g^1,g^2;\alpha,\beta})f\| \geq \|(1 - S_{\tilde{g}^2,g^2;\alpha,\beta})f\| \tag{5.8}$$

where $\mathcal{WH}_{\tilde{g}^2,g^2;\alpha,\beta}$ is the standard biorthogonal WH s-Riesz basis of $\mathcal{WH}_{g^2;\alpha,\beta}$.

Notice the discontinuity of the optimal value $J^*(\alpha, \beta)$, as a function of α, β, at the threshold value $\alpha\beta = 1$ (see Figure 5.1), where the incomplete set $\mathcal{WH}_{g;\alpha,\beta}$ may cross from being incomplete when $\alpha\beta > 1$ to overcomplete when $\alpha\beta < 1$. For the stochastic models presented below, we shall obtain a continuous transition from 1 to 0 (see Figure 5.2).

5.1.2 Stochastic Models

We present two stochastic models: one is nonstationary in terms of second-order statistics, the other is stationary. Since we are working for time-moments defined on the entire real line, the stationary model will require a special class of functions, as we shall see further.

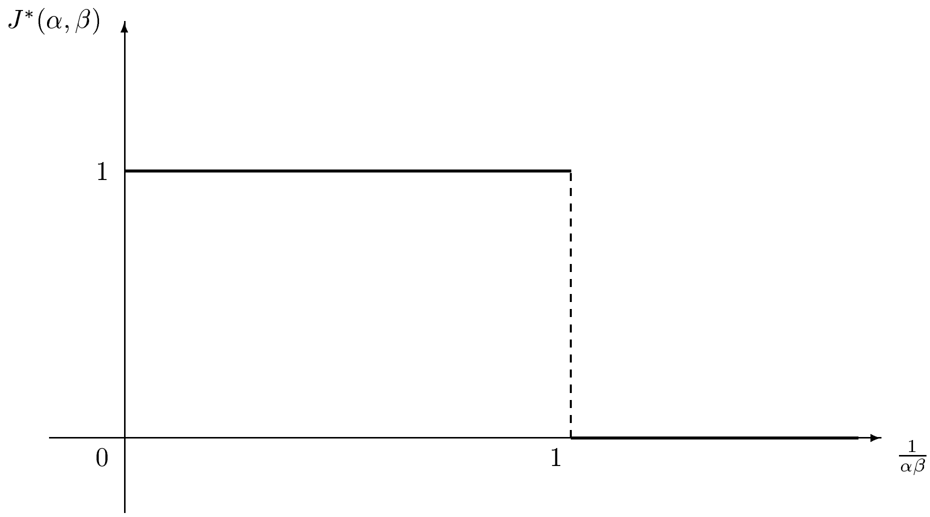

Figure 5.1: Distortion-deficit characteristic for the deterministic model

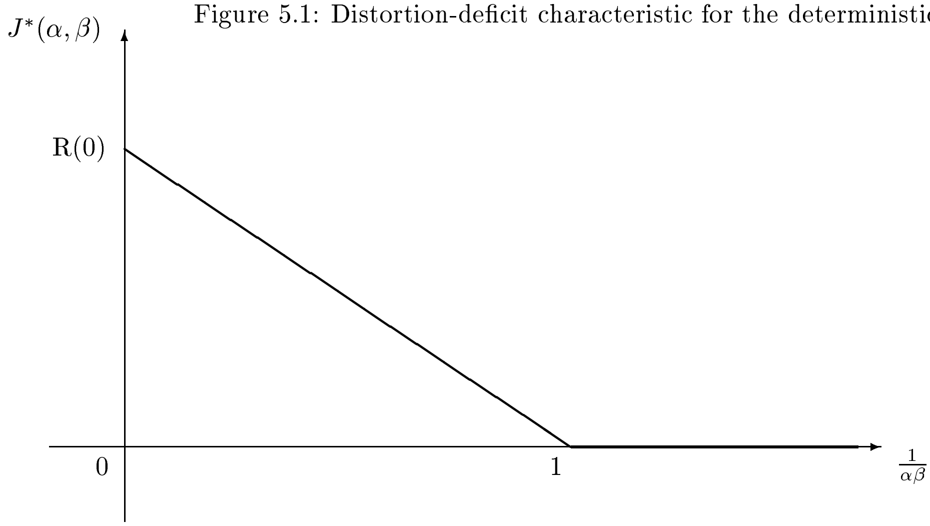

Figure 5.2: Distortion-deficit characteristic for the stationary stochastic model given by (5.85)

The Nonstationary Model. Basically we assume the existence of a probability space (Ω, Σ, μ) and a $L^2(\mathbb{R})$-valued random variable $\mathbf{f} = (f_\omega)_{\omega\in\Omega}$ with the following first- and second-order statistics:

$$\begin{array}{l} \mathbf{Ef}(t) = 0 \quad , \quad \text{a.e. } t \\ \mathbf{Ef}(t)\overline{\mathbf{f}(s)} = R(t,s) \end{array} \tag{5.9}$$

where the expectation $\mathbf{E}$ means $\mathbf{Ef}(t) = \int_\Omega f_\omega(t) d\mu(\omega)$ and $\mathbf{Ef}(t)\overline{\mathbf{f}(s)} = \int_\Omega f_\omega(t)\overline{f_\omega(s)} d\mu(\omega)$. Consider also a WH pair $(g^1, g^2; \alpha, \beta)$. Then the measure of the approximation error is taken as:

$$J(g^1, g^2; \alpha, \beta) = \mathbf{E}\|\mathbf{f} - S_{g^1,g^2;\alpha,\beta}\mathbf{f}\|^2 \tag{5.10}$$

Beside the usual assumptions $A1$ and $A2$ made on the WH pair $(g^1, g^2; \alpha, \beta)$ we also ask the following condition regarding the second-order statistics:

$$\int_{-\infty}^{\infty} R(t,t) dt < \infty \tag{5.11}$$

This condition is necessary and sufficient for $J(g^1, g^2; \alpha, \beta)$ to be finite. We also point out that a stationary model does not obey (5.11) and therefore the analysis needs to be different (see below).

At a more abstract level of formalism, (5.10) and (5.11) can also be understood through the following scheme: Let us denote by $L^2(\Omega; L^2(\mathbb{R}))$ the Hilbert space of $L^2(\mathbb{R})$-valued square integrable functions on Ω with respect to the probability measure μ. Since $\mathcal{E} = 1 - S_{g^1,g^2;\alpha,\beta}$ is a bounded operator on $L^2(\mathbb{R})$, it also lifts to a bounded operator on $L^2(\Omega; L^2(\mathbb{R}))$. For an element $\mathbf{f} \in L^2(\Omega; L^2(\mathbb{R}))$ we have:

$$\|\mathbf{f}\|^2_{L^2(\Omega;L^2(\mathbb{R}))} = \int_{-\infty}^{\infty} R(t,t) dt < \infty$$

and:

$$\|\mathcal{E}(\mathbf{f})\|^2_{L^2(\Omega;L^2(\mathbb{R}))} = J(g^1, g^2; \alpha, \beta)$$

Thus for every nonstationary stochastic model obeying (5.11) the approximation error measure is given by:

$$J(g^1, g^2; \alpha, \beta) = \|\mathcal{E}(\mathbf{f})\|^2_{L^2(\Omega;L^2(\mathbb{R}))} \tag{5.12}$$

(see Figure 5.3). The optimization problem that is to be solved becomes:

$$\inf_{\substack{g^1,g^2 \\ A2 \text{ holds}}} J(g^1, g^2; \alpha, \beta) = \inf_{\substack{g^1,g^2 \\ A2 \text{ holds}}} \mathbf{E}\|(1 - S_{g^1,g^2;\alpha,\beta})\mathbf{f}\|^2 \tag{5.13}$$

for given parameters $\alpha, \beta > 1$ and a given autocovariance function $R(t, s)$.

$$\begin{tikzcd}
L^2(\Omega; L^2(\mathbf{R})) \arrow[rr, "S_{g^1,g^2;\alpha,\beta}"] & & L^2(\Omega; L^2(\mathbf{R})) \\
L^2(\mathbf{R}) \arrow[rr, "S_{g^1,g^2;\alpha,\beta}"] & {} \arrow[u] & L^2(\mathbf{R}) \\
\Omega \arrow[u] & & \Omega \arrow[u]
\end{tikzcd}$$

Figure 5.3: The lifting scheme in the nonstationary case

The Stationary Stochastic Model. As we have mentioned before, we cannot use $L^2(\mathbb{R})$ as space of functions for a stationary random variable, since (5.11) would not be satisfied. Therefore we have to choose a different space of functions. First, let us present the "ingredients" of this model, namely the statistics:

$$\begin{aligned} \mathbf{Ef}(t) &= 0 \\ \mathbf{Ef}(t)\overline{\mathbf{f}(s)} &= R(t-s) \end{aligned} \tag{5.14}$$

We can no longer use $L^2(\mathbb{R})$ as the space in which we consider the signals or realizations, because $\mathbf{E}(\|f\|^2_{L^2}) = \infty$. Instead, we shall be interested in working in weighted L^2-spaces,

$$L^2_w = \{f \ ; \ \int_{-\infty}^{\infty} |f(x)|^2 w(x) dx < \infty\} \tag{5.15}$$

for some non-negative weight function w. It will be convenient to work also with a "periodized" version of L^2_w given by the amalgam space

$$W_\beta(L^2_w, l^\infty) = \{f : \mathbb{R} \to \mathbb{C} \mid \|f\|_{W_\beta(L^2_w, l^\infty)} := \sup_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} w(x)|f(x - n\beta)|^2 dx\} \tag{5.16}$$

This space has better properties with respect to translation (a bounded operator in $W_\beta(L^2_w, l^\infty)$) than L^2_w (in which translation by a finite amount need not be bounded). A special case is given by the choice $\beta = 1$, $w = \mathbf{1}_{[0,1]}$; in this case we obtain the standard amalgam space:

$$W(L^2, l^\infty) := \{f : \mathbb{R} \to \mathbb{C} | \|f\|^2_{W(L^2,l^\infty)} := \sup_n (\int_n^{n+1} |f(x)|^2 dx)^{1/2} < \infty\} \tag{5.17}$$

(see [FouSte85] for a review of properties) In this subsection we shall show that, under certain conditions, $S_{g^1,g^2;\alpha,\beta}$ can be defined as a bounded operator on $W(L^2, l^\infty)$ or $W_\beta(L^2_w, l^\infty)$.

Note that the norm $\|\cdot\|_{W(L^2,l^\infty)}$ is not translation invariant. However one can replace it by a translation invariant equivalent norm, namely:

$$\|f\|_{W(L^2,L^\infty)} := \sup_y (\int_y^{y+1} |f(x)|^2 dx)^{1/2} < +\infty; \tag{5.18}$$

we have:

$$\|f\|_{W(L^2,l^\infty)} \le \|f\|_{W(L^2,L^\infty)} \le \sqrt{2}\|f\|_{W(L^2,l^\infty)} \tag{5.19}$$

Moreover, instead of intervals of length 1 taken in (5.17), we can choose intervals of arbitrary length, say, a and obtain an equivalent norm related to $\|\cdot\|_{W(L^2,l^\infty)}$. Indeed, if $a < 1$ we have:

$$\sqrt{\frac{a}{a+2}}\|f\|_{W(L^2,l^\infty)} \le \sup_n (\int_{na}^{(n+1)a} |f(x)|^2)^{1/2} \le \sqrt{2}\|f\|_{W(L^2,l^\infty)}; \tag{5.20}$$

if $a > 1$ then:

$$\frac{1}{\sqrt{3}}\|f\|_{W(L^2,l^\infty)} \le \sup_n (\int_{na}^{(n+1)a} |f(x)|^2 dx)^{1/2} \le \sqrt{a+2}\|f\|_{W(L^2,l^\infty)} \tag{5.21}$$

Note that $L^p \subset W(L^2, l^\infty)$, for every $p \ge 2$ and also that $(W(L^2, l^1))^* = W(L^2, l^\infty)$ where:

$$W(L^2, l^1) := \{f : \mathbb{R} \to \mathbb{C} \mid \|f\|_{W(L^2,l^1)} := \sum_{n\in\mathbb{Z}} (\int_n^{n+1} |f(x)|^2 dx)^{1/2} < \infty\}$$

and $*$ denotes the dual Banach space (see [FouSte85])

We assume the stochastic model is given by an element $\mathbf{f}$ of $L^2(\Omega; W(L^2, l^\infty))$ (this is the space of $W(L^2, l^\infty)$-valued functions on Ω that are square integrable with respect to the probability measure

μ for a probability space (Ω, Σ, μ)) having the first two orders statistics given by (5.14), i.e.

$$\int_{\Omega} f_{\omega}(t) d\mu(\omega) = 0 \quad , \quad \text{a.e. } t \in \mathbb{R} \tag{5.22}$$

$$\int_{\Omega} f_{\omega}(t)\overline{f_{\omega}(s)} d\mu(\omega) = R(t-s) \quad , \quad \forall t, s \in \mathbb{R} \tag{5.23}$$

where $f_{\omega}(t) := (\mathbf{f}(\omega))(t)$. The assumption $\mathbf{f} \in L^2(\Omega; W(L^2, l^{\infty}))$ yields the following bound on the autocovariance function. Take $t = s$ in (5.23); since the result $R(0)$ is independent of t, we can integrate it over any interval of length 1 and still obtain $R(0)$. Thus:

$$R(0) = \sup_n \int_{\Omega} \int_n^{n+1} |f_{\omega}(t)|^2 dt\, d\mu(\omega) \leq\leq \int_{\Omega} \sup_n \int_n^{n+1} |f_{\omega}(t)|^2 dt\, d\mu(\omega) = \|\mathbf{f}\|^2_{L^2(\Omega; W(L^2, l^{\infty}))}$$

Hence:

$$\|R\|_{\infty} = R(0) \leq \|\mathbf{f}\|^2_{L^2(\Omega; W(L^2, l^{\infty}))}; \tag{5.24}$$

the assumption that $f \in L^2(\Omega; W(L^2, l^{\infty}))$ enables us to control the signal spectral power.

Note however that we cannot control the $L^2(\Omega; W(L^2, l^{\infty}))$-norm of $\mathbf{f}$ by any measure of R. Thus the assumption that the model is given by an element of $L^2(\Omega; W(L^2, l^{\infty}))$ seems to be slighty stronger than just giving a stationary stochastic model on $W(L^2, l^{\infty})$.

Let us consider a WH pair $(g^1, g^2; \alpha, \beta)$ satisfying $A1$ and $A2$. We know that the frame operator $S_{g^1, g^2; \alpha, \beta}$ is well-defined and bounded on $L^2(\mathbb{R})$. Our goal is to extend it to a bounded and well-defined operator on $W(L^2, l^{\infty})$. The next theorem gives sufficient conditions for this to happen. It is strongly inspired by a similar result in [Waln94]. However there is an important difference due to the fact that the set of compactly supported C^{∞} functions is not dense in $W(L^2, l^{\infty})$. Thus we have to deal directly with the $W(L^2, l^{\infty})$ functions. We shall choose g^1 and g^2 to be in the space of functions $W(L^{\infty}, l^1)$, introduced in section 1.3 (1.33) (we recall here its definition:

$$W(L^{\infty}, l^1) := \{f : \mathbb{R} \to \mathbb{C} \mid \|f\|_{W(L^{\infty}, l^1)} := \sum_{n \in \mathbb{Z}} ess \sup_{x \in [n, n+1]} |f(x)| < \infty\} \ ).$$

For this space as for $W(L^2, l^{\infty})$, one can again use a different interval length than 1 and obtain equivalent norms. The translation invariant equivalent norm is $\|f\|_{W(L^{\infty}, L^1)} := \sup_{y \in [0,1]} \sum_n \|f \cdot \mathbf{1}_{[y+n, y+n+1]}\|_{\infty}$. Note that $W(L^{\infty}, l^1)$ is densly imbedded in L^p, for every $p \geq 1$.

THEOREM 5.2 *Suppose $g^1, g^2 \in W(L^\infty, l^1)$.*

a) Let $f \in W(L^2, l^\infty)$ and $\alpha, \beta > 0$. Then $\sum_{m,n} < f, g^1_{mn} > g^2_{mn}$ converges in L^2_{loc}, i.e. for every compact K, there is a function $f_K \in L^2(K)$ such that

$$\lim_{N\to\infty} \lim_{M\to\infty} \|\mathbf{1}_K \sum_{n=-N}^{N} \sum_{m=-M}^{M} < f, g^1_{mn} > g^2_{mn} - f_K\|_{L^2(K)} = 0$$

and the convergence is independent of the order in which we let M and N tend to ∞.

b) If two compact sets K_1 and K_2 have non-empty intersection $K = K_1 \cap K_2$, then $f_{K_1}|_K = f_K = f_{K_2}|_K$. It follows that we can define a unique function on $\mathbb{R}$, which we denote by $S_{g^1,g^2;\alpha,\beta} f$, such that $S_{g^1,g^2;\alpha,\beta} f|_K = f_K$ for all compact K.

c) The series $\sum_{m,n} < f, g^1_{mn} > g^2_{mn}$ converges unconditionally to $S_{g^1,g^2;\alpha,\beta} f$ in the L^2_{loc} topolgy, i.e. for every $\varepsilon > 0$ and compact K there are $N_\varepsilon, M_\varepsilon > 0$ such that for every finite set $S \subset \mathbb{Z}^2 \setminus ([-M_\varepsilon, M_\varepsilon] \times [-N_\varepsilon, N_\varepsilon])$:

$$\| \sum_{(m,n)\in S} < f, g^1_{mn} > g^2_{mn}\|_{L^2(K)} < \varepsilon$$

and it converges also in the weak-$*$ topology of $W(L^2, l^\infty)$, i.e. for every $h \in W(L^2, l^1)$ and $\varepsilon > 0$ there are $M_\varepsilon, N_\varepsilon > 0$ such that for every $N > N_\varepsilon$, $M > M_\varepsilon$

$$| < h, f - \sum_{|m|\le M_\varepsilon} \sum_{|n|\le N_\varepsilon} < f, g^1_{mn} > g^2_{mn} > | < \varepsilon$$

THEOREM 5.3 *For every $\alpha, \beta > 0$ there is some constant $C = C(g^1, g^2; \alpha, \beta)$ such that for every $f \in W(L^2, l^\infty)$, the function defined by $S_{g^1,g^2;\alpha,\beta} f = \sum_{m,n\in\mathbb{Z}} < f, g^1_{mn} > g^2_{mn}$ is in $W(L^2, l^\infty)$ and $\|S_{g^1,g^2;\alpha,\beta} f\|_{W(L^2,l^\infty)} \le C\|f\|_{W(L^2,l^\infty)}$. Therefore $S_{g^1,g^2;\alpha,\beta}$ is a well-defined and bounded operator on $W(L^2, l^\infty)$. Moreover the constant C can be chosen as $C = C_{\alpha,\beta}\|g^1\|_{W(L^\infty,l^1)}\|g^2\|_{W(L^\infty,l^1)}$.*

Proof of Theorem 5.2

a) Consider $g^1, g^2 \in W(L^\infty, l^1)$ and $\alpha, \beta > 0$, $f \in W(L^2, l^\infty)$. We have:

$$\begin{aligned} |c_{mn}| &\le \sum_{l\in\mathbb{Z}} \int_{l\beta}^{(l+1)\beta} |f(x)| \cdot |g^1(x - n\beta)| dx \\ &\le \sum_{l\in\mathbb{Z}} \|g^1 \mathbf{1}_{[(l-n)\beta,(l-n+1)\beta]}\|_\infty \sqrt{\beta} (\int_{l\beta}^{(l+1)\beta} |f(x)|^2 dx)^{1/2} \end{aligned}$$

Using now (5.20) or (5.21) and similar inequalities for $W(L^\infty, l^1)$ we get:

$$|c_{mn}| \leq C\|g^1\|_{W(L^\infty,l^1)} \cdot \|f\|_{W(L^2,l^\infty)}$$

Next we prove the convergence as $M, N \to \infty$ (in either order) on intervals of the form $[\frac{N_0}{\alpha}, \frac{N_0+1}{\alpha}]$. Since every compact is covered by a finite union of such intervals, the conclusion of part a) will then follow immediately. Let us analyze the series $\sum_{m\in\mathbb{Z}} < f, g^1_{mn} > g^2_{mn}$ for fixed $n \in \mathbb{Z}$ on $I = [\frac{N_0}{\alpha}, \frac{N_0+1}{\alpha}]$. We obtain, using the Parseval identity:

$$\begin{aligned} \|\sum_{m\in\mathbb{Z}} < f, g^1_{mn} > g^2_{mn} \cdot \mathbf{1}_I\|^2_{L^2(I)} &= \int_I |\sum_m e^{2\pi i m\alpha x} c_{mn} g^2(x-n\beta)|^2 dx \\ &\leq \sup_{x\in I} |g^2(x-n\beta)|^2 \cdot \frac{1}{\alpha}\sum_m |c_{mn}|^2 \end{aligned}$$

And again by Parseval identity:

$$\begin{aligned} \sum_m |c_{mn}|^2 &= \sum_m |\int_0^{\frac{1}{\alpha}} e^{-2\pi i m\alpha x} \sum_l f(x+\frac{l}{\alpha})\overline{g^1(x+\frac{l}{\alpha}-n\beta)}dx|^2 \\ &= \frac{1}{\alpha}\int_0^{\frac{1}{\alpha}} |\sum_l f(x+\frac{l}{\alpha})\overline{g^1(x+\frac{l}{\alpha}-n\beta)}|^2 dx \end{aligned}$$

Now by the triangle inequality and Cauchy-Schwarz we obtain:

$$(\sum_m |c_{mn}|^2)^{1/2} \leq \frac{1}{\sqrt{\alpha}} \sum_l \|f(\cdot+\frac{l}{\alpha})g^1(\cdot+\frac{l}{\alpha}-n\beta)\|_{L^2(0,\frac{1}{\alpha})} \leq$$

$$\leq \frac{1}{\sqrt{\alpha}}\sum_l \|g^1(\cdot+\frac{l}{\alpha}-n\beta)\|_{L^\infty(0,\frac{1}{\alpha})}\|f(\cdot+\frac{l}{\alpha})\|_{L^2(0,\frac{1}{\alpha})} \leq \tilde{C}\|f\|_{W(L^2,l^\infty)}\|g^1\|_{W(L^\infty,l^1)}$$

Therefore:

$$\|\sum_{m\in\mathbb{Z}} < f, g^1_{mn} > g^2_{mn} \cdot \mathbf{1}_I\|_{L^2(I)} \leq \tilde{C}\sup_{x\in I}|g^2(x-n\beta)| \cdot \|g^1\|_{W(L^\infty,l^1)}\|f\|_{W(L^2,l^\infty)} \tag{5.25}$$

and:

$$\|\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} < f, g^1_{mn} > g^2_{mn} \cdot \mathbf{1}_I\|_{L^2(I)} \leq C_2\|g^2\|_{W(L^\infty,l^1)}\|g^1\|_{W(L^\infty,l^1)}\|f\|_{W(L^2,l^\infty)} \tag{5.26}$$

where C_2 is a constant depending on α and β only. (5.25) proves that for every $n \in \mathbb{Z}$ there is a function $h_n \in L^2(I)$ such that $\|\sum_{|m|\leq M} < f, g^1_{mn} > g^2_{mn} - h_n\|_{L^2(I)} \stackrel{M\to\infty}{\longrightarrow} 0$, i.e. $h_n =$

$\sum_m < f, g^1_{mn} > g^2_{mn}$. Moreover, $\sum_n \|h_n\|_{L^2(I)} \leq \infty$, therefore there is a $f_I \in L^2(I)$ such that $\sum_{|n|\leq N} h_n \overset{n\to\infty}{\longrightarrow} f_I$ in $L^2(I)$. Choose an $\varepsilon > 0$. Then there is a $N^{(1)}_\varepsilon > 0$ such that for every $N > N^{(1)}_\varepsilon$, $\|\sum_{|n|<N} h_n - f_I\|_{L^2(I)} < \frac{\varepsilon}{3}$. Also, since $g^2 \in W(L^\infty, l^1)$, there is a $N^{(2)}_\varepsilon > 0$ such that $\sum_{|n|>N^{(2)}_\varepsilon} \|g^2(\cdot - n\beta)\|_{L^\infty(I)} < \varepsilon(3\tilde{C}\|f\|_{W(L^2,l^\infty)}\|g^1\|_{W(L^\infty,l^1)})^{-1}$. Choose $N_\varepsilon = max(N^{(1)}_\varepsilon, N^{(2)}_\varepsilon)$. On the other hand, for each $|n| \leq N_\varepsilon$ there is a $M_{\varepsilon,n}$ such that:

$$\| \sum_{|m|\leq M} < f, g^1_{mn} > g^2_{mn} - h_n\|_{L^2(I)} \leq \frac{\varepsilon}{3(2N_\varepsilon+1)} \quad , \quad \text{for every } M \geq M_{\varepsilon,n}$$

Choose now $M_\varepsilon = max_{n=-N_\varepsilon,\cdots,N_\varepsilon} M_{\varepsilon,n}$. We get, for every $N > N_\varepsilon$ and $M > M_\varepsilon$:

$$\begin{aligned}
&\| \sum_{|n|\leq N} \sum_{|m|\leq M} < f, g^1_{mn} > g^2_{mn} - f_I\|_{L^2(I)} \\
&\qquad \leq \sum_{N_\varepsilon<|n|\leq N} \|(\sum_{|m|\leq M} e^{2\pi i m\alpha\cdot} < f, g^1_{mn} >) g^2(\cdot - n\beta)\|_{L^2(I)} \\
&\qquad\quad + \sum_{|n|\leq N_\varepsilon} \| \sum_{|m|\leq M} < f, g^1_{mn} > g^2_{mn} - h_n\|_{L^2(I)} + \|\sum_{|n|\leq N_\varepsilon} h_n - f_I\|_{L^2(I)}
\end{aligned}$$

The last two terms are bounded by $\frac{\varepsilon}{3}$. The first one is also bounded by $\frac{\varepsilon}{3}$ as follows:

$$\begin{aligned}
&\sum_{N_\varepsilon<|n|\leq N} \|g^2(\cdot - n\beta) \sum_{|m|\leq M} e^{2\pi i m\alpha\cdot} < f, g^1_{mn} > \|_{L^2(I)} \leq \\
&\leq \sum_{N_\varepsilon<|n|\leq N} \|g^2(\cdot - n\beta)\|_{L^\infty(I)} \| \sum_{|m|\leq M} e^{2\pi i m\alpha\cdot} < f, g^1_{mn} > \|_{L^2(I)} \leq \\
&\leq \sum_{N_\varepsilon<|n|\leq N} \|g^2(\cdot - n\beta)\|_{L^\infty(I)} (\sum_m | < f, g^1_{mn} > |^2)^{1/2} \leq \\
&\leq \sum_{N_\varepsilon<|n|} \|g^2(\cdot - n\beta)\|_{L^\infty(I)} \tilde{C}\|f\|_{W(L^2,l^\infty)}\|g^1\|_{W(L^\infty,l^1)} < \frac{\varepsilon}{3}
\end{aligned}$$

Hence $\|\sum_{|n|\leq N}\sum_{|m|\leq M} < f, g^1_{mn} > g^2_{mn} - f_I\|_{L^2(I)} < \varepsilon$, for every $M > M_\varepsilon$, $N > N_\varepsilon$, which proves the convergence of the series, regardless of the order of summation.

b) Follows immediately from the construction.

c) For the unconditionallity it is sufficient to consider compacts of the form $I = [\frac{N_0}{\alpha}, \frac{N_0+1}{\alpha}]$. Choose an arbitrary $\varepsilon > 0$. Take N_ε such that $\sum_{|n|>N_\varepsilon} \|g^2(\cdot - n\beta)\|_{L^\infty(I)} < \varepsilon(2\tilde{C}\|f\|_{W(L^2,l^\infty)}\|g^1\|_{W(L^\infty,l^1)})^{-1}$ and for every $|n| \leq N_\varepsilon$ find $M_{\varepsilon,n}$ such that $(\sum_{|m|>M_{\varepsilon,n}} | < f, g^1_{mn} > |^2)^{1/2} < \varepsilon(2(2N_\varepsilon+1)\|g^2\|_\infty)^{-1}$. Then set $M_\varepsilon = \max_{|n|\leq N_\varepsilon} M_{\varepsilon,n}$.

Let $S = \cup_n S_n$, where $S_n = \{(m,n) \in S\}$, be the partition of S into subsets of points with the same index n. Then:

$$\|\sum_{(m,n)\in S} < f, g^1_{mn} > g^2_{mn}\|_{L^2(I)} \leq \sum_{|n|>N_\varepsilon} \|g^2(\cdot - n\beta)\|_{L^\infty(I)}(\sum_{(m,n)\in S_n} |< f, g^1_{mn} >|^2)^{1/2} + \\ + \sum_{|n|\leq N_\varepsilon} \|g^2(\cdot - n\beta)\|_{L^\infty(I)}(\sum_{(m,n)\in S_n} |< f, g^1_{mn} >|^2)^{1/2} \quad (5.27)$$

and by a similar computation as before:

$$(\sum_{(m,n)\in S_n} |< f, g^1_{mn} >|^2)^{1/2} \leq (\sum_m |< f, g^1_{mn} >|^2)^{1/2} \leq \tilde{C}\|f\|_{W(L^2,l^\infty)}\|g^1\|_{W(L^\infty,l^1)}$$

Therefore the first term in (5.27) is bounded by $\frac{\varepsilon}{2}$. The second term is bounded as follows, for a fixed $|n| \leq N_\varepsilon$:

$$\|\sum_{(m,n)\in S_n} |< f, g^1_{mn} >|^2\|_{L^2(I)} \leq (\sum_{|m|\geq M_\varepsilon} |< f, g^1_{mn} >|^2)^{1/2} \leq \frac{\varepsilon}{2(2N_\varepsilon+1)\|g^2\|_\infty}$$

Therefore the second term in (5.27) is bounded again by $\frac{\varepsilon}{2}$ and thus the left-hand side of (5.27) is bounded by ε. This proves that the series that defines $S_{g^1,g^2;\alpha,\beta}f$ converges unconditionally in L^2_{loc}.

For the weak-$*$ convergence, take an arbitrary $h \in W(L^2, l^1)$. Choose an $\varepsilon > 0$. Using a similar computation as for (5.26) we get:

$$\|\sum_{|m|\leq M}\sum_{|n|\leq N} < f, g^1_{mn} > g^2_{mn}\|_{W(L^2,l^\infty)} \leq C_3\|g^1\|_{W(L^\infty,l^1)}\|g^2\|_{W(L^\infty,l^1)}\|f\|_{W(L^2,l^\infty)}$$

for any $M, N > 0$ and C_3 depending on α, β only (and thus independent of M, N, f, g^1 or g^2).

Let $N_0 > 0$ be large enough such that

$$\sum_{|n|\geq N_0}(\int_{n/\alpha}^{(n+1)/\alpha} |h(x)|^2 dx)^{1/2} < \frac{\varepsilon}{2\|f\|_{W(L^2,l^\infty)}(\sqrt{2+\frac{1}{\alpha}} + C_3\|g^1\|_{W(L^\infty,l^1)}\|g^2\|_{W(L^\infty,l^1)})}$$

Now choose $N_\varepsilon, M_\varepsilon$ such that for every $N > N_\varepsilon$, $M > M_\varepsilon$,

$$\|f - \sum_{|m|\leq M}\sum_{|n|\leq N} < f, g^1_{mn} > g^2_{mn}\|_{L^2[-\frac{N_0}{\alpha},\frac{N_0+1}{\alpha}]} \leq \frac{\varepsilon}{2\|k\|_{L^2}} \quad (5.28)$$

This is possible since the series converges for every compact K in $L^2(K)$ to f and $W(L^2,l^1) \subset W(L^2,l^2) = L^2(\mathbb{R})$. Then, using Cauchy-Schwarz:

$$|< h, f - \sum_{|m|\le M}\sum_{|n|\le N} < f, g^1_{mn} > g^2_{mn} > | \le \sum_{|n|>N_0} \int_{n/\alpha}^{(n+1)/\alpha} |h(x)| \cdot |f(x) - \sum_{|m|\le M}\sum_{|n|\le N} < f, g^1_{mn} > g^2_{mn}(x)|dx + \int_{-N_0/\alpha}^{(N_0+1)/\alpha} |h(x)| \cdot |f(x) - \sum_{|m|\le M}\sum_{|n|\le N} < f, g^1_{mn} > g^2_{mn}(x)|dx$$

The first term is bounded by $\frac{\varepsilon}{2}$ because of (5.20,5.21), the second term is bounded again by $\frac{\varepsilon}{2}$ because of (5.28). Thus we get the conclusion and the proof is done. □

Proof of Theorem 5.3

The conclusion follows immediately from (5.26). □

REMARK 5.4 If $g^1, g^2 \in W(L^\infty, l^1)$, then g^1 and g^2 are WH Bessel sequence generators. For general WH Bessel sequence generators, the frame operator $S_{g^1,g^2;\alpha,\beta}$ need not be bounded and well-defined on $W(L^2, l^\infty)$, however, as the following example shows:

EXAMPLE 5.5 Consider the following partition of the unit interval $[0,1]$:

$$I_0 = [0, \frac{1}{2}] \quad , \quad I_1 = [\frac{1}{2}, \frac{3}{4}] \quad , \quad I_2 = [\frac{3}{4}, \frac{7}{8}] \quad , \quad \ldots \quad , \quad I_n = [\frac{2^n - 1}{2^n}, \frac{2^{n+1}-1}{2^{n+1}}], \quad \ldots$$

Thus $\cup_{n\ge 0} I_n = [0,1]$ and $I_l \cap I_s = \emptyset$, for $l \ne s$. Consider now the set:

$$S = \cup_{n\ge 0}(n + I_n) = [0, \frac{1}{2}] \cup [\frac{3}{2}, \frac{7}{4}] \cup [\frac{11}{4}, \frac{23}{8}] \cup \cdots$$

where $n + I_n = [n + \frac{2^n-1}{2^n}, n + \frac{2^{n+1}-1}{2^{n+1}}]$. Let the window g^1 be the characteristic function of S, i.e. $g^1 = \mathbf{1}_S$, and choose $g^2 = \mathbf{1}_{[0,1]}$. For $\alpha = \beta = 1$ one can easily check that $\mathcal{WH}_{g^1;\alpha,\beta}$ and $\mathcal{WH}_{g^2;\alpha,\beta}$ are both orthonormal bases for $L^2(\mathbb{R})$, therefore they are WH Bessel sequences and both hypotheses $A1$ and $A2$ are fulfilled. Therefore $S_{g^1,g^2;\alpha,\beta}$ is a well-defined and bounded operator (in fact unitary) on $L^2(\mathbb{R})$.

Consider now the function $f \in W(L^2, l^\infty)$ defined by: $f = \sum_{n\ge 0} 2^{(n+1)/2}\mathbf{1}_{n+I_n}$ and, additionally, the function $\tilde{f} = \sum_{n\ge 0} 2^{(n+1)/2}\mathbf{1}_{I_n}$. Note that $\|f\|_{W(L^2,l^\infty)} = 1$; moreover, for $p < 2$ $f, \tilde{f} \in L^p$;

however, $f, \tilde{f} \notin L^2$. The coefficients of f with respect to $\mathcal{WH}_{g^1;\alpha,\beta}$ are:

$$c_{mn} =< f, g^1_{mn} >= \delta_{n,0} \int_0^1 e^{-2\pi imx} \tilde{f}(x) dx$$

Therefore:

$$\sum_{|m|\leq M} \sum_{|n|\leq N} c_{mn} g^2_{mn}(y) = (\sum_{|m|\leq M} e^{2\pi imy} \int_0^1 e^{-2\pi imx} \tilde{f}(x)\, dx) \mathbf{1}_{[0,1]}(y)$$

By Plancherel's theorem, we have therefore:

$$\begin{aligned} \| \sum_{|m|\leq M} \sum_{|n|\leq N} c_{mn} g^2_{mn} \cdot \|^2_{L^2([0,1])} &= \sum_{|m|\leq M} | \int_0^1 e^{-2\pi imx} \tilde{f}(x)\, dx|^2 \\ &\overset{M\to\infty}{\longrightarrow} \|\tilde{f}\|^2_{L^2[0,1]} = \infty \end{aligned}$$

Thus $S_{g^1,g^2;\alpha,\beta} f$ can be defined in distributional sense (note $(c_{mn})_{m\in\mathbb{Z}} \in l^{p'}$, $\forall n$ and $p' = (1-\frac{1}{p})^{-1}$) but will not be in $W(L^2, l^\infty)$ (in fact it is not even in L^2_{loc}).

REMARK 5.6 The previous example shows that one can have WH Bessel sequences even if $g^1, g^2 \notin W(L^\infty, l^1)$. In fact, one can even find $g^1, g^2 \notin W(L^\infty, l^1)$ for which $S_{g^1,g^2;\alpha,\beta}$ is a bounded operator on $W(L^2, l^\infty)$, as shown in the example below. The condition $g^1, g^2 \in W(L^\infty, l^1)$ in Theorem 5.2 is therefore not necessary.

EXAMPLE 5.7 Consider the same partitions as before. Set:

$$g^1 = \sum_{n\geq 0} \frac{1}{(n+1)^{\alpha+\frac{1}{2}}} \mathbf{1}_{n+I_n}$$

where $0 < \alpha \leq \frac{1}{2}$, and $g^2 = 1_{[0,1]}$. Note that $g^1 \in W(L^\infty, l^2)$, but $g^1 \notin W(L^\infty, l^p)$ for any $p \leq (\alpha+\frac{1}{2})^{-1}$; in particular $g^1 \notin W(L^\infty, l^1)$. We start now analyzing $S_{g^1,g^2;\alpha,\beta}$ for $\alpha = \beta = 1$. Let us consider an arbitrary $f \in W(L^2, l^\infty)$ and denote by $c_{mn} =< f, g^1_{mn} >$ the coefficients of f with respect to the system $\mathcal{WH}_{g^1;\alpha,\beta}$ (they are finite and bounded by $\|f\|_{W(L^2,l^\infty)}$). On the other hand:

$$\|(S_{g^1,g^2;\alpha,\beta} f) \cdot \mathbf{1}_{[N,N+1]}\|^2_{L^2[N,N+1]} = \sum_{m\in\mathbb{Z}} |c_{mN}|^2$$

But $c_{mn} =< f, g^1_{mn} >= \int_0^1 e^{-2\pi imx} [\sum_l f(x+l+n) \overline{g^1(x+l)}] dx$. Therefore:

$$\sum_m |c_{mn}|^2 = \int_0^1 |\sum_l f(x+l+N) \overline{g^1(x+l)}|^2 dx$$

Note: $|\sum_l f(x+l+N)\overline{g^1(x+l)}|^2 = \sum_{l\geq 0}|f(x+l+N)|^2\frac{1}{(l+1)^{1+2\alpha}}\mathbf{1}_{I_l}(x)$, thus:

$$\sum_m |c_{mn}|^2 = \sum_{l\geq 0}\frac{1}{(l+1)^{1+2\alpha}}\int_{I_l}|f(x+l+N)|^2dx \leq \sum_{l\geq 0}\frac{1}{(l+1)^{1+2\alpha}}\|f\|^2_{W(L^2,l^\infty)}$$

so that:

$$\|S_{g^1,g^2;\alpha,\beta}f\|_{W(L^2,l^\infty)} \leq C_\alpha\|f\|_{W(L^2,l^\infty)}$$

which proves that $S_{g^1,g^2;\alpha,\beta}$ is bounded on $W(L^2,l^\infty)$.

REMARK 5.8 We point out that the series that locally defines the operator $S_{g^1,g^2;\alpha,\beta}$ is not strongly convergent in the $W(L^2,l^\infty)$-norm (if it were, we could have written a much shorter proof). Indeed, for example take $g^1 = g^2 = \mathbf{1}_{[0,1]}$ the characteristic function of $[0,1]$, $\alpha = \beta = 1$ and $f = \mathbf{1}_{\mathbb{R}}$ the constant function 1 on the entire real line. Note that $\|f\|_{W(L^2,l^\infty)} = 1$. Then, for each $N > 0$, $\sum_{|n|\leq N}\sum_m < f, g^1_{mn} > g^2_{mn} = \mathbf{1}_{[-N,N+1]}$. Therefore $\|f - \sum_{|n|\leq N}\sum_m < f, g^1_{mn} > g^2_{mn}\|_{W(L^2,l^\infty)} = 1$ for all N.

Summing first over n and then over m still does not lead to strong convergence of the series. For example take the same WH pair as before and $h(x) = \sum_{m\in\mathbb{Z}} e^{2\pi imx}\mathbf{1}_{[m,m+1]}(x)$, i.e. on each interval $[m,m+1]$, the signal consists of a "pure" harmonic pulse $e^{2\pi imx}$. Note that $\|h\|_{W(L^2,l^\infty)} = 1$. Then, for each $M > 0$, $\sum_{|m|\leq M}\sum_n < h, g^1_{mn} > g^2_{mn} = h\cdot\mathbf{1}_{[-M,M+1]}$. Therefore $\|h - \sum_{|m|\leq M}\sum_n < h, g^1_{mn} > g^2_{mn}\|_{W(L^2,l^\infty)} = 1$ and the series does not converge strongly in $W(L^2,l^\infty)$-sense. However, as we proved in part c), it converges in the sense of tempered distributions.

Although the converse of Theorem 5.3 is not true, the following result offers a necessary condition to have a bounded WH pair on $W(L^2,l^\infty)$.

DEFINITION 5.9 *A function $f:\mathbb{R}\to\mathbb{C}$ has* persistency length *a if there is a $\delta > 0$ and a compact set K congruent to $[0,a]$ mod a, such that for every $x\in K$, $|f(x)|\geq\delta$.*

THEOREM 5.10 *Let $(g^1,g^2;\alpha,\beta)$ be the given data. Suppose the following:*

1. *For every $f\in W(L^2,l^\infty)$, the series $\sum_{mn} < f, g^1_{mn} > g^2_{mn}$ converges unconditionally in L^2_{loc};*

2. *The frame operators are bounded operators on $W(L^2,l^\infty)$;*

3. *g^2 has persistency length $\frac{1}{\alpha}$.*

Then $g^1 \in W(L^\infty, l^2)$.

From this theorem we get immediately the following corollary:

COROLLARY 5.11 *Suppose that for every $f \in W(L^2, l^\infty)$ the series $\sum_{mn} < f, g_{mn} > g_{mn}$ converges unconditionally in L^2_{loc}, that the frame operator associated to $(g, g; \alpha, \beta)$ is bounded on $W(L^2, l^\infty)$ and that g has persistency length $\frac{1}{\alpha}$. Then $g \in W(L^\infty, l^2)$.* □

The proof of Theorem 5.10 is based on the following lemma which is interesting in itself:

LEMMA 5.12 *Let $g \in W(L^2, l^\infty)$ and $\alpha, \beta > 0$ be such that the analysis operator $T : f \mapsto \{< f, g_{mn} > \}_{(m,n)\in\mathbb{Z}^2}$ is well-defined and bounded between $W(L^2, l^\infty)$ and $l^{2,\infty}(\mathbb{Z}^2) = \{c = (c_{mn})_{m,n\in\mathbb{Z}} \mid \|c\|^2_{l^{2,\infty}} := \sup_n \sum_m |c_{mn}|^2 < \infty\}$. Then $g \in W(L^\infty, l^2)$.*

Proof of Lemma 5.12

We know there exists a constant $C > 0$ such that for every $f \in W(L^2, l^\infty)$, $\sum_m |< f, g_{mn} >|^2 \leq C\|f\|^2_{W(L^2,l^\infty)}$. Take $f = e^{-i \, arg \, g}$. Obviously $f \in W(L^2, l^\infty)$ and $\|f\|_{W(L^2,l^\infty)} = 1$. For $m = n = 0$, $< f, g_{mn} >= \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} |g(x)|dx \leq C$. Therefore $g \in L^1(\mathbb{R})$.

Next we show that $g \in L^\infty(\mathbb{R})$. Suppose the contrary, that for every $D > 0$ there is a measurable subset J of an interval of the form $[\frac{N_0}{\alpha}, \frac{N_0+1}{\alpha}]$ such that $|J| > 0$ and $|g(x)| > D$ for every $x \in J$. Take $f = \frac{1}{\sqrt{|J|}} e^{-i \, arg \, g}\mathbf{1}_J$. Note that $\|f\|_{W(L^2,l^\infty)} \leq \|f\|_{L^2(\mathbb{R})} = 1$ and for $n = 0$, $< f, g_{mn} >= \frac{1}{\sqrt{|J|}} \int_{N_0/\alpha}^{(N_0+1)/\alpha} |g(x)|\mathbf{1}_J(x)e^{-2\pi i m\alpha x}dx$. Then:

$$\sum_{m\in\mathbb{Z}} |< f, g_{mn} >|^2 = \frac{1}{\alpha}\|\frac{1}{\sqrt{|J|}} g \cdot \mathbf{1}_J\|^2_{L^2[\frac{N_0}{\alpha}, \frac{N_0+1}{\alpha}]} = \frac{1}{\alpha|J|}\int_J |g(x)|^2dx > D^2$$

which contradicts $\sum_m |< f, g_{mn} >|^2 \leq C\|f\|_{W(L^2,l^\infty)}$. Therefore $g \in L^\infty(\mathbb{R})$.

Using the Parseval identity we obtain (as in the proof of theorem 5.2):

$$\sum_m |< f, g_{mn} >|^2 = \frac{1}{\alpha}\int_0^{\frac{1}{\alpha}} |\sum_{l\in\mathbb{Z}} f(x + n\beta + \frac{l}{\alpha})\overline{g(x + \frac{l}{\alpha})}|^2dx$$

For $n = 0$ we need to check that $\int_0^{1/\alpha} |\sum_l f(x + \frac{l}{\alpha})\overline{g(x + \frac{l}{\alpha})}|^2dx \leq C\|f\|_{W(L^2,l^\infty)}$. To avoid messy computation (as in the proof of the theorem 5.2, point b) we may take $\alpha = 1$. For each $n \in \mathbb{Z}$ denote by J_n the measurable subset of $[n, n+1]$ defined by $J_n = \{x \in [n, n+1] \mid |g(x)| \geq \frac{1}{2}\|g\|_{L^\infty[n,n+1]}\}$. If $|J_n| \leq \varepsilon$, define $J_{n,\varepsilon} = J_n$; if $|J_n| > \varepsilon$, then take a subset $J_{n,\varepsilon}$ of J_n with $|J_{n,\varepsilon}| = \varepsilon$. Note that, by the definition of J_n, $|J_{n,\varepsilon}| > 0$ for all n. Let N_ε be an integer such that for every $|n| < N_\varepsilon$, $|J_{n,\varepsilon}| \geq \frac{\varepsilon}{2}$. Obviously $\lim_{\varepsilon \to 0} N_\varepsilon = \infty$. Take $f = \sum_{|n| \leq N_\varepsilon} \mathbf{1}_{J_{n,\varepsilon}} e^{i\, arg\, g}$. Then $\|f\|^2_{W(L^2,l^\infty)} \leq \varepsilon$ and $|\sum_{l \in \mathbb{Z}} f(x+l)\overline{g(x+l)}|^2 \geq \sum_{|n| \leq N_\varepsilon} |g(x+n)|^2 \mathbf{1}_{J_{n,\varepsilon}}(x+n)$ which implies:

$$\int_0^1 |\sum_l f(x+l)\overline{g(x+l)}|^2 dx \geq \frac{\varepsilon}{8} \sum_{|n| \leq N_\varepsilon} \|g\|^2_{L^\infty[n,n+1]}$$

Using now the boundedness of the analysis operator T, we obtain that

$$\sum_{|n| \leq N_\varepsilon} \|g\|^2_{L^\infty[n,n+1]} \leq 8C$$

Since $\lim_{\varepsilon \to 0} N_\varepsilon = \infty$ we get $\sum_{n \in \mathbb{Z}} \|g\|^2_{L^\infty[n,n+1]} \leq 8C$ which means $g \in W(L^\infty, l^2)$. Q.E.D. □

Now we are prepared to prove the theorem 5.10.

Proof of Theorem 5.10

We know that $f \mapsto \sum_{m,n} < f, g^1_{mn} > g^2_{mn}$ is bounded on $W(L^2, l^\infty)$ and the series converges unconditionally in L^2_{loc}. We claim that $f \mapsto \sum_m < f, g^1_{mn} > g^2_{mn}$ is uniformly bounded on $W(L^2, l^\infty)$ for every n. To see this we prove first that for every compact K there is a constant $C(K)$ such that for every n, $\| \sum_m < f, g^1_{mn} > g^2_{mn}\|_{L^2(K)} \leq C(K)\|f\|_{W(L^2,l^\infty)}$.

Indeed, for every fixed f, the sequence $\sum_{m=-M}^M < f, g^1_{mn} > g^2_{mn}$ converges in L^2_{loc}, for $M \to \infty$. Thus it is bounded. On the other hand the partial sums of operators $S_{M,n} := \sum_{m=-M}^M < \cdot, g^1_{mn} > g^2_{mn}$ are bounded operators, therefore by the uniform boundedness principle they are also uniformly bounded, i.e. for every M, $\| \sum_{m=-M}^M < \cdot, g^1_{mn} > g^2_{mn}\|_{B(W(L^2,l^\infty),L^2_K)} \leq C_n$ for some $C > 0$. Next, for every $\varepsilon > 0$ and for every $f \in W(L^2, l^\infty)$ with $\|f\|_{W(L^2,l^\infty)} = 1$ there is a M_0 such that $\| \sum_{|m| > M_0} < f, g^1_{mn} > g^2_{mn}\|_{L^2(K)} < \varepsilon$. Hence

$$\| \sum_m < f, g^1_{mn} > g^2_{mn}\|_{L^2(K)} \leq \| \sum_{|m| \leq M_0} < f, g^1_{mn} > g^2_{mn}\|_{L^2(K)} + \| \sum_{|m| > M_0} < f, g^1_{mn} > g^2_{mn}\|_{L^2(K)} < \varepsilon + C_n$$

Since ε was arbitrary, we get that $f \mapsto S_n(f) := \sum_m < f, g^1_{mn} > g^2_{mn}$ is a bounded operator in $B(W(L^2, l^\infty), L^2(K))$. Next we apply again the uniform boundedness principle to the sequence of operators S_n. Each is bounded from $W(L^2, l^\infty)$ to $L^2(K)$ as we have seen. For every fixed $f \in W(L^2, l^\infty)$ the series $\sum_n S_n(f)$ converges on $L^2(K)$ therefore each term is bounded by the same constant. Thus we obtain a constant $C(K)$ such that $\|S_n\|_{B(W(L^2,l^\infty),L^2(K))} < C(K)$ for every n.

Now we return to the operator $f \mapsto \sum_m < f, g^1_{mn} > g^2_{mn}$ on $W(L^2, l^\infty)$. Notice that

$$\|S_n\|_{B(W(L^2,l^\infty),L^2(K+\beta))} = \|S_{n+1}\|_{B(W(L^2,l^\infty),L^2(K))} < C(K)$$

Thus if we take $K = [0, \beta]$ we get immediately that $\| \sum_m < f, g^1_{mn} > g^2_{mn}\|_{W(L^2,l^\infty)} \leq C\|f\|_{W(L^2,l^\infty)}$ for every n.

Let K and $\delta > 0$ be the compact set, respectively the positive constant from the definition of persistency for g^2; remember that K is congruent to $[0, \frac{1}{\alpha}]$ modulo $\frac{1}{\alpha}$. Then, for every n:

$$\begin{aligned}\|\sum_m < f, g^1_{mn} > g^2_{mn}\|_{L^2(K_\delta+n\beta)} &= \|g^2(\cdot)\sum_m < f, g^1_{mn} > e^{2\pi i m\alpha(\cdot+n\beta)}\|_{L^2(K_\delta)} \\ &\geq \delta(\sum_m |< f, g^1_{mn} >|^2)^{1/2}\end{aligned}$$

and thus: $(\sum_m |< f, g^1_{mn} >|^2)^{1/2} \leq \frac{C}{\delta}\|f\|_{W(L^2,l^\infty)}$, for every $f \in W(L^2, l^\infty)$ and $n \in \mathbb{Z}$.

Now we apply the previous lemma and obtain the conclusion. Q.E.D. □

So far we extended the frame operator from $L^2(\mathbb{R})$ to $W(L^2, l^\infty)$. Next we show that, under certain conditions, $W(L^2, l^\infty)$ is equivalent to the space $W_\beta(L^2_w, l^\infty)$ introduced earlier. this will then imply that $S_{g^1,g^2;\alpha,\beta}$ is defined as a bounded operator on $W_\beta(L^2_w, l^\infty)$ as well. The connection between the two norms in $W(L^2, l^\infty)$ and $W_\beta(L^2_w, l^\infty)$ is given by the following result:

LEMMA 5.13 *Suppose the weight $w : \mathbb{R} \to \mathbb{R}_+$ satisfies the following condition*

$$\text{(C)} \quad w \in W(L^\infty, l^1) \quad \text{and } w \text{ has persistency length } \beta \tag{5.29}$$

Then the norm $\|\cdot\|_{W(L^2,l^\infty)}$ is equivalent to $\|\cdot\|_{W_\beta(L^2_w,l^\infty)}$ and thus the two Banach spaces are identical: $W(L^2, l^\infty) = W_\beta(L^2_w, l^\infty)$.

Proof

The only thing we have to prove is that there are constants $D_2 > D_1 > 0$ such that for every f, $D_1\|f\|^2_{W_\beta(L^2_w,l^\infty)} \leq \|f\|^2_{W(L^2,l^\infty)} \leq D_2\|f\|^2_{W_\beta(L^2_w,l^\infty)}$.

As pointed out before in (5.20) and (5.21), we may choose any translation step in dealing with the norm $\|\cdot\|_{W(L^2,l^\infty)}$. For convenience we choose β. Then $\|f\|^2_{W(L^2,l^\infty)} \equiv \sup_n \int_0^\beta |f(x+n\beta)|^2dx$. On the one hand, for $f \in W(L^2,l^\infty)$ we obtain:

$$\int_{-\infty}^{\infty} w(x)|f(x+n\beta)|^2dx = \sum_k \int_0^\beta w(x+k\beta)|f(x+(k+n)\beta)|^2dx \leq$$

$$\leq \sum_k \sup_{x\in[0,\beta]} w(x+k\beta)\int_0^\beta |f(x+(k+n)\beta)|^2dx \leq \|w\|_{W(L^\infty,l^1)} \sup_n \int_{n\beta}^{(n+1)\beta} |f(x)|^2dx$$

Therefore there is a $D_1 > 0$ such that $D_1\|f\|^2_{W_\beta(L^2_w,l^\infty)} \leq \|f\|^2_{W(L^2,l^\infty)}$. Hence $W(L^2,l^\infty) \subset W_\beta(L^2_w,l^\infty)$.

For the other inequality let $\delta > 0$ and K be the constant and the compact set from the definition of persistency length of w. Then $w(x) \geq \delta$, for every $x \in K$. For every $f \in W_\beta(L^2_w,l^\infty)$ we have:

$$\int_{-\infty}^{\infty} w(x)|f(x+n\beta)|^2dx \geq \delta\int_K |f(x+n\beta)|^2dx = \delta\int_0^\beta |f(x+(n+l_x)\beta)|^2dx$$

where l_x is the integer associated to $x \in [0,\beta]$ such that $x + l_x\beta \in K$; l_x is bounded by $|l_x| \leq N_\delta$ since K is compact. We partition $[0,\beta] = \cup_{|j|\leq N_\delta} S_j$ where $S_j = \{y \in [0,\beta] | l_y = j\}$. Let $u = \sup_n \int_{n\beta}^{(n+1)\beta} |f(x)|^2dx$ and n_0 be an integer such that $\int_0^\beta |f(x+n_0\beta)|^2dx \geq \frac{1}{2}u$. Then there is a $|j| \leq N_\delta$ such that $\int_{S_j} |f(x+n_0\beta)|^2dx \geq \frac{u}{2(2N_\delta+1)}$. For $n_1 = n_0 - j$ we get:

$$\int_{-\infty}^{\infty} w(x)|f(x+n_1\beta)|^2dx \geq \delta\int_{S_j} |f(x+n_0\beta)|^2dx \geq \frac{\delta}{2(2N_\delta+1)} \sup_n \int_{n\beta}^{(n+1)\beta} |f(x)|^2dx$$

Thus there is a $D_2 > 0$ such that $D_2\|f\|^2_{W_\beta(L^2_w,l^\infty)} \geq \|f\|^2_{W(L^2,l^\infty)}$. Therefore $W_\beta(L^2_w,l^\infty) \subset W(L^2,l^\infty)$ and the norms are equivalent. Q.E.D. □

REMARK 5.14 A consequence of this theorem is that if w_1, w_2 both satisfy (C) then $W_\beta(L^2_{w_1},l^\infty)$ and $W_\beta(L^2_{w_2},l^\infty)$ are equivalent.

This lemma together with Theorem 5.3 shows that if $g^1, g^2 \in W(L^\infty,l^1)$ and the weight w satisfies condition (C) then $S_{g^1,g^2;\alpha,\beta}$ is bounded on $W_\beta(L^2_w, l^\infty)$. The lifting from $L^2(\mathbb{R})$ to $W(L^2, l^\infty)$ or $W_\beta(L^2_w, l^\infty)$ is shown in figure 5.4.

$$\begin{tikzcd}
\begin{matrix} L^2(\Omega; W_\beta(L^2_w, l^\infty)) \\ \| \\ L^2(\Omega; W(L^2, l^\infty)) \end{matrix} \arrow[rr, "S_{g^1,g^2;\alpha,\beta}"] & & \begin{matrix} L^2(\Omega; W_\beta(L^2_w, l^\infty)) \\ \| \\ L^2(\Omega; W(L^2, l^\infty)) \end{matrix} \\
& \arrow[u, phantom, "\uparrow"] & \\
L^2(\mathbf{R}) \arrow[rr, "S_{g^1,g^2;\alpha,\beta}"] & & L^2(\mathbf{R}) \\
\Omega \arrow[u] & & \Omega \arrow[u]
\end{tikzcd}$$

,

Figure 5.4: The Lifting Scheme in the Stationary Case

The picture is now the following. We would like to work with $f \in L^2_w$, because $f \in L^2(\mathbb{R})$ is not possible for stationary signals. However, extending $S_{g^1,g^2;\alpha,\beta}$ to L^2_w is tricky because L^2_w is not well-adapted to the study of translations. Therefore, we introduce $W_\beta(L^2_w, l^\infty)$ instead. We can impose the slightly stronger restriction $f \in W_\beta(L^2_w, l^\infty)$; on this smaller space $S_{g^1,g^2;\alpha,\beta}$ is well defined. We still measure our approximation error in L^2_w:

$$J(g^1, g^2; \alpha, \beta) = \mathbf{E}\|\mathbf{f} - S_{g^1,g^2;\alpha,\beta}\mathbf{f}\|^2_w \tag{5.30}$$

This is finite and bounded:

$$\begin{aligned} J((g^1, g^2; \alpha, \beta)) &\le \int_\Omega d\mu(\omega) \sup_n \|T^n_\beta(1 - S_{g^1,g^2;\alpha,\beta})\mathbf{f}_\omega\|^2_w \\ &\le (1 + \|S_{g^1,g^2;\alpha,\beta}\|_{B(W_\beta(L^2_w,l^\infty))})\|f\|^2_{L^2(\Omega;W_\beta(L^2_w,l^\infty))} \end{aligned}$$

because $S_{g^1,g^2;\alpha,\beta}$ commutes with the translates $T^n_\beta f = f(\cdot - n\beta)$. Note that $\|S_{g^1,g^2;\alpha,\beta}\|_{B(W_\beta(L^2_w,l^\infty))} = C_{\alpha,\beta,w}\|g^1\|_{W(L^\infty,l^1)}\|g^2\|_{W(L^\infty,l^1)}$ which turns the previous relation into:

$$J((g^1, g^2; \alpha, \beta)) \le (1 + C_{\alpha,\beta,w}\|g^1\|_{W(L^\infty,l^1)}\|g^2\|_{W(L^\infty,l^1)})\|f\|^2_{L^2(\Omega;W_\beta(L^2_w,l^\infty))} \tag{5.31}$$

All the above are summarized by the following theorem:

THEOREM 5.15 *Suppose $g^1, g^2 \in W(L^\infty, l^1)$, and suppose that $w : \mathbb{R} \to \mathbb{R}_+$ is a nonnegative-valued function in $w \in W(L^\infty, l^1)$ with persistency length β. Then, for every stochastic model $\mathbf{f} \in L^2(\Omega; W(L^2, l^\infty))$, the approximation error given by a WH pair $((g^1, g^2; \alpha, \beta))$ is bounded above as in (5.31).* $\square$

REMARK 5.16 *One may wonder, after all, whether there is a realization of a stationary stochastic model on $W(L^2, l^\infty)$. The answer is positive as the following example shows. In this example we assume a mild condition on the autocovariance function, namely $R \in L^2(\mathbb{R})$.*

EXAMPLE 5.17 *Suppose $R \in L^2(\mathbb{R})$ where $R(u) = \mathbf{E}[\mathbf{f}(t)\overline{\mathbf{f}(t-u)}]$. Then the Fourier transform of R is positive, $\hat{R} \geq 0$. Define the following probability space:*

$$\Omega = \mathbb{R} \times \{-1, 1\} d\mu(\omega, q) = \frac{1}{2\sqrt{2\pi}R(0)} \hat{R}(q|\omega|) d\omega = \begin{cases} \frac{1}{2\sqrt{2\pi}R(0)} \hat{R}(|\omega|) d\omega & , \quad q = +1 \\ \frac{1}{2\sqrt{2\pi}R(0)} \hat{R}(-|\omega|) d\omega & , \quad q = -1 \end{cases}$$

Consider now the stationary stochastic model:

$$\mathbf{f} : \Omega \to W(L^2, l^\infty) \ , \ f_{\omega,q}(x) = R(0) e^{iq|\omega|x + i\frac{\pi}{2} sgn(\omega)}$$

Then, some direct computations show that:

$$\mathbf{Ef}(x) = \sum_q \int_{-\infty}^{\infty} d\mu(\omega, q) f_{\omega,q}(x) = 0$$

$$\mathbf{E}[\mathbf{f}(t)\overline{\mathbf{f}(s)}] = \sum_q \int_{-\infty}^{\infty} d\mu(\omega, q) f_{\omega,q}(t) \overline{f_{\omega,q}(s)} = R(t-s)$$

5.2 The Optimization Problems

In this section we analyze certain optimization problems. Earlier we introduced different stochastic models. Both nonstationary and stationary models have first and second order statistics given by:

$$\begin{aligned} \mathbf{Ef}(\cdot) &= 0 \\ \mathbf{E}[\mathbf{f}(t)\overline{\mathbf{f}(s)}] &= R(t, s) \end{aligned} \tag{5.32}$$

The criterion that we want to minimize has the following form:

$$J(g^1, g^2; \alpha, \beta) = \mathbf{E}\|\mathbf{f} - \sum_{m,n} < \mathbf{f}, g^1_{mn} > g^2_{mn}\|^2_X \tag{5.33}$$

where X is either $L^2(\mathbb{R})$ (for finite energy, nonstationary signals) or $L^2_w(\mathbb{R})$ (for bounded power, stationary signals). We therefore have to minimize (5.33) subject to the assumptions $A1$ and $A2$ made about the windows g^1 and g^2. We can look for the infimum of (5.33) under different additional assumptions. The following statements represent four problems that we formulate and solve in this section:

Problem 1 (suboptimal 1)

We fix α, β and g^1 and search for g^2 that minimizes (5.33) subject to the assumption $A2$, i.e.

$$\inf_{g^2,\ A2 \text{ holds}} J(g^1, g^2; \alpha, \beta) \quad , \quad \alpha, \beta \text{ and } g^1 \text{ given} \tag{5.34}$$

Problem 2 (suboptimal 2)

We fix α, β and g^2 and look for g^1 that minimizes (5.34) subject to the assumption $A2$, i.e.

$$\inf_{g^1,\ A2 \text{ holds}} J(g^1, g^2; \alpha, \beta) \quad , \quad \alpha, \beta \text{ and } g^2 \text{ given} \tag{5.35}$$

Problem 3 (optimal)

We fix only α, β and we look for g^1, g^2 subject to $A2$ that minimize (5.33), i.e.

$$\inf_{\substack{g^1, g^2 \\ \text{A2 holds}}} J(g^1, g^2; \alpha, \beta) \quad , \quad \alpha, \beta \text{ are fixed} \tag{5.36}$$

Problem 4 (optimal-tight)

We fix α, β and we look for $g^1 = g^2 = g$ subject to $A2$, that minimizes (5.33), i.e.

$$\inf_{g, A2 \text{ holds}} J(g, g; \alpha, \beta) \quad , \quad \alpha, \beta \text{ are fixed} \tag{5.37}$$

Notice that the criterion is quadratic in g^1, g^2. Thus the first two optimization problems should not be (and in fact are not) difficult to solve. However, once they are solved, the (suboptimal) criterion becomes highly nonlinear in the remaining function g^1 or g^2. Despite of this nonlinear form, we shall be able to solve the optimal problems and to parametrize all the solutions.

Since our method is based on the Zak transform, we make one additional assumption, namely we ask that the sampling ration $\frac{1}{\alpha\beta}$ be a rational number. Thus $A1$ is replaced by:

$$A1'.\ \ \alpha \cdot \beta = \frac{p}{q} > 1\ ,\ \text{with } p, q \in \mathbb{N}\ ,\ \text{integers} \tag{5.38}$$

The computation proceeds in two steps. In the first step we do not need the Zak transform and consequently neither $A1'$. The second step will make use of the Zak transform.

We point out that, beside the constraint $A2$ we may require some other qualitative constraints like the time-frequency localization of the windows g^1 and/or g^2.

For the nonstationary model $\mathbf{f} \in L^2(\Omega; L^2(\mathbb{R}))$ we obtain:

$$\begin{aligned} J(g^1, g^2; \alpha, \beta) \quad = \quad & \mathbf{E}\left[\int_{-\infty}^{\infty} dx(\mathbf{f}(x) - \sum_{m,n} < \mathbf{f}, g^1_{mn} > g^2_{mn}(x)) \right. \\ & \left. (\overline{\mathbf{f}(x)} - \sum_{m,n} < g^1_{mn}, \mathbf{f} > \overline{g^2_{mn}(x)})\right] = T_1 - T_2 - T_3 + T_4 \end{aligned} \tag{5.39}$$

where:

$$T_1 = \mathbf{E}\int_{-\infty}^{\infty} dx \mathbf{f}(x)\overline{\mathbf{f}(x)} = \int_{-\infty}^{\infty} R(x,x)dx$$

$$T_3 = \bar{T}_2 = \mathbf{E}\sum_{m,n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy \mathbf{f}(x)\overline{g^1_{mn}(x)}g^2_{mn}(y)\overline{\mathbf{f}(y)} =$$

$$= \mathbf{E}\sum_{m,n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy \mathbf{f}(x)\overline{\mathbf{f}(y) g^1(x - n\beta)} g^2(y - n\beta) e^{2\pi i m\alpha(y-x)} =$$

$$= \frac{1}{\alpha}\sum_{m,n}\int_{-\infty}^{\infty} dx\, R(x + \frac{m}{\alpha}, x) g^2(x - n\beta)\overline{g^1(x + \frac{m}{\alpha} - n\beta)}$$

$$T_4 = \mathbf{E}\sum_{m,n}\sum_{m',n'}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, dz\, \mathbf{f}(y)\overline{g^1_{mn}(y)} g^2_{mn}(x)\overline{g^2_{m'n'}(x)} g^1_{m'n'}(z)\overline{\mathbf{f}(z)} =$$

$$= \frac{1}{\alpha^2}\sum_{m,n}\sum_{m',n'}\int_{-\infty}^{\infty} dx\, R(x + \frac{m}{\alpha}, x + \frac{m'}{\alpha})\overline{g^1(x + \frac{m}{\alpha} - n\beta)} g^2(x - n\beta)\overline{g^2(x - n'\beta)} g^1(x + \frac{m'}{\alpha} - n'\beta)$$

Note that in the computation of T_2, T_3, T_4 we used the Parseval formula and next we took the expectation (thus we avoided the use of the Poisson summation formula).

Similarly, for the stationary model $\mathbf{f} \in L^2(\Omega; W(L^2, l^\infty))$ we obtain:

$$\begin{aligned} J(g^1, g^2; \alpha, \beta) &= \mathbf{E} \int_{-\infty}^{\infty} dx\, w(x) |\mathbf{f}(x) - \sum_{m,n} < f, g^1_{mn} > g^2_{mn}(x)|^2 \\ &= T_5 - T_6 - T_7 + T_8 \end{aligned} \tag{5.40}$$

where:

$$T_5 = R(0) \int_{-\infty}^{\infty} w(x) dx$$

$$T_7 = \bar{T}_6 = \frac{1}{\alpha} \sum_{m,n} R(\frac{m}{\alpha}) \int_{-\infty}^{\infty} dx\, w(x) g^2(x - n\beta) \overline{g^1(x + \frac{m}{\alpha} - n\beta)}$$

$$T_8 = \frac{1}{\alpha^2} \sum_{m,n} \sum_{m',n'} R(\frac{m - m'}{\alpha}) \int_{-\infty}^{\infty} dx\, w(x) \overline{g^1(x + \frac{m}{\alpha} - n\beta)} \\ g^2(x - n\beta) \overline{g^2(x - n'\beta)} g^1(x + \frac{m'}{\alpha} - n'\beta)$$

In the next few pages, we shall manipulate these expressions $J(g^1, g^2; \alpha, \beta)$ to write them in a different form. To do this we use the Zak transform. As we mentioned before, we assume $\alpha\beta = \frac{p}{q} > 1$ with p and q relatively prime. The Zak transforms of the two windows g^1, g^2 are denoted by G^1, respectively G^2 and are defined as follows:

$$G^1(t, s) = \sqrt{\beta} \sum_{k \in \mathbb{Z}} e^{2\pi i k t} g^1(\beta(s + k)) \quad , \quad G^2(t, s) = \sqrt{\beta} \sum_{k \in \mathbb{Z}} e^{2\pi i k t} g^2(\beta(s + k))$$

The inversion formulae in time and frequency domain are:

$$g(x) = \frac{1}{\sqrt{\beta}} \int_0^1 G(t, \frac{x}{\beta}) dt \quad , \quad \hat{g}(\xi) = \sqrt{\frac{\beta}{2\pi}} \int_0^1 e^{-i\beta s \xi} G(-\frac{\beta\xi}{2\pi}, s) ds \tag{5.41}$$

For more information on Zak transform we refer the reader to [Jans82, Jans88]. We recall the two quasi-periodicity relations of a Zak transform $G(t, s)$:

$$G(t + 1, s) = G(t, s) \qquad G(t, s + 1) = e^{-2\pi i t} G(t, s) \tag{5.42}$$

We also denote by $\Gamma^1(t,s)$ the $p \times q$ matrix whose (j,k) entry is $G^1(t+\frac{k}{q}, s+j\frac{q}{p})$, $j = 0,\dots,p-1$, $k = 0,\dots,q-1$, i.e.

$$\Gamma^1(t,s) = \begin{bmatrix} G^1(t,s) & G^1(t+\frac{1}{q},s) & \cdots & G^1(t+\frac{q-1}{q},s) \\ G^1(t,s+\frac{q}{p}) & G^1(t+\frac{1}{q},s+\frac{q}{p}) & \cdots & G^1(t+\frac{q-1}{q},s+\frac{q}{p}) \\ \vdots & \vdots & & \vdots \\ G^1(t,s+(p-1)\frac{q}{p}) & G^1(t+\frac{1}{q},s+(p-1)\frac{q}{p}) & \cdots & G^1(t+\frac{q-1}{q},s+(p-1)\frac{q}{p}) \end{bmatrix} \tag{5.43}$$

Similarly, the (j,k) entry of the $p \times q$ matrix $\Gamma^2(t,s)$ is given by $G^2(t+\frac{k}{q}, s+j\frac{q}{p})$.

Next we plug the Zak transforms G^1, G^2 into the expressions of the terms $T_1,\dots,T_8$.

5.2.1 The Nonstationary Model - Computations with Zak Transform

For the nonstationary model we obtain:

$$T_3 = \frac{1}{\beta}\sum_{m,n}\int_{-\infty}^{\infty}dx\int_0^1 dt_1\int_0^1 dt_2 R(x+\frac{m}{\alpha},x)G^2(t_1,\frac{x}{\beta}-n)\overline{G^1(t_2,\frac{x}{\beta}+\frac{m}{\alpha\beta}-n)}$$

Performing summation over n, via Parseval identity, we get $t_1 = t_2 = t$ and only one integral (in t) from 0 to 1 (we denote this, symbolically by *"$\delta(t_1-t_2)$ in a weak sense "*; note again that we do not use the Poisson summation formula, which would yield the same result but under stronger conditions). We make a change of variable $x = \beta(s+k)$, $0 \le s \le 1$, $k \in \mathbb{Z}$ and we get:

$$T_3 = \frac{1}{\alpha}\sum_m\sum_k\int_0^1 ds\int_0^1 dt R(\beta(s+k+m\frac{q}{p}),\beta(s+k))G^2(t,s+k)\overline{G^1(t,s+k+m\frac{q}{p})}$$

With $m = m_1p + r$, $0 \le r < p$ we obtain:

$$T_3 = \frac{1}{\alpha}\int_0^1 ds\int_0^1 dt\sum_{m_1,k}\sum_{r=0}^{p-1}R(\beta(s+k+m_1q+r\frac{q}{p}),\beta(s+k))e^{2\pi itm_1q}G^2(t,s)\overline{G^1(t,s+r\frac{q}{p})}$$

Let

$$\rho_{r_1,r_2}(t,s) = \sum_m e^{2\pi imqt}\sum_k R(\beta(s+k+mq+r_1\frac{q}{p}),\beta(s+k+r_2\frac{q}{p})) \tag{5.44}$$

and denote by $\mathbf{M}$ the $p \times p$ matrix whose (r_1, r_2) entry is $\rho_{r_1,r_2}(t,s)$, $0 \le r_1, r_2 \le p-1$. Note the following properties of the functions $\rho_{r_1,r_2}(t,s)$:

$$\begin{aligned} \rho_{r_1+p,r_2+p}(t,s) = \rho_{r_1,r_2}(t,s) \quad &, \quad \rho_{r_1,r_2}(t, s+\frac{q}{p}) = \rho_{r_1+1,r_2+1}(t,s) \\ \rho_{r_1,r_2}(t+\frac{1}{q}, s) = \rho_{r_1,r_2}(t,s) \quad &, \quad \rho_{r_1,r_2}(t,s+1) = \rho_{r_1,r_2}(t,s) \\ \rho_{r_1+p,r_2}(t,s) = e^{-2\pi iqt}\rho_{r_1,r_2}(t,s) \quad &, \quad \rho_{r_1,r_2+p}(t,s) = e^{2\pi iqt}\rho_{r_1,r_2}(t,s) \\ \overline{\rho_{r_1,r_2}(t,s)} = \rho_{r_2,r_1}(t,s) \quad &\Rightarrow \quad \bar{\mathbf{M}}^t = \mathbf{M} \end{aligned} \tag{5.45}$$

i.e. for fixed (t,s) $\mathbf{M}(t,s)$ is self-adjoint as a matrix (we shall also use $\mathbf{M}^*$ for $\bar{\mathbf{M}}^t(t,s)$).

Then, the previous expression of T_3 turns into:

$$T_3 = \frac{1}{\alpha}\int_0^1 ds \int_0^1 dt\ G^2(t,s)\sum_{r=0}^{p-1}\overline{G^1(t, s+r\frac{q}{p})}\rho_{r,0}(t,s)$$

Notice now that the integrand is 1-periodic in s. Then:

$$T_3 = \frac{1}{\alpha p}\sum_{k=0}^{p-1}\int_{k\frac{q}{p}}^{k\frac{q}{p}+1} ds \int_0^1 dt\ G^2(t,s)\sum_{r=0}^{p-1}\overline{G^1(t,s+r\frac{q}{p})}\rho_{r,0}(t,s) =$$

$$= \frac{1}{\alpha p}\int_0^1 ds \int_0^1 dt \sum_{k,r=0}^{p-1} G^2(t, s+k\frac{q}{p})\overline{G^1(t, s+(r+k)\frac{q}{p})}\rho_{r+k,k}(t,s)$$

Remark now that we can replace $r+k$ by r, running from 0 to $p-1$ again, because $\overline{G^1(t,s+l\frac{q}{p})}\rho_{l,k}(t,s)$ is p-periodic in l. Then:

$$T_3 = \frac{1}{p\alpha}\int_0^1 ds \int_0^1 dt \sum_{k,r=0}^{p-1} G^2(t,s+k\frac{q}{p})\overline{G^1(t,s+r\frac{q}{p})}\rho_{r,k}(t,s)$$

Similarly we obtain that the integrand is $\frac{q}{p}$-periodic in s. Therefore it is $\frac{1}{p}$-periodic in s ($\frac{1}{p}$ is the greatest common divisior between 1 and $\frac{q}{p}$ in $\frac{1}{p}\mathbb{Z}$). Then using that ρ's are $\frac{1}{q}$-periodic in t:

$$\begin{aligned} T_3 &= \frac{1}{\alpha}\int_0^{1/p} ds \int_0^{1/q} dt \sum_{l=0}^{q-1}\sum_{k,r=0}^{p-1} G^2(t+\frac{l}{q}, s+k\frac{q}{p})\overline{G^1(t+\frac{l}{q}, s+r\frac{q}{p})}\rho_{r,k}(t,s) \\ &= \frac{1}{\alpha}\int_0^{1/p} ds \int_0^{1/q} dt\ Tr\{\Gamma^2(t,s)\Gamma^{1^*}(t,s)\mathbf{M}(t,s)\} \end{aligned} \tag{5.46}$$

For T_4 we obtain:

$$\begin{aligned} T_4 &= \frac{1}{\alpha\beta^2}\sum_{m,n}\sum_{m',n'}\int_{-\infty}^{\infty}dx\int_0^1 dt_1\int_0^1 dt_2\int_0^1 dt_3\int_0^1 dt_4\ R(x+\frac{m}{\alpha},x+\frac{m'}{\alpha}) \\ & \overline{G^1(t_1,\frac{x}{\beta}+m\frac{q}{p}-n)}G^2(t_2,\frac{x}{\beta}-n)\overline{G^2(t_3,\frac{x}{\beta}-n')}G^1(t_4,\frac{x}{\beta}+m'\frac{q}{p}-n') \end{aligned}$$

Performing the summation over n and n' we get $\delta(t_1-t_2)\delta(t_3-t_4)$ in weak sense. Next for $x=\beta(s+k)$, $m=m_1p+r_1$, $m'=m_1^{'}p+r_2$ we get:

$$\begin{aligned} T_4 &= \frac{1}{\alpha\beta}\sum_{m_1,m_1{'},k}\int_0^1 ds\int_0^1 dt_1\int_0^1 dt_4\sum_{r_1,r_2=0}^{p-1}R(\beta(s+k+m_1q+r_1\frac{q}{p}),\beta(s+k+m_1^{'}q+r_2\frac{q}{p})) \\ & \overline{G^1(t_1,s+r_1\frac{q}{p})}G^2(t_1,s)\overline{G^2(t_4,s)}G^1(t_4,s+r_2\frac{q}{p})e^{2\pi i m_1 q t_1}e^{-2\pi i m_1^{'} q t_4} \end{aligned}$$

Notice that after we perform the summation over k in $R(\cdot,\cdot)$, the sum will depend on $m_1-m_1^{'}$ only. Thus we get $\rho_{r_1,r_2}(t_1,s)\sum_{m_1^{'}}exp(2\pi i m_1^{'}(qt_1-qt_4))$. Replacing $t_1=\tau_1+\frac{l_1}{q}$, $t_4=\tau_2+\frac{l_2}{q}$ and using the Parseval relation again we get:

$$\begin{aligned} T_4 &= \frac{1}{\alpha\beta^2 q}\int_0^1 ds\int_0^{1/q}dt\sum_{l_1,l_2=0}^{q-1}\sum_{r_1,r_2=0}^{p-1}\rho_{r_1,r_2}(t,s)\overline{G^1(t+\frac{l_1}{q},s+r_1\frac{q}{p})}G^2(t+\frac{l_1}{q},s) \\ & \overline{G^2(t+\frac{l_2}{q},s)}G^1(t+\frac{l_2}{q},s+r_2\frac{q}{p}) \end{aligned}$$

Using 1-periodicity in s of the integrand we obtain:

$$\begin{aligned} T_4 &= \frac{1}{\alpha^2\beta qp}\int_0^1 ds\int_0^{1/q}\sum_{l_1,l_2=0}^{q-1}\sum_{r_1,r_2,r=0}^{p-1}\rho_{r+r_1,r+r_2}(t,s)\overline{G^1(t+\frac{l_1}{q},s+(r+r_1)\frac{q}{p})} \\ & G^2(t+\frac{l_1}{q},s+r\frac{q}{p})\overline{G^2(t+\frac{l_2}{q},s+r\frac{q}{p})}G^1(t+\frac{l_2}{q},s+(r+r_2)\frac{q}{p}) \end{aligned}$$

Next we notice that the integrand is $\frac{q}{p}$-periodic in s and $r+r_1$, $r+r_2$ can be replaced by r_1, respectively r_2 again. Hence the integrand is $\frac{1}{p}$-periodic in s and we end up with:

$$\begin{aligned} & T_4 = \frac{1}{\alpha^2\beta q}\int_0^{1/p}ds\int_0^{1/q}dt\sum_{l_1,l_2=0}^{q-1}\sum_{r_1,r_2,r=0}^{p-1}G^2(t+\frac{l_1}{q},s+r\frac{q}{p})\overline{G^1(t+\frac{l_1}{q},s+r_1\frac{q}{p})} \\ & \rho_{r_1,r_2}(t,s)G^1(t+\frac{l_2}{q},s+r_2\frac{q}{p})\overline{G^2(t+\frac{l_2}{q},s+r\frac{q}{p})} = \frac{1}{\alpha p}\int_0^{1/p}ds\int_0^{1/q}dt\ Tr\{\Gamma^2{\Gamma^1}^*\mathbf{M}\Gamma^1{\Gamma^2}^*\} \qquad (5.47) \end{aligned}$$

For T_1 we get immediately that:

$$\int_0^1 ds \int_0^{1/q} dt \rho_{r,r}(t,s) = \frac{1}{\beta q}\int_{-\infty}^{\infty} R(x,x)dx = \frac{1}{\beta q} T_1$$

Thus:

$$T_1 = \beta q \int_0^{1/p} ds \int_0^{1/q} dt\ Tr\{\mathbf{M}\} \tag{5.48}$$

Putting together (5.46), (5.47) and (5.48) we obtain:

$$J(g^1, g^2; \alpha, \beta) = \beta q \int_0^{1/p} ds \int_0^{1/q} dt\ Tr\{(I - \frac{1}{p}\Gamma^2 \Gamma^{1*})\mathbf{M}(I - \frac{1}{p}\Gamma^1\Gamma^{2*})\} \tag{5.49}$$

where $\Gamma^1 = \Gamma^1(t,s)$, $\Gamma^2 = \Gamma^2(t,s)$, $\mathbf{M} = \mathbf{M}(t,s)$ and I is the $p \times p$ identity matrix.

5.2.2 The Stationary Model - Computations with Zak Transform

For the stationary model we can rewrite $J(g^1, g^2; \alpha, \beta)$ in a similar way. We obtain (see 5.40):

$$\begin{aligned} T_7 &= \frac{1}{\alpha}\sum_{m,n,k} R(\frac{m}{\alpha}) \int_0^1 ds \int_0^1 dt_1 \int_0^1 dt_2 w(\beta(s+k)) G^2(t_1, s+k-n)\overline{G^1(t_2, s+k+m\frac{q}{p}-n)} \\ &= \frac{1}{\alpha}\int_0^1 ds \int_0^1 dt \sum_{r=0}^{p-1}\sum_{m'} e^{2\pi i m' q t} R(\frac{m'p+r}{\alpha}) \sum_k w(\beta(s+k)) G^2(t,s)\overline{G^1(t, s+r\frac{q}{p})} \end{aligned}$$

Let us denote:

$$\rho_{r_1,r_2}(t) = \sum_m e^{2\pi i m q t} R(\frac{mp + r_1 - r_2}{\alpha}) \tag{5.50}$$

and

$$W(s) = \sum_k w(\beta(s+k)) \tag{5.51}$$

We point out that (5.50) is consistent with (5.44). Since ρ does not depend on s we have $\rho_{r_1+1,r_2+1}(t) = \rho_{r_1,r_2}(t)$. Notice again the integrand is 1-periodic in s and ρ is $\frac{1}{q}$-periodic in t. Then:

$$T_7 = \frac{1}{\alpha p}\int_0^1 ds \int_0^{1/q} dt \sum_{r,r_1=0}^{p-1}\sum_{l=0}^{q-1} W(s + r_1\frac{q}{p}) G^2(t + \frac{l}{q}, s + r_1\frac{q}{p}) \overline{G^1(t+\frac{l}{q}, s+(r+r_1)\frac{q}{p})}\rho_{r+r_1,r_1}(t)$$

Again the integrand is $\frac{q}{p}$-periodic in s (and also $\frac{1}{p}$-periodic) and we can replace $r+r_1$ by r_1. In the end we get:

$$T_7 = \frac{1}{\alpha}\int_0^{1/p} ds \int_0^{1/q} dt\ Tr\{\mathbf{W}(s)\Gamma^2(t,s)\Gamma^{1*}(t,s)\mathbf{M}(t)\} \tag{5.52}$$

where $\mathbf{W} = \mathbf{W}(s)$ is a $p\times p$ diagonal matrix whose (r,r) entry is $W(s+r\frac{q}{p})$, $r=0,\ldots,p-1$, i.e

$$\mathbf{W}(s) = \begin{bmatrix} W(s) & & & \\ & W(s+\frac{q}{p}) & & \\ & & \ddots & \\ & & & W(s+(p-1)\frac{q}{p}) \end{bmatrix} \tag{5.53}$$

and $\mathbf{M}(t)$ is the $p\times p$ matrix whose (r_1,r_2) entry is $\rho_{r_1,r_2}(t)$. For T_8 we get:

$$T_8 = \frac{1}{\alpha^2\beta}\sum_{m,n}\sum_{m',n',k} R(\frac{m-m'}{\alpha})\int_0^1 ds\ w(\beta(s+k))\int_0^1 dt_1\int_0^1 dt_2\int_0^1 dt_3\int_0^1 dt_4$$
$$\overline{G^1(t_1,s+k-n+m\frac{q}{p})}G^2(t_2,s+k-n)\overline{G^2(t_3,s+k-n')}G^1(t_4,s+k-n'+m'\frac{q}{p})$$

Repeating the scheme as before we get:

$$T_8 = \frac{1}{\alpha^2\beta^2 q}\int_0^{1/p} ds\int_0^{1/q} dt\ Tr\{\mathbf{W}\Gamma^2\Gamma^{1*}\mathbf{M}\Gamma^1\Gamma^{2*}\} \tag{5.54}$$

Since $\int_0^{1/p} ds\int_0^{1/q} dt\ Tr\{\mathbf{W}\mathbf{M}\} = \frac{1}{q\beta}R(0)\int_{-\infty}^{\infty} w(x)dx$ we get:

$$T_5 = \beta q\int_0^{1/p} ds\int_0^{1/q} dt\ Tr\{\mathbf{W}\mathbf{M}\}$$

Therefore, the criterion (5.40) becomes:

$$J(g^1,g^2;\alpha,\beta;R,w) = \beta q\int_0^{1/p} ds\int_0^{1/q} dt\ Tr\{\mathbf{W}(I-\frac{1}{p}\Gamma^2\Gamma^{1*})\mathbf{M}(I-\frac{1}{p}\Gamma^1\Gamma^{2*})\} \tag{5.55}$$

Notice that for $(t,s)\in[0,\frac{1}{q}]\times[0,\frac{1}{p}]$ the entries of the matrices Γ^1 and Γ^2 are independent. Moreover, if we denote by $L^2([0,\frac{1}{q}]\times[0,\frac{1}{p}];\mathbb{C}^{p\times q})$ the Hilbert space of the $p\times q$ matrix - valued functions defined on the rectangle $[0,\frac{1}{q}]\times[0,\frac{1}{p}]$ endowed with the scalar product

$$<\Gamma^1,\Gamma^2> = \int_0^{1/p} ds\int_0^{1/q} dt\ Tr\{\Gamma^1\Gamma^{2*}\}\quad,\quad \forall\Gamma^1,\Gamma^2\in L^2([0,\frac{1}{q}]\times[0,\frac{1}{p}];\mathbb{C}^{p\times q})$$

then $g\in L^2(\mathbb{R})\mapsto\Gamma\in L^2([0,\frac{1}{q}]\times[0,\frac{1}{p}];\mathbb{C}^{p\times q})$ is a unitary map.

5.2.3 Miscelaneous Results

We have now recast $J(g^1, g^2; \alpha, \beta)$ into the forms (5.49) and (5.55) and we can use these to analyze the optimization problems (5.34)–(5.37). But first we need to make a couple of comments about the matrices $\mathbf{W}$ and $\mathbf{M}$ as well about the assumption $A2$.

The condition (C) (see Lemma 5.13) implies there are two positive constants $0 < A_1 \leq B_1 < \infty$ such that:

$$A_1 \leq \mathbf{W}(s) \leq B_1 \tag{5.56}$$

for every $s \in [0, \frac{1}{p}]$. About the matrix $\mathbf{M}$ we can only claim it is nonnegative for every (t, s). However, if we want to get an uniform bound as in (5.56) we need to require some extra conditions.

LEMMA 5.18 *For the nonstationary case, for every (t, s), $\mathbf{M}(t, s) \geq 0$ (as a matrix). If in the stationary case R is continuous and in $W(L^\infty, l^1)$ then for every (t, s), $\mathbf{M}(t, s) \geq 0$ as well.*

Proof

Take $z_1, \ldots, z_{p-1} \in \mathbb{C}$ arbitrarily complex numbers. We have to prove that:

$$< z, \mathbf{M}z >= \sum_{r_1, r_2 = 0}^{p-1} z_{r_1} \bar{z}_{r_2} \rho_{r_1, r_2} \geq 0$$

In the nonstationary case a straightforward computation shows that:

$$< z, \mathbf{M}z >= \sum_{l=0}^{q-1} \mathbf{E}[|\sum_{r=0}^{p-1} z_r \sum_m e^{2\pi i m q t} \mathbf{f}(\beta(s + l + mq + r\frac{q}{p}))|^2] \geq 0 \tag{5.57}$$

In the stationary case, we use the Fourier transform of R and the Poisson summation formula (see [Gro96]) to obtain:

$$< z, \mathbf{M}z >= \frac{\sqrt{2\pi}\alpha}{p} \sum_m [|\sum_{r=0}^{p-1} z_r e^{i\lambda r/\alpha}|^2 \hat{R}(\lambda)]|_{\lambda = \frac{2\pi m}{p} - \frac{2\pi}{\beta} t} \geq 0 \tag{5.58}$$

Thus (5.57) and (5.58) prove the assertion. $\Box$

Concerning the uniform boundedness, the following result gives sufficient conditions for $\mathbf{M}$ to be bounded:

THEOREM 5.19

1. *Consider the nonstationary stochastic model (5.9). Assume* $R \in W(L^\infty(\Box), l^1(\mathbb{Z}^2))$ *where:*

$$W(L^\infty(\Box), l^1(\mathbb{Z}^2)) := \{F : \mathbb{R}^2 \to \mathbb{C} \mid \|F\|_{W(L^\infty(\Box),l^1(\mathbb{Z}^2))} := \sum_{m,n\in\mathbb{Z}} ess \sup_{0\le x,y\le 1} |F(x+m, y+n)| < \infty\}$$

Then:

a) For every $\alpha, \beta > 0$ *with* $\alpha \cdot \beta \in \mathbb{Q}$ *there is a constant* $B_2(\alpha,\beta)$ *independent of* R *such that:*

$$\mathbf{M}(t,s) \le B_2(\alpha,\beta)\|R\|_{W(L^\infty(\Box),l^1(\mathbb{Z}^2))} \quad , \quad \forall t,s$$

b) Assume the function $x \mapsto R(x,x)$ *has persistency length* β*. Then there is a* $\alpha_0 > 0$ *such that for every* $0 < \alpha < \alpha_0$ *with* $\alpha\beta \in \mathbb{Q}$ *there is a constant* $A_2(\alpha,\beta)$ *such that:*

$$\mathbf{M}(t,s) \ge A_2(\alpha,\beta) \quad , \quad \forall t,s$$

2. *Consider the stationary stochastic model (5.14). Assume* $R \in W(L^\infty, l^1)$*. Then:*

a) For every $\alpha, \beta > 0$ *with* $\alpha\beta \in \mathbb{Q}$ *there is a constant* $B_2(\alpha,\beta)$ *independent of* R *such that:*

$$\mathbf{M}(t,s) \le B_2(\alpha,\beta)\|R\|_{W(L^\infty,l^1)} \quad , \quad \forall t,s$$

b) There is a $\alpha_0 > 0$ *such that for every* $0 < \alpha < \alpha_0$ *and* $\beta > 0$ *with* $\alpha\beta \in \mathbb{Q}$ *there is a constant* $A_2(\alpha,\beta)$ *such that:*

$$\mathbf{M}(t,s) \ge A_2(\alpha,\beta) \quad , \quad \forall t,s$$

Proof

In order to obtain the upper bound we need to prove the uniform boundedness of $\rho_{r_1,r_2}(t,s)$. For the nonstationary model:

$$|\rho_{r_1,r_2}(t,s)| \le \sum_{m,k} |R(\beta(s+k+mq+r_1\frac{q}{p}), \beta(s+k+r_2\frac{q}{p}))| \le B_2(\alpha,\beta)\|R\|_{W(L^\infty(\Box),l^1(\mathbb{Z}^2))}$$

with the constant $B_2(\alpha,\beta)$ obtained from adapting the norm on $W(L^\infty(\square), l^1(\mathbb{Z}^2))$ to the translation step β.

For the stationary case, we obtain:

$$|\rho_{r_1,r_2}(t)| \leq \sum_m |R(\frac{mp+r_1-r_2}{\alpha}| \leq \sum_m |R(\frac{m}{\alpha})| \leq B_2(\alpha)\|R\|_{W(L^\infty,l^1)}$$

For part b) we write:

$$\mathbf{M} = M^0 + \sum_{m\neq 0} e^{2\pi i m q t} \begin{bmatrix} \rho^m_{00} & \cdots & \rho^m_{0,p-1} \\ \vdots & & \vdots \\ \rho^m_{p-1,0} & \cdots & \rho^m_{p-1,p-1} \end{bmatrix}$$

In the nonstationary case, the diagonal elements of M^0 are:

$$M^0_{rr} = \sum_k R(\beta(s+k+r\frac{q}{p}), \beta(s+k+r\frac{q}{p})) \geq \delta > 0$$

for $\delta > 0$ from the definition of of persistency length of $x \mapsto R(x,x)$. Notice that:

$$\begin{aligned} \sum_{m\neq 0} |\rho^m_{r_1,r_2}| &= \sum_{m\neq 0}\sum_k |R(\beta(s+k+mq+r_1\frac{q}{p}), \beta(s+k+r_2\frac{q}{p}))| \\ &\leq B_3 \sum_{m\neq 0}\sum_k ess \sup_{0\leq x,y\leq 1} |R(x+k+m\beta q, y+k)| = U \end{aligned}$$

But $\beta q = \frac{p}{\alpha} \geq \frac{1}{\alpha_0}$. Thus the difference between the two arguments is at least $\frac{1}{\alpha_0}$ and

$$U \leq B_3 \sum_{\substack{m,k \\ |m-k|\geq \frac{1}{\alpha_0}}} ess \sup_{\leq x,y\leq 1} |R(x+m, y+k)|$$

If we choose α_0 small enough, we can make U smaller than $\frac{\delta}{2p}$. Then $\mathbf{M}$ is a diagonally dominant matrix and the lower bound follows immediately.

In the stationary case we prove that $\mathbf{M}$ is a diagonally dominant matrix for α small enough, in the following way. Firstly,

$$\mathbf{M}_{rr} = \rho_{rr}(t) = \sum_m e^{2\pi i m q t} R(\frac{mp}{\alpha}) \geq R(0) - \sum_{m\neq 0} |R(\frac{m}{\alpha})|$$

and for $r_1 \neq r_2$:

$$\rho_{r_1,r_2}(t) \leq \sum_m |R(\frac{mp+r_1-r_2}{\alpha})| \leq \sum_{m\neq 0} |R(\frac{m}{\alpha})|$$

Thus for α_0 chosen such that for every $\alpha < \alpha_0$, $\sum_{m\neq 0} |R(\frac{m}{\alpha})| < \frac{1}{2p}R(0)$ we obtain the statement. Q.E.D. □

In the optimization problems (5.34)–(5.37) we look for optimizers that satisfy the assumption $A2$, namely the sets $\{g^1_{mn}; m,n \in \mathbb{Z}\}$ and $\{g^2_{mn}; m,n \in \mathbb{Z}\}$ to be s-Riesz bases. In terms of the matrices Γ^1 and Γ^2 this condition turns into an algebraic easily verifiable relation as the following proposition shows:

PROPOSITION 5.20 *The set $\{g_{mn}; m,n \in \mathbb{Z}\}$ with $g_{mn}(x) = e^{2\pi i m\alpha x}g(x-n\beta)$, $\alpha\beta = \frac{p}{q} > 1$ is a s-Riesz basis if and only if there are constants $0 < A \leq B < \infty$ such that:*

$$A \leq \frac{1}{p}\Gamma^*\Gamma \leq B \tag{5.59}$$

for every $(t,s) \in [0,\frac{1}{q}] \times [0,\frac{1}{p}]$. The matrix Γ is defined similarily to (5.43). The optimal s-Riesz basis bounds are the largest A and smallest B that satisfy (5.59), i.e.:

$$A_{optimal} = \inf_{t,s}\frac{1}{p}\lambda_{min}(\Gamma(t,s)^*\Gamma(t,s)) \quad , \quad B_{optimal} = \sup_{t,s}\frac{1}{p}\lambda_{max}(\Gamma(t,s)^*\Gamma(t,s)) \tag{5.60}$$

where $\lambda_{min}(M)$ denotes the smallest eigenvalue of the matrix M, and $\lambda_{max}(M)$ the largest one.

The standard biorthogonal s-Riesz basis generator $\tilde{g}$ is given by:

$$\tilde{\Gamma} = p\Gamma(\Gamma^*\Gamma)^{-1} \tag{5.61}$$

Proof

Let $f \in L^2(\mathbb{R})$. Then using a similar computation as before gives:

$$< f, S_{g,g;\alpha,\beta}f >= \sum_{m,n} |< f, g_{mn} >|^2 = \frac{1}{p}\int_0^{1/p} ds \int_0^1 dt \|\Gamma^*\Phi\|^2$$

where:

$$\Phi = \Phi(t,s) = [F(t,s) \quad F(t,s+\frac{q}{p}) \quad \cdots \quad F(t,s+(p-1)\frac{q}{p})]^t$$

and $F(t,s)$ is the Zak transform of f. Since $f \in L^2(\mathbb{R}) \mapsto \Phi \in L^2([0,1]\times[0,\frac{1}{p}];\mathbb{C}^p)$ is a unitary operator, (5.59) and (5.60) follow immediately.

On the other hand, making similar computations as for the terms T_3, T_4 we get:

$$\|f-\sum_{m,n}<f,g_{mn}>\tilde{g}_{mn}\|^2=\|f\|^2-\sum_{m,n}<f,\tilde{g}_{mn}><g_{mn},f>$$
$$-\sum_{m,n}<f,g_{mn}><\tilde{g}_{mn},f>+\sum_{m,n}\sum_{m',n'}<f,g_{mn}><\tilde{g}_{mn},\tilde{g}_{m'n'}><g_{m'n'},f>$$

and next:

$$\|f-S_{g,\tilde{g};\alpha,\beta}f\|^2=p\int_0^{1/p}ds\int_0^1 dt\|(I-\frac{1}{p}\tilde{\Gamma}\Gamma^*)\Phi\|^2$$

Now it is easy to see that $S_{g,\tilde{g};\alpha,\beta}$ is an orthogonal projection in $L^2(\mathbb{R})$ if and only if $\frac{1}{p}\tilde{\Gamma}\Gamma^*$ is an orthogonal projection in $\mathbb{C}^p$. And this happens if and only if $\tilde{\Gamma}$ is given by (5.61). Q.E.D. □

5.2.4 Optimization Algorithm

Now we are ready to analyze the optimization problems (5.34)–(5.37). The relations (5.49) and (5.55) give a decomposition of the criterion in terms of finite dimensional matrices, and in both cases we have to minimize a trace. In the nonstationary model, this is:

$$Tr\{(I-\frac{1}{p}\Gamma^2{\Gamma^1}^*)\mathbf{M}(I-\frac{1}{p}\Gamma^1{\Gamma^2}^*)\} \tag{5.62}$$

whereas in the stationary case it has the following form:

$$Tr\{\mathbf{W}(I-\frac{1}{p}\Gamma^2{\Gamma^1}^*)\mathbf{M}(I-\frac{1}{p}\Gamma^1{\Gamma^2}^*)\} \tag{5.63}$$

Notice that if we denote by $X_1=\frac{1}{\sqrt{p}}\mathbf{W}^{-1/2}\Gamma^1$, $X_2=\frac{1}{\sqrt{p}}\mathbf{W}^{1/2}\Gamma^2$ and $\Sigma=\mathbf{W}^{1/2}\mathbf{M}\mathbf{W}^{1/2}$ then (5.63) turns into:

$$j(X_1,X_2)=\ Tr\{(I-X_2X_1^*)\Sigma(I-X_1X_2^*)\} \tag{5.64}$$

In the nonstationary case we set $X_1=\frac{1}{\sqrt{p}}\Gamma^1$, $X_2=\frac{1}{\sqrt{p}}\Gamma^2$ and $\Sigma=\mathbf{M}$ and (5.62) becomes also (5.64).

The optimization problems stated at the beginning of thi section turn into the following four problems:

Problem 1' (suboptimal 1')

Given X_1 with $A_1 \leq X_1^* X_1 \leq B_1$ for every (t,s), find the infimum of (5.64) over X_2 subject to $0 < A_2 \leq X_2^* X_2 \leq B_2 < \infty$ for every (t,s) with A_2, B_2 independent of (t,s).

Problem 2' (suboptimal 2')

Given X_2 with $A_2 \leq X_2^* X_2 \leq B_2$ for every (t,s), find the infimum of (5.64) over X_1 subject to $0 < A_1 \leq X_1^* X_1 \leq B_1 < \infty$ for every (t,s) with A_1, B_1 independent of (t,s).

Problem 3' (optimal')

Given Σ, find the infimum of (5.64) over X_1 and X_2 subject to $0 < A_1 \leq X_1^* X_1 \leq B_1 < \infty$, $0 < A_2 \leq X_2^* X_2 \leq B_2 < \infty$ for every (t,s) with A_1, A_2, B_1, B_2 independent of (t,s).

Problem 4' (optimal'-tight)

Given Σ, solve the Problem 3' under the additional constraint $X_1 = X_2$.

Notice that Problem 4' is not exactly equivalent to the Problem 4 stated before, however the difference will involve a simple rescaling as we shall see in a moment.

The following lemma solves these problems:

LEMMA 5.21 *Consider the optimization problem:*

$$j^* = inf\ Tr\{(I - X_2 X_1^*)\Sigma(I - X_1 X_2^*)\} \tag{5.65}$$

a) For fixed X_1 and given $\Sigma > 0$, the optimizer X_2 is uniquely given by:

$$X_2^{opt} = \Sigma X_1 (X_1^* \Sigma X_1)^{-1} \tag{5.66}$$

and the optimum value of (5.65) is given by:

$$j^* = \ Tr\{\Sigma - \Sigma X_1 (X_1^* \Sigma X_1)^{-1} X_1^* \Sigma\} \tag{5.67}$$

If $0 < A \leq \Sigma \leq B < \infty$ and $0 < A_1 \leq X_1^* X_1 \leq B_1 < \infty$ then

$$\frac{AA_1}{(BB_1)^2} \leq (X_2^{opt})^* X_2^{opt} \leq \frac{BB_1}{(AA_1)^2} \tag{5.68}$$

b) For fixed X_2 and given Σ, the optimizer X_1 is uniquely given by:

$$X_1^{opt} = X_2 (X_2^* X_2)^{-1} \tag{5.69}$$

and the optimum of (5.65) is then:

$$j^* = Tr\{\Sigma - X_2(X_2^* X_2)^{-1} X_2 \Sigma\} \tag{5.70}$$

If $0 \leq A_2 \leq X_2^* X_2 \leq B_2 < \infty$*, then:*

$$\frac{1}{B_2} \leq (X_1^{opt})^* X_1^{opt} \leq \frac{1}{A_2} \tag{5.71}$$

c) For a given Σ*, the optimizers* X_1*,* X_2 *of (5.65) are given by:*

$$X_1^o = F \cdot L \tag{5.72}$$

$$X_2^o = F \cdot L^{*-1} \tag{5.73}$$

where F *is the* $p \times q$ *matrix whose columns are the first* q *eigenvectors of* Σ*, i.e.*

$$F = [f_1|\dots|f_q] \quad , \quad \Sigma f_j = \sigma_j f_j \ \text{ and } \ \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_q \geq \dots \sigma_p \tag{5.74}$$

and L *is an arbitrary invertible* $q \times q$ *matrix. The optimum of (5.65) is then:*

$$j^* = \sum_{j=q+1}^{p} \sigma_j \tag{5.75}$$

If we choose L *such that* $\|L\| \leq b$ *and* $\|L^{-1}\| \leq a$ *then:*

$$\frac{1}{a^2} \leq (X_1^o)^* X_1^o \leq b^2 \quad , \quad \frac{1}{b^2} \leq (X_2^o)^* X_2^o \leq a^2 \tag{5.76}$$

d) If Σ *is given, then the optimizers* $X = X_1 = X_2$ *of (5.65) are:*

$$X = F \cdot U \tag{5.77}$$

where F *is as in (5.74) and* U *is an arbitrary unitary matrix. The optimal value is given again by (5.75) and* $X_1^* X_1 = X_2^* X_2 = I_{q \times q}$ *(the* $q \times q$ *identity matrix).*

Proof

a) It is easy to check that for $X_2 = X_2^{opt} + Z$ we get:

$$j(X_1, X_2^{opt} + Z) = j(X_1, X_2^{opt}) + Tr\{Z X_1^* \Sigma X_1 Z^*\} \geq j(X_1, X_2^{opt})$$

Thus the minimum j is obtained for $Z = 0$, hence (5.66). Remark that if $\Sigma > 0$ and $X_1^* X_1 > 0$, then the minimizer is unique. (5.67) and (5.68) are obtained by straightforward computations.

b) As before, for $X_1 = X_1^{opt} + Z$ we obtain:

$$j(X_1^{opt} + Z, X_2) = j(X_1^{opt}, X_2) + Tr\{X_2 Z^* \Sigma Z X_2^*\} \geq j(X_1^{opt}, X_2)$$

The conclusions are obtained similarily.

c) We have to look for the minimum of (5.67) over X_1 or, equivalently, the minimum of (5.70) over X_2. The first way is more algebraically difficult; the second way, however, takes us more easily to the result.

First we note that $P_2 = X_2(X_2^* X_2)^{-1} X_2^*$ is the orthogonal projection onto $Ran\, X_2$. Hence, what we look for is a rank $q = dim\, Ran\, X_2$ orthogonal projection P_2 that maximizes the $Tr\{P_2 \Sigma\}$. Let $f_1, \ldots, f_p$ be a set of orthonormal eigenvectors of Σ whose eigenvalues are $\sigma_1 \geq \ldots \geq \sigma_p$. Then:

$$Tr\{P_2 \Sigma\} = \sum_{j=1}^{p} < f_j, P_2 \Sigma f_j >= \sum_{j=1}^{p} \sigma_j < f_j, P_2 f_j >= \sum_{j=1}^{p} p_j \sigma_j$$

where $0 \leq p_j \leq 1$ and $\sum_{j=1}^{p} p_j = q$. Then the largest value is achieved for $p_1 = \cdots = p_q = 1$ and $p_{q+1} = \cdots = p_p = 0$, which means P_2 is the projection onto $span\{f_1, \ldots, f_q\}$. This gives (5.73) and (5.74). (5.72) is obtained from (5.69). The other part of the conclusion is straightforward.

d) The conclusion is immediately drawn from the part c). Q.E.D. □

REMARK 5.22 In the stationary case, if $W(s) \not\equiv 1$ then $X_1 = X_2$ gives $\Gamma^1 = \mathbf{W}\Gamma^2$. In terms of Zak transform this means $G^1(t,s) = W(s) \cdot G^2(t,s)$ and in the time-domain this implies:

$$g^1(x) = g^2(x) W(\frac{x}{\beta}) \tag{5.78}$$

In a physical implementation, (5.78) is equivalent with a prescaling of the signal by $W(\frac{x}{\beta})$.

This lemma completely solves the first two suboptimal problems. For the optimal problems (5.36) and (5.37) we have to choose invertible L and unitary U for every (t,s). Despite this apparently simple requirement, the (practical) optimal problem is far from a “good” solution. What we mean by a “good” solution is a pair of well-localized windows in time-frequency domain. As a bad example of what may happen, consider the following example:

EXAMPLE 5.23 Consider the weight $w = \mathbf{1}_{[0,\beta]}$ (which gives $\mathbf{W}(s) = I_{p\times p}$ for every s) and the autocovariance function $\hat{R}(\xi)$ given by:

$$\hat{R}(\xi) = |H(i\xi)|^2 \quad , \quad H(s) = \frac{158.1s(s^2+60s+300^2)}{(s^2+20s+100^2)(s^2+200s+1000^2)} \tag{5.79}$$

If we choose $L(t,s) = I_{q\times q}$ for every $(t,s) \in [0,\frac{1}{q}) \times [0,\frac{1}{p})$ then for the optimal $g^1 = g^2 = g^{opt}$ we get the solutions shown in Figures 5.5, 5.6 and 5.7 for three sampling ratios: $\frac{p}{q} = 2, 5, 7$. We notice how poorly localized in time-frequency domain they are.

The boundary conditions are the main obstruction in choosing freely the invertible L. Indeed, the Zak transform obeys the conditions (5.42). However, in terms of the matrix Γ, the boundary conditions have a very messy form. The relation $t \mapsto t+\frac{1}{q}$ is simple, however when we try to connect Γ at $s+\frac{1}{p}$ to the matrix at s we get a very messy albeit linear relation, that cannot be written as a matrix product.

Unfortunately, the optimal solution cannot be well-localized in time-frequency domain. A Balian-Low type phenomenon happens with the optimal solution. We shall prove next an amalgam nonlocalization theorem, similar to the one that holds for WH Riesz basis generators (see [BHW95]).

THEOREM 5.24 *Suppose $\alpha, \beta > 0$, $\alpha\beta = \frac{p}{q} \in \mathbb{Q}$ and the stochastic model is given such that the matices $\mathbf{M}(t,s)$ (in the nonstationary case) or $\mathbf{W}(s)$ and $\mathbf{M}(t)$ (in the stationary case) are bounded, invertible, with bounded inverse, for a.e. (t,s). Then, if (g^1,g^2) denotes an optimizer of (5.36), none of the functions $g^1, \hat{g}^1, g^2, \hat{g}^2$ belongs to $W(C_0,l^1)$ (i.e. $g^1, \hat{g}^1, g^2, \hat{g}^2 \in L^2(\mathbb{R}) \setminus W(C_0,l^1)$) where:*

$$W(C_0,l^1) := \{f \mid f \text{ continuous and } \; f \in W(L^\infty,l^1)\} \tag{5.80}$$

Proof

Note first that $\mathbf{M}(t,s)$, respectively $\mathbf{W}(s)$ and $\mathbf{M}(t)$, are 1-periodic in t and 1-periodic in s, by construction. Thus $\Sigma(t+1,s) = \Sigma(t,s)$, $\Sigma(t,s+1) = \Sigma(t,s)$ which implies the same periodicity for the eigenvectors, so that $F(t+1,s) = F(t,s)$, $F(t,s+1) = F(t,s)$. Next, since $G^1(t+1,s) = G^1(t,s)$ and $G^1(t,s+1) = e^{-2\pi i t}G^1(t,s)$ we obtain for $\Gamma^1(t,s)$:

$$\Gamma^1(t+1,s) = \Gamma(t,s) \quad , \quad \Gamma^1(t,s+1) = \Gamma^1(t,s)\cdot D(t)$$

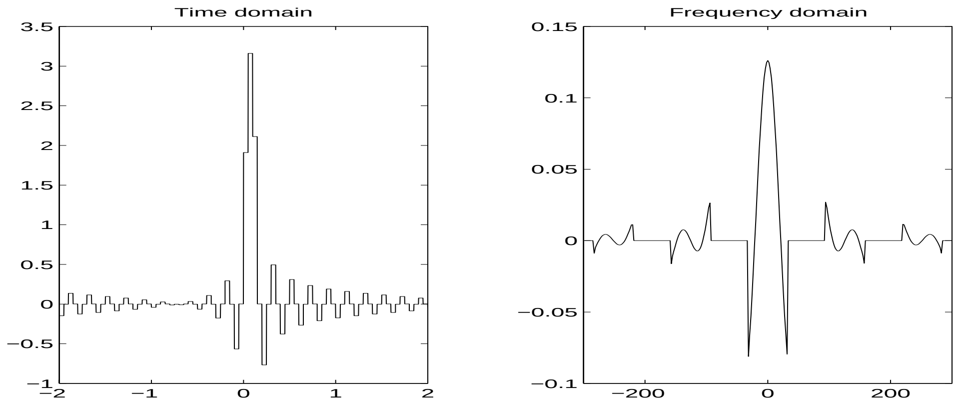

Figure 5.5: The optimal solution for $p = 2$, $q = 1$ and $L(t,s) = I$

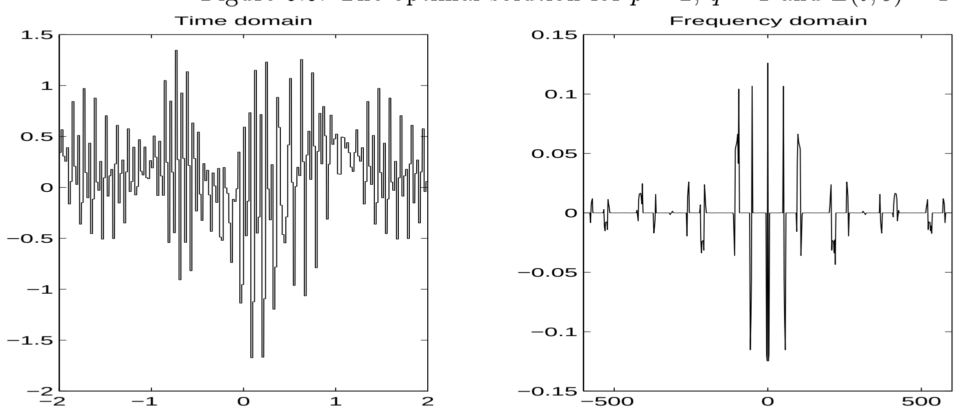

Figure 5.6: The optimal solution for $p = 5$, $q = 1$ and $L(t,s) = I$

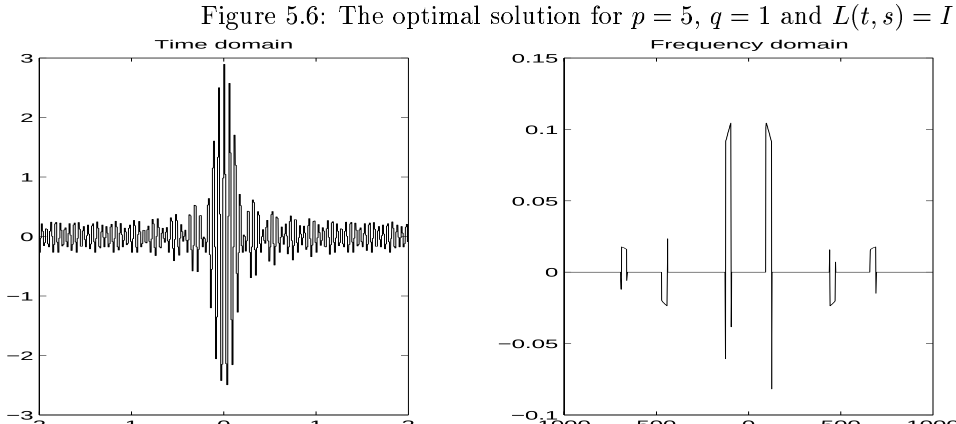

Figure 5.7: The optimal solution for $p = 9$, $q = 1$ and $L(t,s) = I$

where $D(t)$ is a $q \times q$ diagonal matrix whose (r,r)-entry is given by $e^{-2\pi i(t+\frac{r-1}{q})}$, $r = 1, 2 \ldots q$. In the nonstationary case $\Gamma^1(t,s) = \sqrt{p}F(t,s) \cdot L(t,s)$, whereas in the stationary case $\Gamma^1(t,s) = \sqrt{p}W^{1/2}(s) \cdot F(t,s) \cdot L(t,s)$. In either case we obtain for $L(t,s)$ the following relations:

$$L(t+1,s) = L(t,s) \quad , \quad L(t,s+1) = L(t,s) \cdot D(t) \tag{5.81}$$

If we denote $f(t,s) = det\, L(t,s)$, (5.81) implies the following:

$$f(t+1,s) = f(t,s) \quad , \quad f(t,s+1) = f(t,s) \cdot e^{-2\pi iqt} \tag{5.82}$$

Let us return now to g^1. Because $\Gamma^{1*}\Gamma^1 = pL^*F^*FL$ (by construction) and $F^*F = I_{q\times q}$ we see that $det(\Gamma^{1*}\Gamma^1) = p^q\, det(L^*L)$. It then follows from Proposition 5.20 that there are constants $a, b > 0$ such that $0 < a \leq |f(t,s)| \leq b < \infty$, for a.e. (t,s). Suppose now $g^1 \in W(C_0, l^1)$. Then $G^1(t,s)$ is continuous at every (t,s). This can happen only if all entries in $L(t,s)$ are continuous. Thus $f(t,s)$ should be continuous which, together with $0 < a \leq f(t,s) \leq b < \infty$ implies there is a continuous real-valued function $\varphi(t,s)$ such that $f(t,s) = |f(t,s)| \cdot e^{i\varphi(t,s)}$. If this is so, then (5.82) implies:

$$\varphi(t+1,s) = \varphi(t,s) + 2\pi N \quad , \quad \varphi(t,s+1) = \varphi(t,s) - 2\pi qt + 2\pi M \tag{5.83}$$

for some fixed integers $M, N \in \mathbb{Z}$. Evaluating φ on the four corners of the unit square in the (t,s)-plane we obtain:

$$0 = (\varphi(0,0) - \varphi(0,1)) + (\varphi(0,1) - \varphi(1,1)) + (\varphi(1,1) - \varphi(1,0)) + (\varphi(1,0) - \varphi(0,1))$$
$$= -2\pi M - 2\pi N - 2\pi q + 2\pi M + 2\pi N = -2\pi q$$

Contradiction! The contradiction comes from the assumption $g^1 \in W(C_0, l^1)$. Similarly we prove the statement for the other three functions $\hat{g}^1, g^2, \hat{g}^2$. $\square$

This theorem proves that we cannot expect to find well-localized optimal solutions. Instead we might look for suboptimal windows that decay fast in time-frequency domain. In section 5.4 we present examples that achieve a distortion within 3% larger than the optimal value.

We end this section by establishing an asymptotic formula for the optimal J^* given by:

$$J^* = \beta q \int_0^{1/p} ds \int_0^{1/q} dt \sum_{j=q+1}^{p} \sigma_j(t,s) \tag{5.84}$$

Assume the stationary case with $w = \mathbf{1}_{[0,\beta]}$. This implies $\mathbf{W}(s) = I_{p\times p}$ for every s. Furthermore, we assume the support of R is included in $[-\frac{1}{\alpha}, \frac{1}{\alpha}]$. Then we get $\rho_{r_1,r_2}(t) = R(0)\delta_{r_1,r_2}$. Thus $\mathbf{M}(t) = R(0)I_{p\times p}$, for every t. Plugging into (5.84), we obtain:

$$J^* = \beta q \int_0^{1/p} ds \int_0^{1/q} dt R(0)(p-q) = \beta R(0)(1-\frac{q}{p}) \tag{5.85}$$

This corresponds to the graph in Figure 5.2. Note the continuous transition from $\beta R(0)$ $= \mathbf{E}\int_0^\beta |f(x)|^2 dx$ to 0, unlike the deterministic case in Figure 5.1.

5.3 Distortion and Rate

In this section we start analyzing the one-channel compression problem using windowed Fourier transform. The block diagram is drawn in Figure 5.8.

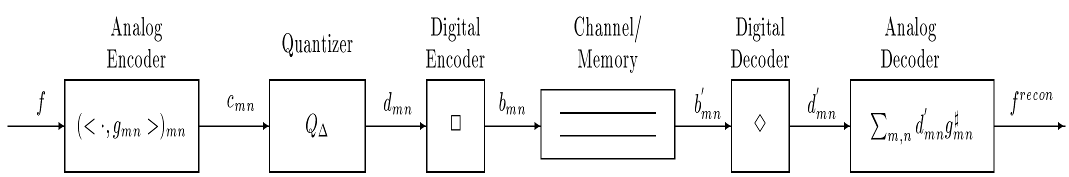

Figure 5.8: The one-channel compression block diagram

The *analog encoder* is a bank of filters that compute the coefficients $c_{mn} =< f, g_{mn} >$ of the signal f, with respect to the vectors g_{mn}. Next the coefficients are passed through the *quantizer* Q_Δ. We quantize separately the real and imaginary part of the coefficients. The quantizer is assumed to have a uniform interlevel distance Δ. Thus $l = Q_\Delta(x)$ means $x \in [(l-\frac{1}{2})\Delta, (l+\frac{1}{2})\Delta)$, so that $l = \lfloor \frac{x}{\Delta} + \frac{1}{2} \rfloor$. Hence d_{mn}, the quantization output, carries the interlevel label to which the real or imaginary part of c_{mn} belongs: $d_{mn} = Q_\Delta(c_{mn})$. The *digital encoder* is an entropic encoder, based on the variance of c_{mn}; it converts the meaningful labels into sequences of bits $b_{mn} = D(d_{mn})$, for $(m,n) \in \mathcal{S}$, where $\mathcal{S}$ is the set of meaningful coefficients (we shall return later to how $\mathcal{S}$ is to be determined). Next comes the *channel* (in a transmission problem) or the *memory block* (in a pure compression problem). In any case we assume this block to be ideal, i.e. $b'_{mn} = b_{mn}$. The *digital decoder* simply inverts the digital decoder: $d'_{mn} = D^{-1}(b'_{mn})$. The last block is the analog decoder which reconstructs the original signal f based on the information it has gotten, namely the coefficients (labels) d'_{mn}. For reconstruction we use a Weyl-Heisenberg synthesis operator with the window $g^{\#}$. Thus $f^{recon} = \sum_{m,n} d'_{mn} g^{\#}_{mn}$ and putting all these relations together we get:

$$f^{recon} = \sum_{(m,n)\in\mathcal{S}} Q_\Delta(< f, g_{mn} >) g^{\#}_{mn} \tag{5.86}$$

Our goal is to analyze the reconstruction error $\|f - f^{recon}\|_X$ and to estimate the memory capacity (the number of bits used) or the channel rate, based on a certain stochastic model.

First we have to choose which coefficients are going to be encoded. As we mentioned before, the digital encoder will convert into a binary sequence only those coefficients whose variances are larger than a threshold. Note that the variances can be computed a priori once we know the statistics of the signal. Therefore the entire scheme can be designed based on this statistics. It is shown (see [Dav72]) that the most economic way of encoding a discrete sequence of quantized zero-mean random variables $(x_i)_i$ is to allocate to each variable x_i a number of bits R_i in the following way: if $\mathbf{E}[|x_i|^2] < \frac{\Delta^2}{12}$ then $R_i = 0$, otherwise it is given by:

$$\mathbf{E}[|x_i|^2] = \frac{\Delta^2}{12} 2^{2R_i} \tag{5.87}$$

Thus the total number of bits used by the entropic encoder would be:

$$N_{total} = \frac{1}{2} \sum_{i, \mathbf{E}[|x_i|^2] \geq \Delta^2/12} \log_2\left(\frac{12}{\Delta^2} \mathbf{E}[|x_i|^2]\right) \tag{5.88}$$

Next we analyze this scheme for the two stochastic models presented before (nonstationary and stationary). We assume from now on that we deal with real-valued signals and the window g is normalized to 1 in L^2-sense, i.e. $\|g\|_{L^2(\mathbb{R})} = 1$.

5.3.1 The Nonstationary Model: The Memory Capacity

In this model the signal has finite energy, $f \in L^2(\mathbb{R})$. The variance of the coefficient c_{mn} is given by:

$$\sigma^2_{mn} := \mathbf{E}[|c_{mn}|^2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(x,y) e^{-2\pi i m\alpha x} \overline{g(x - n\beta)} e^{2\pi i m\alpha y} g(y - n\beta) dx\, dy \tag{5.89}$$

Since we deal with real-valued signals (as we already have), the variances of the real and respectively imaginary part of c_{mn} are given by:

$$\mu_{mn}^2 := \mathbf{E}[|Re(c_{mn})|^2] = \frac{1}{4}\int\int R(x,y)(e^{-2\pi i m\alpha x}\overline{g(x-n\beta)} + e^{2\pi i m\alpha x}g(x-n\beta))$$
$$(e^{-2\pi i m\alpha y}\overline{g(y-n\beta)} + e^{2\pi i m\alpha y}g(y-n\beta))dx\,dy \tag{5.90}$$

$$\nu_{mn}^2 := \mathbf{E}[|Im(c_{mn})|^2] = \frac{1}{4}\int\int R(x,y)(e^{-2\pi i m\alpha x}\overline{g(x-n\beta)} - e^{2\pi i m\alpha x}g(x-n\beta))$$
$$(e^{-2\pi i m\alpha y}\overline{g(y-n\beta)} - e^{2\pi i m\alpha y}g(y-n\beta))dx\,dy \tag{5.91}$$

Notice that $\mu_{mn}^2 + \nu_{mn}^2 = \sigma_{mn}^2$. We have the following result:

LEMMA 5.25 *Suppose $g \in L^2(\mathbb{R})$ and $\int_{-\infty}^{\infty} R(x,x)dx < \infty$. Then for every $\varepsilon > 0$ there are $M_\varepsilon, N_\varepsilon > 0$ such that for every $(m,n) \in \mathbb{Z}^2 \setminus ([-M_\varepsilon, M_\varepsilon] \times [-N_\varepsilon, N_\varepsilon])$ we have $\sigma_{mn}^2 < \varepsilon$.*

Proof

Fix $\varepsilon > 0$. Since $|R(x-y)| \leq \sqrt{R(x,x)}\sqrt{R(y,y)}$ we obtain

$$\sigma_{mn}^2 \leq \int (R(x,x))^{1/2}|g(x-n\beta)|dx$$

The translation operator T_t, $f \mapsto f(\cdot - t) = T_t f$, on $L^2(\mathbb{R})$ is weakly convergent to zero when $t \to \infty$. Therefore there is a N_ε such that for every $|n| > N_\varepsilon$ and $m \in \mathbb{Z}$, $\sigma_{mn}^2 < \varepsilon$. On the other hand, for every fixed n, a Riemann-Lebesgue lemma argument proves the existence of a $M_{n,\varepsilon}$ such that $\sigma_{mn}^2 < \varepsilon$, for every $|m| > M_{n,\varepsilon}$. Choose $M_\varepsilon = \max_{|n| \leq N_\varepsilon} M_{n,\varepsilon}$. Then for every $(m,n) \notin [-M_\varepsilon, M_\varepsilon] \times [-N_\varepsilon, N_\varepsilon]$, $\sigma_{mn}^2 < \varepsilon$. Q.E.D. $\square$

This lemma proves that only a finite number of coefficients will be encoded. Notice also that $\mathbf{E}[|Re(c_{mn})|^2] \leq \mathbf{E}[|c_{mn}|^2]$, $\mathbf{E}[|Im(c_{mn}|^2] \leq \mathbf{E}[|c_{mn}|^2]$. Then the total number of bits used by the entropic encoder is (asymptotically) given by:

$$Memory = \frac{1}{2}\sum_{\substack{m,n \\ \mu_{mn}^2 \geq \Delta^2/12}} \log_2(\frac{12}{\Delta^2}\mu_{mn}^2) + \frac{1}{2}\sum_{\substack{m,n \\ \nu_{mn}^2 \geq \Delta^2/12}} \log_2(\frac{12}{\Delta^2}\nu_{mn}^2) \tag{5.92}$$

5.3.2 The Stationary Case: The Rate

In the stationary model, signals are assumed to be in $W(L^2, l^\infty)$. An easy computation shows that for $R \in L^2(\mathbb{R})$, $g \in L^{4/3}(\mathbb{R})$ and real-valued signals, the variances of the coefficient c_{mn}, $Re\, c_{mn}$ and $Im\, c_{mn}$, respectively, are:

$$\sigma_{mn}^2 = \mathbf{E}[|c_{mn}|^2] = \sqrt{2\pi} \int_{-\infty}^{\infty} d\xi \hat{R}(\xi) |\hat{g}(\xi - 2\pi m\alpha)|^2 \leq \sqrt{2\pi} \|R\|_{L^2} \|\hat{g}\|_{L^4}^2 \tag{5.93}$$

$$\begin{aligned}\mu_{mn}^2 = \mathbf{E}[|Re(c_{mn})|^2] =& \frac{\sqrt{2\pi}}{4} \int_{-\infty}^{\infty} d\xi \hat{R}(\xi) |e^{2\pi i mn\alpha\beta} \hat{g}(\xi - 2\pi m\alpha) \\ &+ e^{-2\pi i mn\alpha\beta} \overline{\hat{g}(-\xi - 2\pi m\alpha)}|^2\end{aligned} \tag{5.94}$$

$$\begin{aligned}\nu_{mn}^2 = \mathbf{E}[|Im(c_{mn})|^2] =& \frac{\sqrt{2\pi}}{4} \int_{-\infty}^{\infty} d\xi \hat{R}(\xi) |e^{2\pi i mn\alpha\beta} \hat{g}(\xi - 2\pi m\alpha) \\ &- e^{-2\pi i mn\alpha\beta} \overline{\hat{g}(-\xi - 2\pi m\alpha)}|^2\end{aligned} \tag{5.95}$$

The scheme works in the following way: in a β time interval, say $[N\beta, (N+1)\beta)$, the transmitter has to send the meaningful coefficients $c_{m,N}$ (or $c_{m,N-d}$ for some fixed delay $d > 0$). The meaningful coefficients are those given by $\mu_{mn}^2 \geq \frac{\Delta^2}{12}$ or $\nu_{mn}^2 \geq \frac{\Delta^2}{12}$ (when each real and imaginary part is quantized separately). Using again the weakly convergence to zero of the translation operator T_t when $t \to \infty$, there exists a $M > 0$ such that for every $|m| > M$, $\sigma_{mn}^2 \leq \frac{\Delta^2}{12}$. Thus we have to send only a finite number of quantized values. This yields the following rate:

$$Rate = \frac{1}{2\beta} \sum_{\substack{m,n \\ \mu_{mn}^2 \geq \frac{\Delta^2}{12}}} \log_2\left(\frac{12}{\Delta^2} \mu_{mn}^2\right) + \frac{1}{2\beta} \sum_{\substack{m,n \\ \nu_{mn}^2 \geq \frac{\Delta^2}{12}}} \log_2\left(\frac{12}{\Delta^2} \nu_{mn}^2\right) \tag{5.96}$$

Later we shall return to this formula to establish the asymptotic behaviour when $\Delta \to 0$.

Returning to the analysis of the general scheme 5.8, we have to analyze the distortion given by this compression scheme. The distortion measures the reconstruction error as:

$$Distortion = \mathbf{E}\|f - f^{recon}\|_X^2$$

Let $\mathcal{S}_1 = \{(m,n)|\mu_{mn}^2 \geq \frac{\Delta^2}{12}\}$, $\mathcal{S}_2 = \{(m,n)|\nu_{mn}^2 \geq \frac{\Delta^2}{12}\}$. Then the reconstructed signal has the following form:

$$f^{recon} = \sum_{(m,n)\in\mathcal{S}_1} Q_\Delta(Re(< f, g_{mn} >))g_{mn}^{\#} + i \sum_{(m,n)\in\mathcal{S}_2} Q_\Delta(Im(< f, g_{mn} >))g_{mn}^{\#}$$

Then:

$$\begin{aligned}
\sqrt{Distortion} &\leq (\mathbf{E}\|f - \sum_{m,n} c_{mn} g_{mn}^{\#}\|_X^2)^{1/2} \\
&+ (\mathbf{E}\| \sum_{(m,n)\notin\mathcal{S}_1} Re(< f, g_{mn} >)g_{mn}^{\#} + i \sum_{(m,n)\notin\mathcal{S}_2} Im(< f, g_{mn} >)g_{mn}^{\#}\|_X^2)^{1/2} \\
&+ (\mathbf{E}\| \sum_{(m,n)\in\mathcal{S}_1} (Re(< f, g_{mn} >) - Q_\Delta(Re(< f, g_{mn} >)))g_{mn}^{\#} \qquad (5.97) \\
&+ i \sum_{(m,n)\in\mathcal{S}_2} (Im(< f, g_{mn} >) - Q_\Delta(Im(< f, g_{mn} >)))g_{mn}^{\#}\|_X^2)^{1/2} \\
&= \sqrt{J} + \sqrt{J_\varepsilon} + \sqrt{J_q}
\end{aligned}$$

where J represents *the stochastic approximation error* due to the incompleteness of the set $\{g_{mn}; m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$; J_ε is *the truncation error* and represents those coefficients that are excluded from encoding; J_q is *the quantization error* and is due to the uncertainty introduced by the quantizer. Our problem is to bound and control each term. The stochastic error has been studied in the previous section. We analyze now the other two terms.

5.3.3 The Nonstationary Model: General Relations

We start with the quantization error. Suppose $g^{\#}$ is a s-Riesz basis generator with Riesz basis bounds $A^{\#}$, $B^{\#}$. Then:

$$\begin{aligned}
J_q \leq\ & B^{\#}(\sum_{(m,n)\in\mathcal{S}_1} \mathbf{E}[|Re(< f, g_{mn} >) - Q_\Delta(Re(< f, g_{mn} >))|^2] \\
& + \sum_{(m,n)\in\mathcal{S}_2} \mathbf{E}[|Im(< f, g_{mn} >) - Q_\Delta(Im(< f, g_{mn} >))|^2]
\end{aligned}$$

For an arbitrary distribution of $< f, g_{mn} >$, the difference $|Re(< f, g_{mn} >) - Q_\Delta(Re(< f, g_{mn} >))| \leq \frac{\Delta}{2}$ which implies $\mathbf{E}[|Re(< f, g_{mn} >) - Q_\Delta(Re(< f, g_{mn} >))|^2] \leq \frac{\Delta^2}{4}$. The same relation holds true for the imaginary part too. However, if we assume the signal f is gaussian, the upper bound becomes $\frac{\Delta^2}{12}$ instead of $\frac{\Delta^2}{4}$. The same thing is obtained if we assume the $< f, g_{mn} >$ is uniformly distributed on each quantization interlevel. Anyway in general we obtain:

$$J_q \leq B^{\#}\frac{\Delta^2}{4}(\#\mathcal{S}_1 + \#\mathcal{S}_2) \tag{5.98}$$

More information on the autocovariance function is required to progress further from this point.

For the truncation error, we use the same estimate with the help of the upper bound of the s-Riesz basis generated by $g^{\#}$. Then:

$$J_\varepsilon \leq B^{\#}(\sum_{(m,n)\notin\mathcal{S}_1} \mathbf{E}[|Re(c_{mn})|^2] + \sum_{(m,n)\notin\mathcal{S}_2} \mathbf{E}[|Im(c_{mn})|^2] \tag{5.99}$$

Suppose we have a symmetric distribution of the real and imaginary part of the coefficients. Then we can replace (5.99) by:

$$J_\varepsilon \leq 2B^{\#}\sum_{(m,n)\notin\mathcal{S}} \mathbf{E}[|c_{mn}|^2] \quad , \quad \mathcal{S} = \{(m,n); \mathbf{E}[|c_{mn}|^2] \geq \frac{\Delta^2}{12}\} \tag{5.100}$$

Suppose $\mathcal{S} = [-M, M] \times [-N, N]$ for some $M, N > 0$. Then, using (5.89) we obtain:

$$\begin{aligned} J_\varepsilon \quad \leq \quad & \frac{B^{\#}}{\alpha}\sum_m \int_{-\infty}^{\infty} dx\ R(x, x+\frac{m}{\alpha}\sum_{|n|\geq N}\overline{g(x-n\beta)}g(x+\frac{m}{\alpha}-n\beta) \\ & +\frac{B^{\#}}{\alpha}\int\int dx\, dy\ R(x,y)\sum_{|m|\geq M} e^{2\pi i m\alpha(y-x)}\sum_{|n|<N}\overline{g(x-n\beta)}g(y-n\beta) \end{aligned} \tag{5.101}$$

This is all we can say for this case. Future work will study the upper bounds (5.98) and (5.101) under some additional assumptions. An aparently interesting assumption is to assume the following factorization of the autocovariance function: $R(t,s) = u(t-s)v(\frac{t+s}{2})$ for some u and $v \in L^1(\mathbb{R})$.

5.3.4 The Stationary Model: Asymptotic Analysis

For the stationary model, we establish first an upper bound similar to the s-Riesz basis bound, The following lemma gives this bound:

LEMMA 5.26 *Suppose $g, w \in W(L^\infty, l^1)$. Then $T_g^* : l^{2,\infty} \to L^2_w$ defined by $T_g^*(c) = \sum_{m,n\in\mathbb{Z}} c_{mn}g_{mn}$ is well defined and bounded by:*

$$\begin{aligned} \|T_g^*\|^2_{B(l^{2,\infty},L^2_w)} \quad & \leq \quad \frac{1}{\alpha}\sum_n \|\sum_k w(\cdot+\frac{k}{\alpha})|g(\cdot+\frac{k}{\alpha}-n\beta)|^2\|_{L^\infty(0,\frac{1}{\alpha})} \\ & \leq \quad C_{\alpha,\beta}\|w\|_{W(L^\infty,l^1)}\|g\|_{W(L^\infty,l^1)}\|g\|_\infty \end{aligned} \tag{5.102}$$

Proof

Take $(c_{mn})_{m,n\in\mathbb{Z}} \in l^{2,\infty}$. Recall $\|c\|^2_{l^{2,\infty}} = \sup_n \sum_m ||c_{mn}|^2$. Then:

$$\begin{aligned}
\|T_g^* c\|_w^2 &= \int_{-\infty}^{\infty} w(x)|\sum_n (\sum_m c_{mn} e^{2\pi i m\alpha x})g(x-n\beta)|^2 dx \\
&\leq \sum_n \int w(x)|\sum_m c_{mn} e^{2\pi i m\alpha x}|^2 |g(x-n\beta)|^2 dx \\
&\leq \sum_n \int_0^{1/\alpha} dx (\sum_k w(x+\frac{k}{\alpha})|g(x+\frac{k}{\alpha}-n\beta)|^2)|\sum_m c_{mn} e^{2\pi i m\alpha x}|^2 \\
&\leq \sum_n \|\sum_k w(\cdot+\frac{k}{\alpha}|g(\cdot+\frac{k}{\alpha}-n\beta)|^2\|_{L^\infty(0,\frac{1}{\alpha})}\frac{1}{\alpha}\sum_m |c_{mn}|^2,
\end{aligned}$$

from which we obtain (5.102). Q.E.D. □

Let us denote $B^{2,\infty} = \sum_n \|\sum_k w(\cdot+\frac{k}{\alpha})|g^\#(\cdot+\frac{k}{\alpha}-n\beta)|^2\|_{L^\infty(0,\frac{1}{\alpha})}$. Then for the quantization error we get a result similar to (5.98):

$$J_q \leq B^{2,\infty}\frac{\Delta^2}{4}\sup_n(\#\mathcal{S}_{1n}+\#\mathcal{S}_{2n}) \tag{5.103}$$

where $\mathcal{S}_{1n} = \{(m,n)\in\mathcal{S}_1\}$, $\mathcal{S}_{2n} = \{(m,n)\in\mathcal{S}_2\}$. Assuming symmetry between the distribution of real and imaginary parts of the coefficients c_{mn} we get:

$$J_q \leq 2B^{2,\infty}\frac{\Delta^2}{4}(\#\mathcal{S}) \tag{5.104}$$

where $\mathcal{S} = \{m|\mathbf{E}[|c_{mn}|^2] \geq \Delta^2/12\}$. We give now a rough evaluation of the cardinality of $\mathcal{S}$ based on (5.93) and the following assumptions: $\hat{R}(\xi)$ is concentrated in a band of size $2b_R$ (2 because $\hat{R}$ is even in frequency domain - recall we assumed real-valued signals) and the support of $\hat{g}$ is much narrower than $2b_R$. Then the number of coefficients is roughly constant and it is given by:

$$\#\mathcal{S} \approx \frac{2b}{2\pi\alpha} = \frac{b}{\pi\alpha}$$

Thus:

$$J_q \approx \frac{bB^{2,\infty}}{2\pi\alpha}\Delta^2 \sim C\ \Delta^2 \tag{5.105}$$

which says that J_q decays to 0 as Δ^2 when $\Delta \to 0$.

For the truncation error, using the previous lemma again we obtain a first estimate similar to (5.99) where instead of $B^{\#}$we have to use $B^{2,\infty}$:

$$J_\varepsilon \leq B^{2,\infty}(\sup_n \sum_{m \notin \mathcal{S}_{1n}} \mathbf{E}[|Re(c_{mn})|^2] + \sup_n \sum_{m \notin \mathcal{S}_{2n}} \mathbf{E}[|Im(c_{mn})|^2])$$

Next, assuming again a symmetry in the distribution of the real and imaginary part we obtain:

$$J_\varepsilon \leq 2B^{2,\infty} \sup_n \sum_{m \notin \mathcal{S}} \mathbf{E}[|c_{mn}|^2] = 2B^{2,\infty} \sum_{m \notin \mathcal{S}} \sigma_{mn}^2$$

with $\mathcal{S} = \{m | \sigma_{mn}^2 > \frac{\Delta^2}{12}\}$. Assuming $\mathcal{S} = [-M, M]$ we obtain:

$$J_\varepsilon \leq 2\sqrt{2\pi} B^{2,\infty} \sum_{|m| \geq M} \int_{-\infty}^{\infty} \hat{R}(\xi) |\hat{g}(\xi - 2\pi m\alpha)|^2 d\xi \tag{5.106}$$

The assumptions made before to obtain (5.105) would now give $J_\varepsilon = 0$. Thus if we assume that both the autocovariance function and the window are band-limited, we get rid of the truncation error provided we take into account all the (finite) non-zero coefficients.

Another (more realistic) model of R and g is to assume that both decay in frequency domain as:

$$|\hat{R}(\xi)| \leq \frac{C_1}{(1+|\xi|)^a} \quad , \quad |\hat{g}(\xi)| \leq \frac{C_2}{(1+|\xi|)^b} \tag{5.107}$$

with $a, b > 1$. The assumption on $\hat{R}$ is particularily useful when we assume that our signal is the output of a linear system excited by white noise. Then $\hat{R}(\xi) = |H(i\xi)|^2$ where $H(s)$ is the linear system transfer function. We shall give an asymptotic estimation of the rate and the truncation and quantization errors.

We start by estimating the variance σ_{mn}^2:

$$\begin{aligned}
\sigma_{mn}^2 &= \sqrt{2\pi} \int_{-\infty}^{\infty} |\hat{R}(\xi + \pi m\alpha)| \cdot |\hat{g}(\xi - \pi m\alpha)|^2 d\xi \\
&\leq C' \int_{-\infty}^{\infty} \frac{d\xi}{(1+|\xi+\pi m\alpha|)^a (1+|\xi - \pi m\alpha|)^{2b}} \\
&\leq \frac{C'}{(\pi m\alpha)^{2b}} \int_{-\infty}^{0} \frac{d\xi}{(1+|\xi+\pi m\alpha|)^a} + \frac{C'}{(\pi m\alpha)^a} \int_0^{\infty} \frac{d\xi}{(1+|\xi - \pi m\alpha|)^{2b}} \\
&\leq \frac{C'}{(\pi m\alpha)^{2b}} \int_{-\infty}^{\infty} \frac{d\xi}{(1+|\xi|)^a} + \frac{C'}{(\pi m\alpha)^a} \int_{-\infty}^{\infty} \frac{d\xi}{(1+|\xi|)^{2b}} \leq \frac{C}{m^r}
\end{aligned} \tag{5.108}$$

where $r = \min(a, 2b)$ and an estimate of C is:

$$C = 2C_1C_2\sqrt{2\pi}(\frac{1}{(a-1)(\pi\alpha)^{2b}} + \frac{1}{(2b-1)(\pi\alpha)^{a}})$$

Next we estimate M_Δ such that for $|m| > M_\Delta$, $\sigma_{mn}^2 < \frac{\Delta^2}{12}$. Using (5.107) we obtain for M_Δ the following estimate:

$$M_\Delta = \frac{(12C)^{1/r}}{\Delta^{2/r}} \tag{5.109}$$

Therefore we have to encode at most $2M_\Delta + 1$ coefficients. This gives the following estimate for the quantization error J_q (see (5.104):

$$J_q \leq 2B^{2,\infty}\frac{\Delta^2}{4}(2M_\Delta + 1) \approx C_q\Delta^{2(1-\frac{1}{r})} \tag{5.110}$$

with an estimate of C_q given by $C_q = (12C)^{1/r}B^{2,\infty}$.

For the truncation error we use the following estimate (see (5.106)):

$$J_\varepsilon \leq 2\sqrt{2\pi}B^{2,\infty}\sum_{|m|\geq M_\Delta}\frac{C}{m^r} \approx \tilde{C}'' \int_{M_\Delta}^{\infty}\frac{dx}{x^r} = \frac{C''}{M_\Delta^{r-1}}$$

Using now (5.109) we obtain:

$$J_\varepsilon \leq C_\varepsilon\Delta^{2(1-\frac{1}{r})} \tag{5.111}$$

with an estimate of C_ε given by $C_\varepsilon = 2C\sqrt{2\pi}B^{2,\infty}/(r-1)$. We notice that J_ε and J_q are both of the same order in Δ. Moreover, for $a > 1$ and $b > \frac{1}{2}$ they both decay to zero as $\Delta \to 0$. Thus by choosing a sufficiently small Δ we can make $J_\varepsilon + J_q < J$. The moral of this computation was to show that asymptotically (i.e. for $\Delta \to 0$), the dominant term in the distortion (5.97) is given by the stochastic approximation error J. We end this section by finding an asymptotic approximation of the rate, under the same assumptions as before. We use (5.96) and again we replace μ_{mn} and ν_{mn} by σ_{mn} and we get:

$$Rate \leq \frac{1}{\beta}\sum_{|m|\leq M_\Delta}\log_2(\frac{12}{\Delta^2}\sigma_{mn}^2) = \frac{2}{\beta}\sum_{1\leq m\leq M_\Delta}(\log_2\frac{12C}{\Delta^2} - r\log_2 m)$$

Note that M_Δ has been chosen so that $\log_2 \frac{12C}{\Delta^2} = r \log_2 M_\Delta$. Then, when we approximate the sum by an integral we get:

$$Rate \leq \frac{2r}{\beta} \int_1^{M_\Delta} (\log_2 M_\Delta - \log_2 x) dx \approx \frac{2r}{\beta \ln 2} M_\Delta$$

Thus:

$$Rate \leq \frac{(12C)^{1/r} 2r}{\beta \ln 2} \Delta^{-2/r} \tag{5.112}$$

We notice that on each coefficient we spent an average of $\frac{2r}{\ln 2}$ bits. We also notice that the upper bound of the rate goes to ∞ when $\Delta \to 0$, a very natural conclusion since we are going to send more and more coefficients.

5.4 Case Study

Here are the numerical results for the suboptimal problem. We used a stationary 4-pole Markov process given by (see also (5.79))

$$\hat{R}(\xi) = |H(i\xi)|^2 \quad , \quad H(s) = \frac{158.1 s (s^2 + 60s + 300^2)}{(s^2 + 20s + 100^2)(s^2 + 200s + 1000^2)}$$

with the characteristic function of $[0, \beta]$ as weight. Here $\beta = 0.1$, as in the example 5.23. In figure 5.9 we represent the autocovariance function.

We took the gaussian:

$$g^1(x) = e^{-1000x^2} \tag{5.113}$$

as the first window. Its plot is given in figure 5.10.

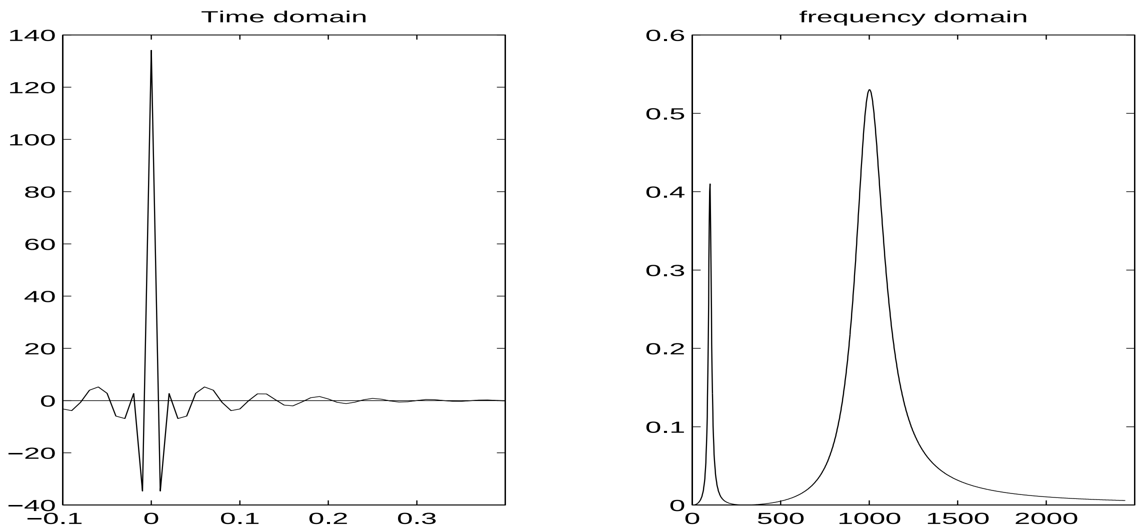

Figure 5.9: The autocovariance function of the stationary process

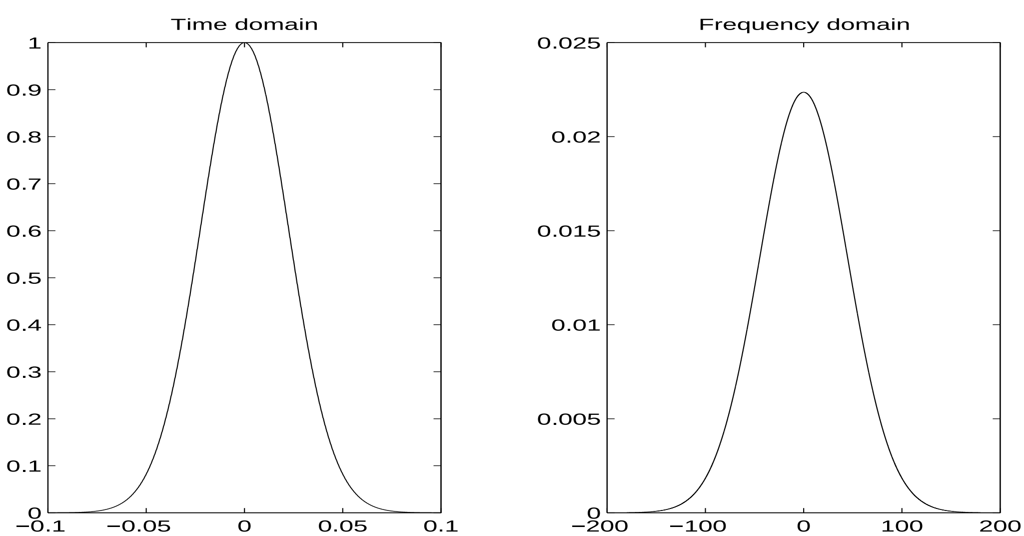

Figure 5.10: The gaussian window

We used Lemma 5.21, (5.66), to find the optimal g^2 that minimizes the distortion (5.55). The following table summarizes the numerical results. Note the maximal distortion (obtained for $\alpha\beta = \infty$ or $g^2 = 0$) is $J_{max} = R(0) = 134.4$:

p/q	$A_{g^1} \cdot 10^2$	$B_{g^1} \cdot 10^2$	J	J_{opt}	$(J - J_{opt})/J_{opt}$
6/5	0.5149	8.333	22.71	21.57	5.2%
5/4	0.6509	8	26.98	26.09	3.4%
4/3	0.9002	7.5	33.22	32.46	2.3%
3/2	1.444	6.669	43.68	42.76	2.15%
5/3	1.983	6.01	52.35	51.2	2.2%
2/1	2.827	5.136	66.27	64.92	2.07%
3/1	3.815	4.109	92.51	83.95	10.1%
4/1	3.907	4.02	98.74	97.49	1.2%
5/1	3.91	4.017	107.3	104.5	2.6%
6/1	3.91	4.017	117.5	106.2	10.6%
7/1	3.91	4.017	119.4	108.8	9.74%
8/1	3.91	4.017	107.2	106.2	0.94%
9/1	3.91	4.017	121	111.5	8.5%
10/1	3.91	4.017	127.9	113.5	12.68%

Table 5.1: Numerical results for the suboptimal problem with gaussian window

Next we represent the solutions of the suboptimal problem when g^1 is the gaussian (5.113).

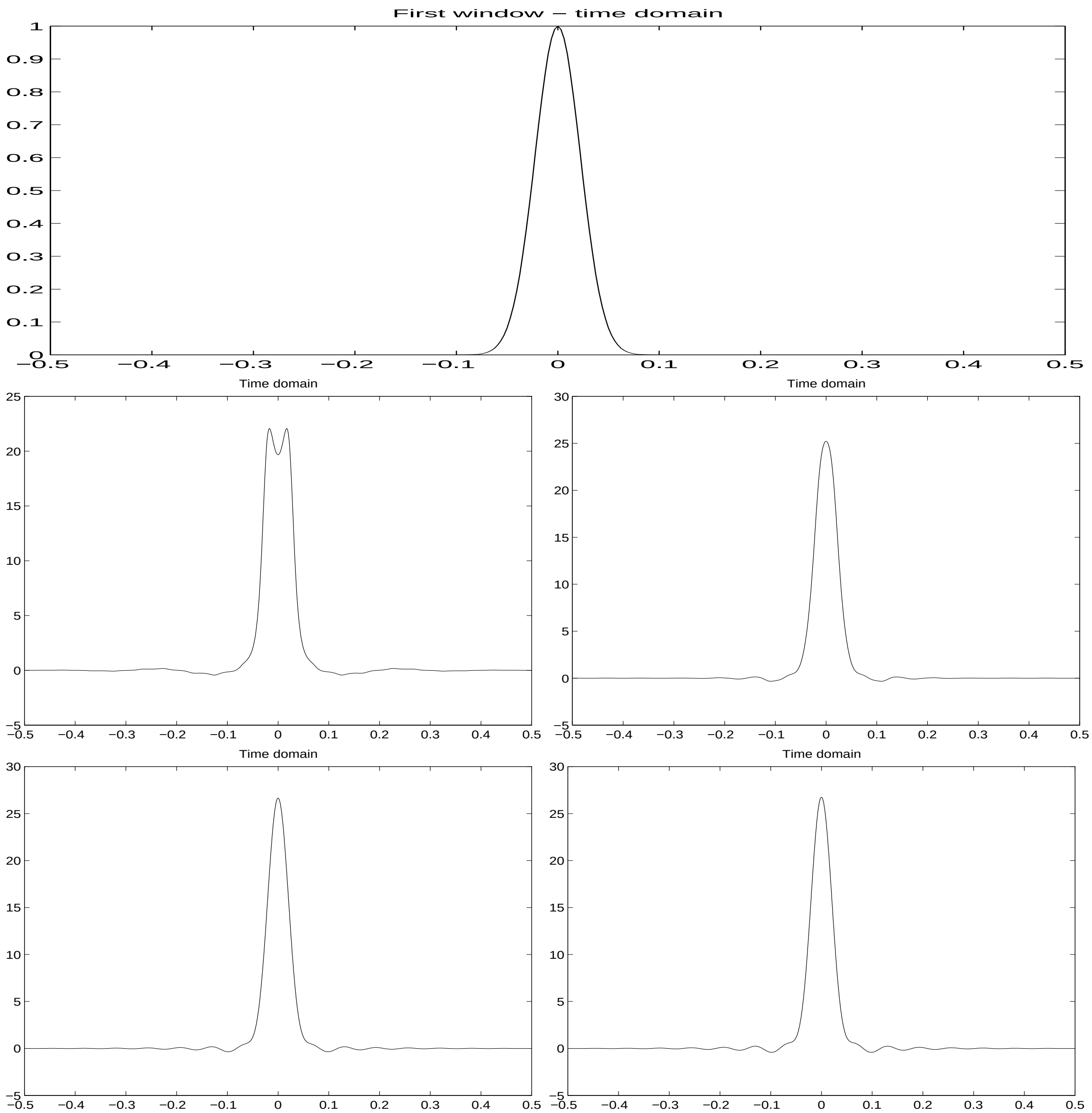

Figure 5.11: The gaussian g^1 given by (5.113) (top) and the suboptimal g^2 found in time domain for various choices of p and q: Middle left: $p = 2, q = 1$; Middle right: $p = 3, q = 1$; Bottom left: $p = 6, q = 1$; Bottom right: $p = 9, q = 1$.

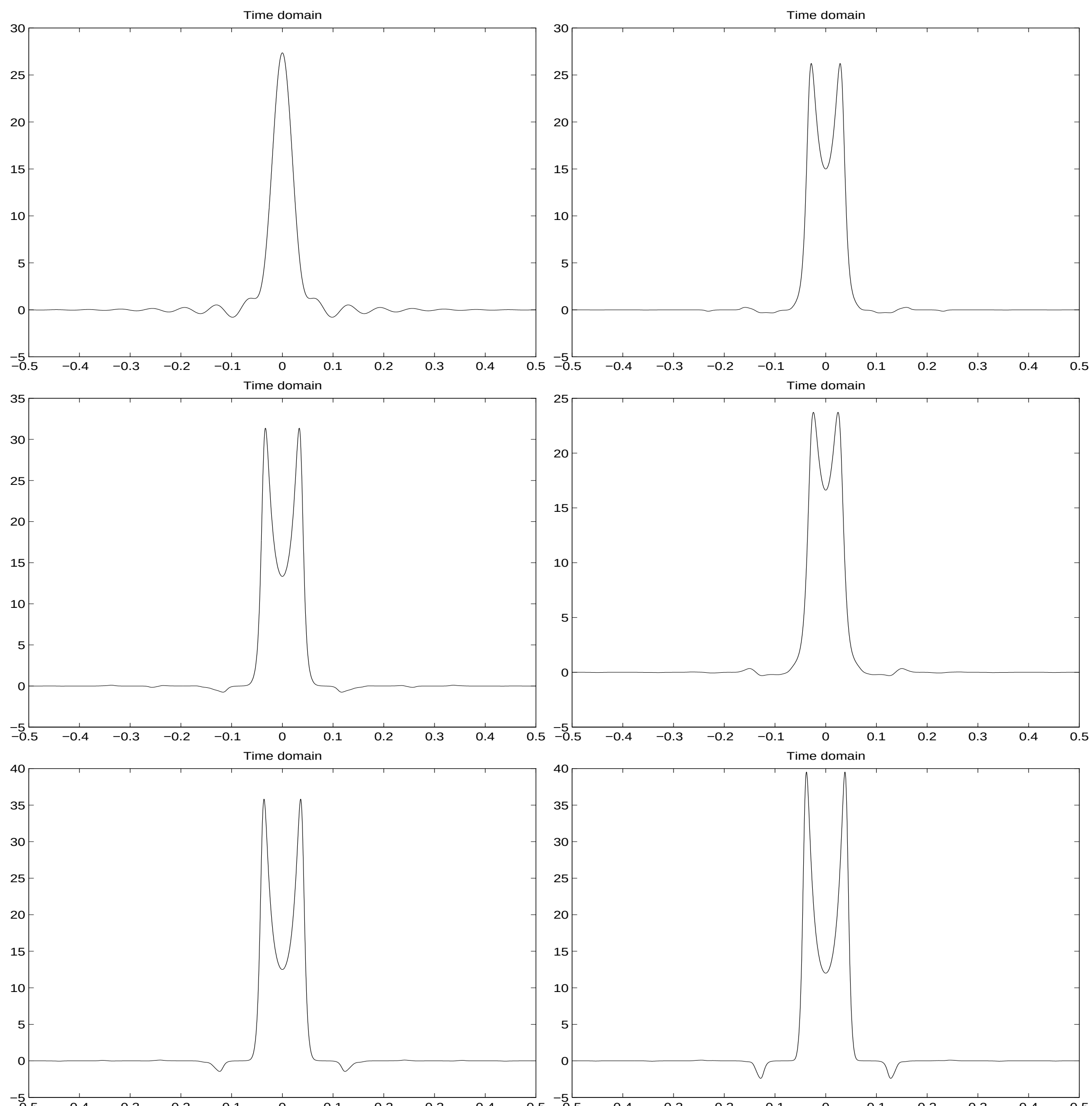

Figure 5.12: (continued) Top left: $p = 10, q = 1$; Top right: $p = 3, q = 2$; Middle left: $p = 4, q = 3$; Middle right: $p = 5, q = 3$; Bottom left: $p = 5, q = 4$; Bottom right: $p = 6, q = 5$

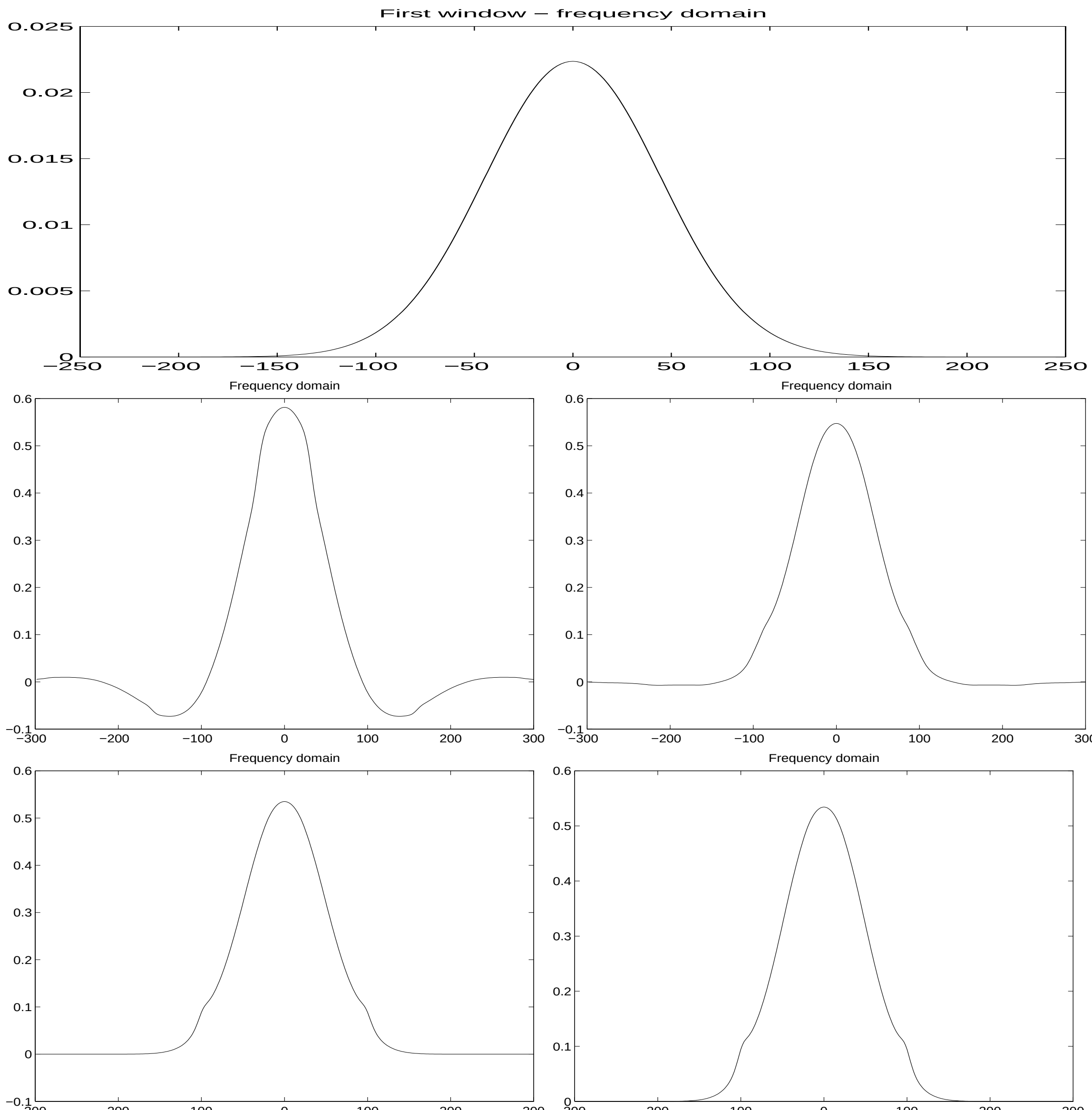

Figure 5.13: The gaussian g^1 given by (5.113) (top) and the suboptimal g^2 found in frequency domain for various choices of p and q: Middle left: $p = 2, q = 1$; Middle right: $p = 3, q = 1$; Bottom left: $p = 6, q = 1$; Bottom right: $p = 9, q = 1$.

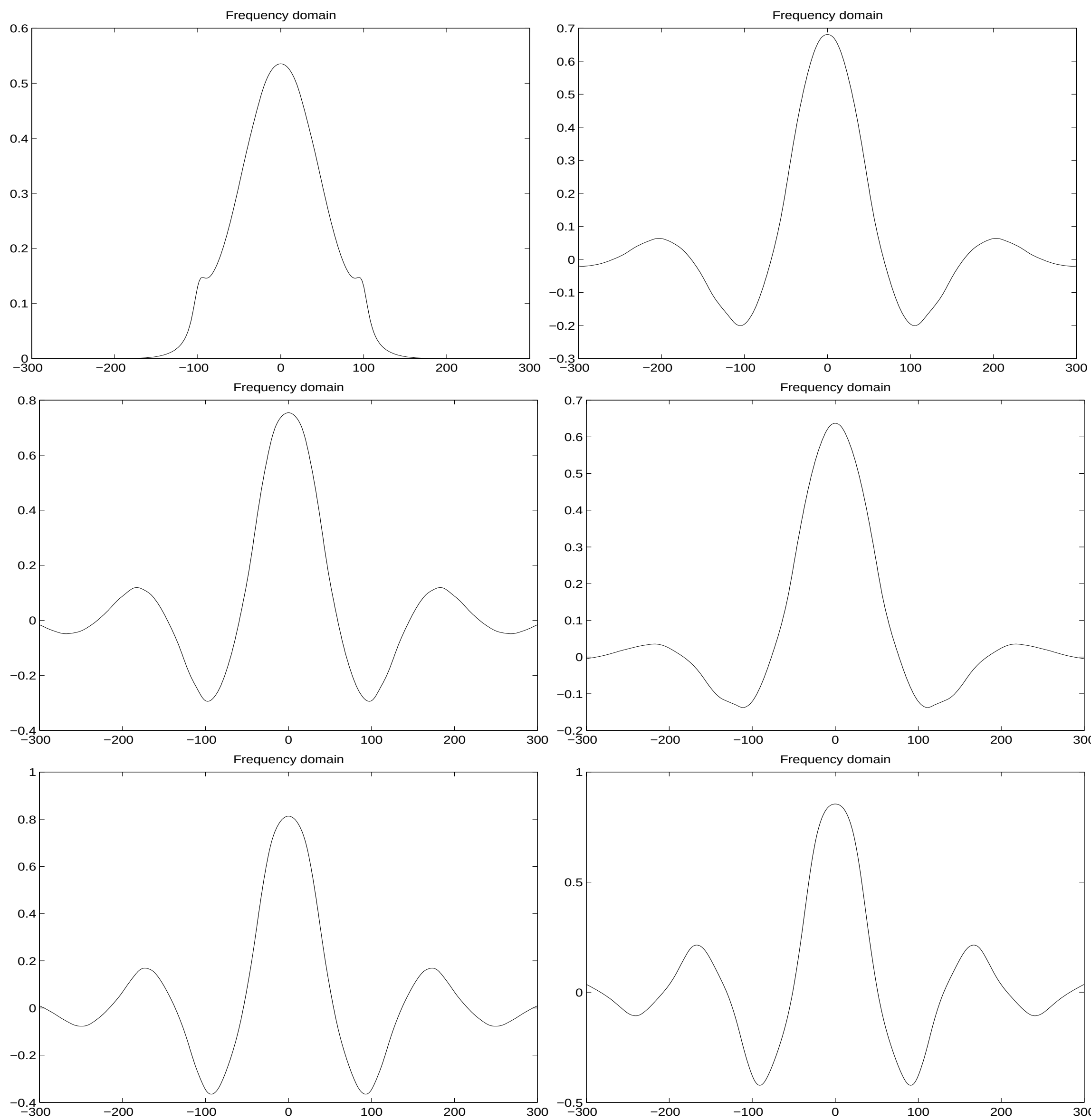

Figure 5.14: (continued) Top left: $p = 10, q = 1$; Top right: $p = 3, q = 2$; Middle left: $p = 4, q = 3$; Middle right: $p = 5, q = 3$; Bottom left: $p = 5, q = 4$; Bottom right: $p = 6, q = 5$

In the next figure we compare the distortion for optimal and sub-optimal cases. In the suboptimal case we used the gaussian window defined in (5.113) for g^1, and we found the optimal g^2 for this fixed f^1.

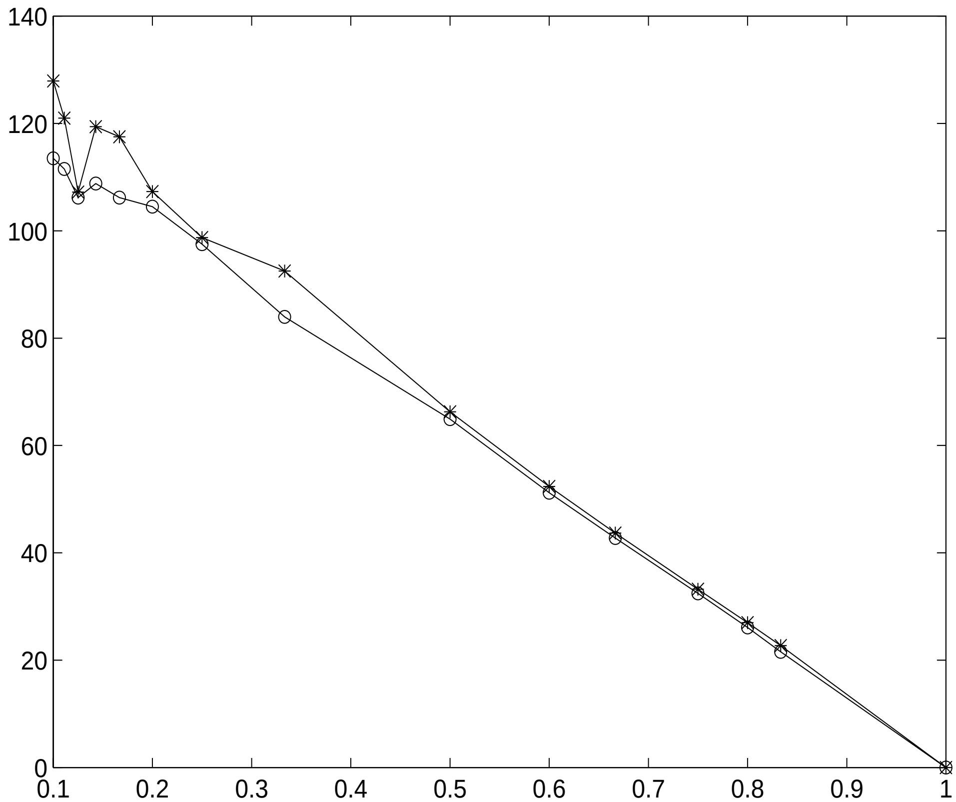

Figure 5.15: The Optimal and Suboptimal Distortions versus $\frac{1}{\alpha\beta}$

We note that significant differences (higher than 3%) between the optimal and suboptimal distortions, appear only for half of the deficit ratios we examined, namely for $\frac{1}{\alpha\beta} = \frac{5}{6}, \frac{4}{5}, \frac{1}{3}, \frac{1}{6}, \frac{1}{7}, \frac{1}{9}, \frac{1}{10}$. Next we shall show a design procedure to find better suboptimal solutions. We set it as our task to find well-localized solutions in time-frequency domain that achieve a distortion that is less than 3% above the optimal value.

Step1. In the first step we find a couple of optimal solutions to give a "hint" where the optimal solution should lay in the time-frequency domain. To do this we use the *Projection Algorithm* based on the following choice for $L(t,s)$ in (5.72) and (5.73):

$$L = \frac{1}{\sqrt{p}} F^* \mathbf{W}^{-1/2} \Gamma^{nice} \tag{5.114}$$

where Γ^{nice} is the matrix (5.43) associated to a well-localized function g^{nice}. The optimal window g^1 obtained in this way, represents the projection of the "nice" function g^{nice} into the eigenspace spaned by the first q eigenvectors of Σ, (see 5.74).

We start with two chices for the function g^{nice}. Both of these are "bump" functions, plotted in Figures 5.16 and 5.17 and defined by:

$$g^{bump_1}(x) = \frac{1}{\sqrt{\beta}} \begin{cases} 1 & , \quad |x| \le \frac{\beta}{4} \\ e^{-10*(|x|-\frac{\beta}{4})^2} & , \quad |x| > \frac{\beta}{4} \end{cases}$$

$$g^{bump_2}(x) = \begin{cases} 1 & , \quad |x| \le \frac{\beta}{4} \\ 1 + cos(\pi(\frac{|x|}{\beta} - \frac{1}{4})) & , \quad \frac{\beta}{4} < |x| \le \frac{3\beta}{4} \\ 0 & , \quad |x| > \frac{3\beta}{4} \end{cases}$$

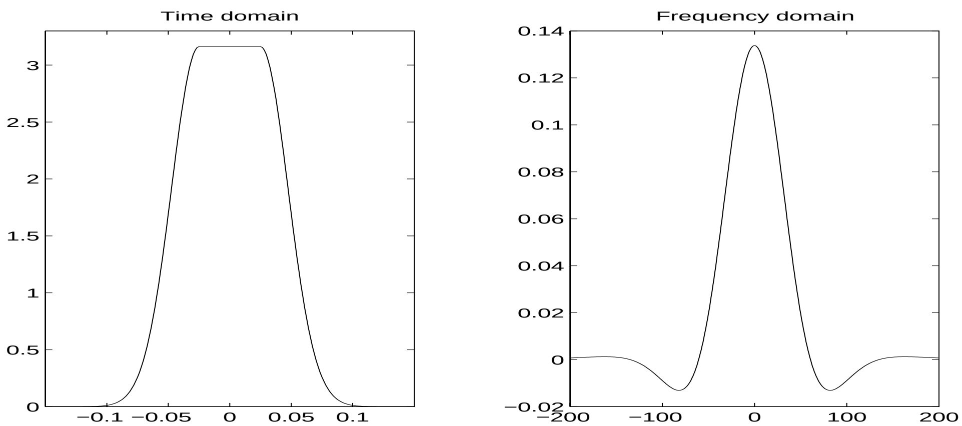

Figure 5.16: The bump g^{bump_1} window in time and frequency domain

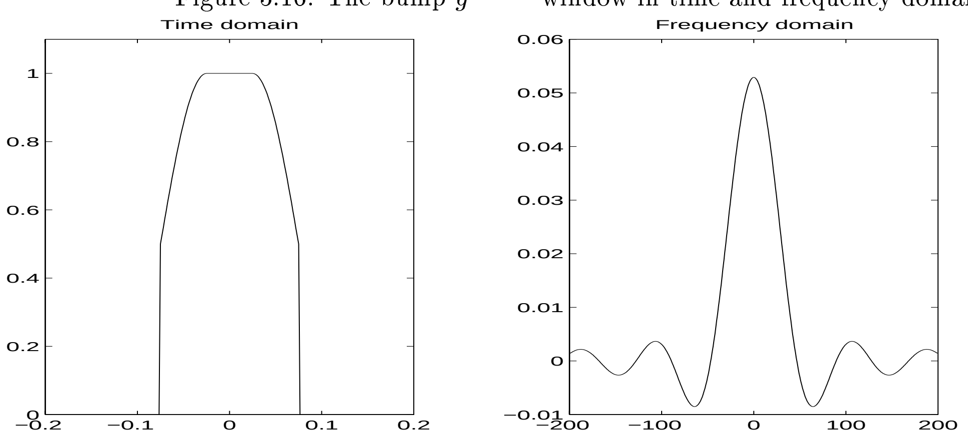

Figure 5.17: The bump g^{bump_2} window in time and frequency domain

Next we perform the projection onto the eigenspace spanned by the first q eigenvectors of Σ. This results in a function g^1. For both choices of g^{nice}, the spaces spanned by the $\{g^1_{mn}; m, n \in \mathbb{Z}\}$ are the same, but the g^1 themselves need not be, and in fact are not. Figures 5.18, 5.19 show the plots of g^1 starting from either g^{bump_1} or g^{bump_2}, as well as the plots of the corresponding optimal g^2; their Fourier transforms can be found in Figures 5.20, 5.21.

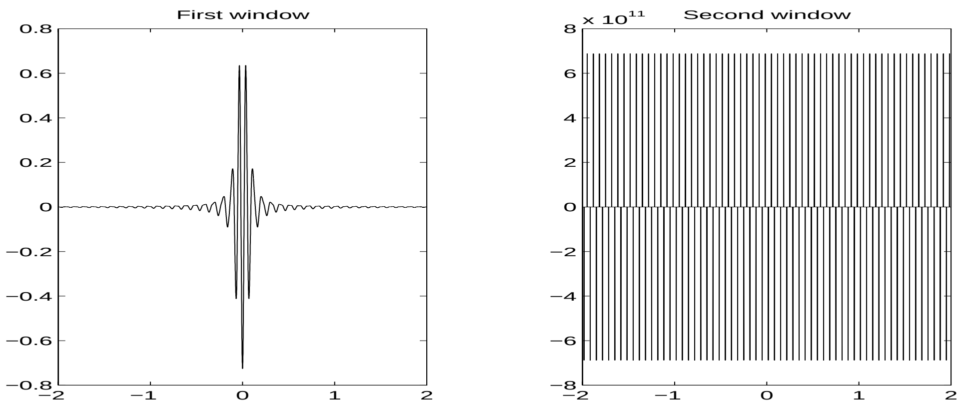

Figure 5.18: The optimal windows for $p = 3$, $q = 1$ in time domain with the bump g^{bump_1}

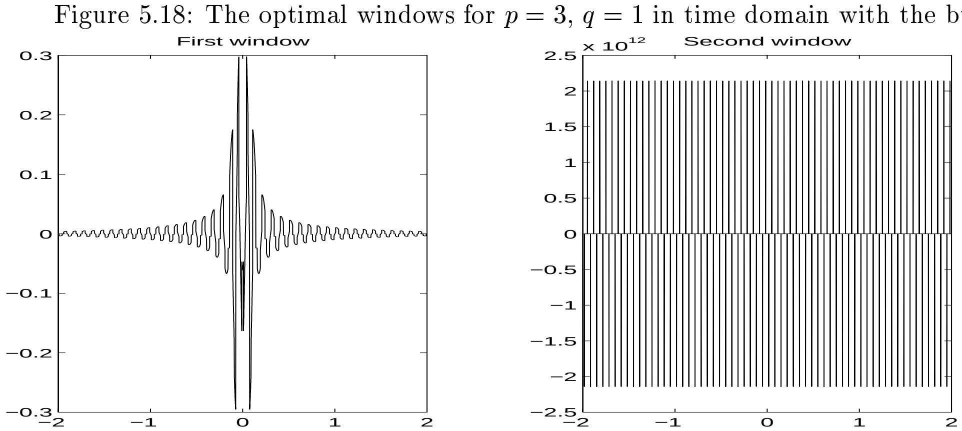

Figure 5.19: The optimal windows for $p = 3$, $q = 1$ in time domain with the bump g^{bump_1}

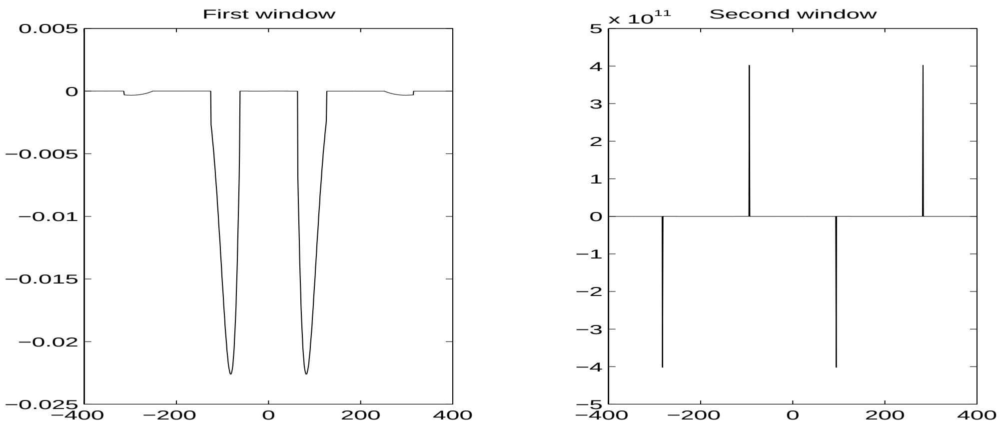

Figure 5.20: The optimal windows for $p = 3$, $q = 1$ in frequency domain with the bump g^{bump_1}

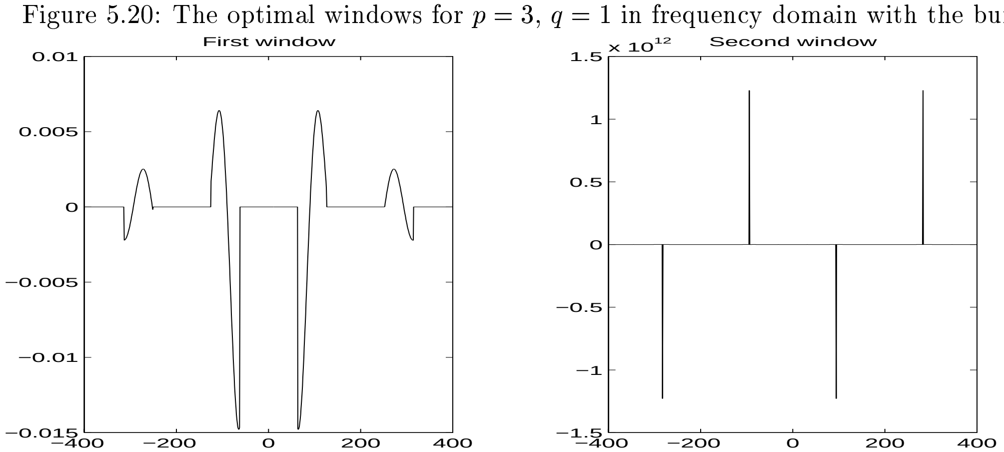

Figure 5.21: The optimal windows for $p = 3$, $q = 1$ in frequency domain with the bump g^{bump_2}

Step 2. These are obviously not very good chices for g^1, g^2, even if they are "optimal" from the point of view of our criterion. It is not even clear that they give rise to s-Riesz bases. Let us look a little closer at their behaviour. For instance, if we concentrate on the "big wiggles" in g^1 obtained from g^{bump_2}, then we see that they correspond to rather sharp frequency cut-offs, as shown in Figure 5.22

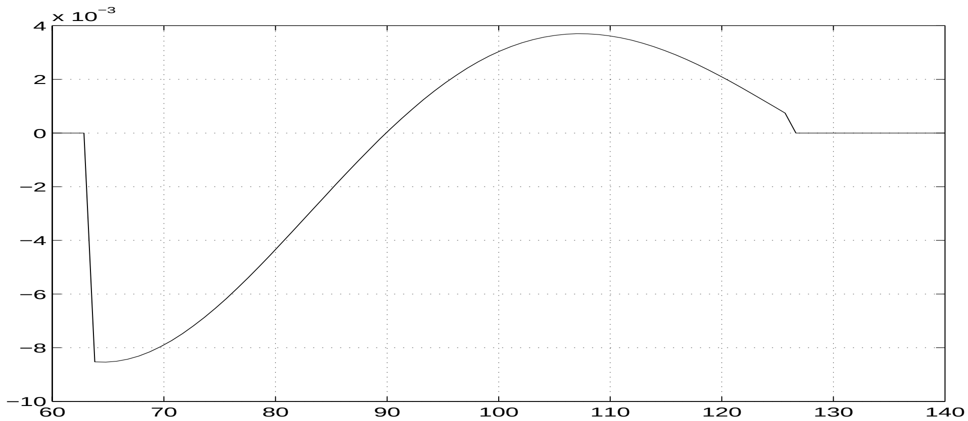

Figure 5.22: Zoomin-out the optimal $\hat{g}^1$ for $p = 3$ and $q = 1$

The frequency band is about $[65 \ldots 125]$.

In this step we shall try to construct other (non-optimal) g^1 that approximate this behaviour, and find the corresponding g^2. As Figure 5.22 shows, the optimal $\hat{g}^1$ is concentrated mostly on the band $[-125 \ldots -65] \cup [65 \ldots 125]$. Let us now design a well-localized window that satisfies this localization constraint. However, an analytic computation with Zak transform shows that this interval is too small for g^1 to give rise to a s-Riesz basis with a well-localized biorthogonal (note $2\pi\alpha = 188.5$ and $\frac{2\pi\alpha}{3} > 60 = 125 - 65$). To fix this problem, we allow ourselves to construct 2 symmetric lobes inside the "forbidden" band $[-65 \ldots 65]$. Rather than constructing strict cut-offs, we choose a (more convenient) gaussian form for the lobes within $[-125, -65] \cup [65, 125]$ as well as for the "inside lobes". Exploring numerically the parameter space for these gaussians, we found that the following choice gives the best ratio of the Riesz basis bounds:

$$g^1(x) = \cos(80x)e^{-\frac{11^2 x^2}{2}} + 0.1\cos(\frac{\pi\alpha}{3}x)e^{-\frac{3^2 x^2}{2}} \tag{5.115}$$

For this window, the suboptimal solution g^2 of the problem 1 (5.34) is obtained using again (5.66). These two functions are plotted in Figure 5.23 and 5.24.

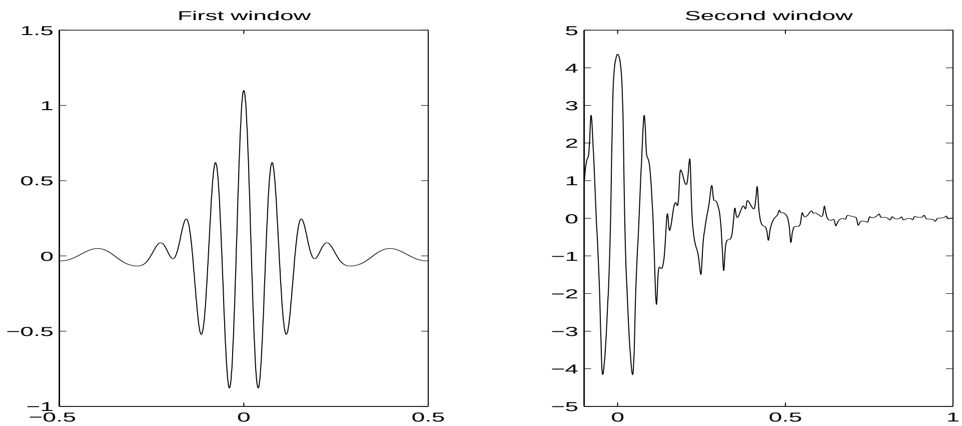

Figure 5.23: The suboptimal solutions g^1 and g^2 in time domain

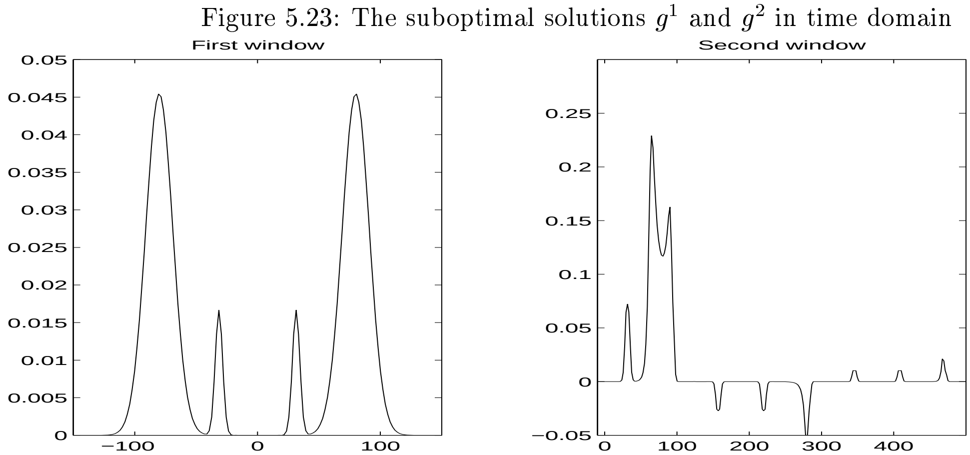

Figure 5.24: The suboptimal solutions g^1 and g^2 in frequency domain

The distortion achieved by this pair is $J_1 = 84.64$ which is with 0.82% larger than the optimal value of 83.95. However, this solution is still not very good: g^2 has too many oscilations and spikes. We correct this in the next step.

Step 3. Finally, we replace the g^2 that fits optimally with the g^1 designed in Step 2, with a "nicer" window in such a way not to significanly increase the distortion. We do this by filtering out the high-frequency components. We choose the cut-off frequency as $f_{max} = 125$ similar to the allowed band in the second step. The function g^2 obtained in this way is graphed in Figure 5.25. Using (5.55) we can compute the distortion when we use the functions g^1 of Figure 5.23 (5.24) and g^2 of Figure 5.25. We obtain for the distortion the value $J_2 = 84.642$ which is only 0.82% larger than the optimal value. The Riesz basis bounds for this window are $A_2 = 0.794$ and $B_2 = 4.895$ whereas the Riesz basis bounds for g^1 (5.115) are $A_1 = 0.0227$ and $B_1 = 0.1399$.

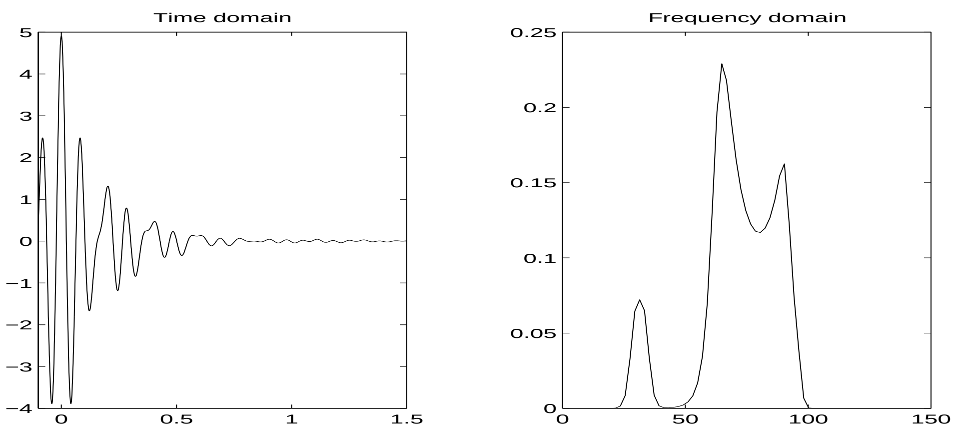

Figure 5.25: The window g^2 after filtering out the high frequency components

This case study shows that we can design reasonable windows g^1 and g^2 that will perform almost optimally from the rate-distortion point of view, even when our first gaussian guess gives results that are relatively far from optimal.

Our study of the distortion that results from using incomplete WH sets originated from a multiple description compression problem. In this framework, we consider the decomposition of a signal into a (possibly redundant) WH system, that is however split into two subsystems, each of which is an incomplete WH set. The coefficients corresponding to both these subsets are then sent over separate channels. At the other end, the receiver reconstructs from the two sets of coefficients if both channels function, or from only one set, if one channel is disfunctional. In this type of situation, it is important to choose the WH set such that the distortion is as small as possible if only one channel is used. The problem of finding such an appropriate g^1 is exactly what we have addressed in this chapter. It is intresting to note that for such a multiple description framework, it is advantageous to use a redundant frame for the original WH system. This is therefore a situation in which redundancy is useful even when one tries to optimize the rate/distortion, a fact that seems counterintuitive. We shall address this question in detail elsewhere.

Chapter 6

Conclusions

In this thesis I presented some aspects of the coherent sets theory in Hilbert space and some applications in signal processing. The general theory focused on three important types of coherent sets: Fourier sets, Weyl-Heisenberg sets and wavelet sets. A Hilbert coherent set is based on a square-integrable unitary representation of a locally compact group. Once we are given such a representation, one chooses a generator (an admissible vector of the Hilbert space) and a discrete subset of the locally compact group. Then the coherent set (associated to the given generator and the discrete subset) is given by discretizing the continuous orbit passing through the generator, with respect to the discrete subset.

Fourier sets are associated with the sampling theory. In this case the Hilbert space is the Paley-Wiener space of band-limited functions. Signal oversampling represents an overcomplete Fourier set in the space of band-limited signals, which means a frame. The irregular sampling is equivalent to nonharmonic Fourier series. Thus the analysis of Fourier sets is intimately connected with the theory of nonharmonic Fourier series and irregular sampling.

Weyl-Heisenberg sets are obtained from a function (called window) by translations and modulations given by a discrete subset of the time-frequency plane. The continuous transform is known in practice as the windowed Fourier transform.

Wavelet sets are obtained starting again from a function (called wavelet) and then translating and dilating it with parameters taken from a discrete subset of the time-scale plane.

My analysis concentrates around three problems: stability, localization and density.

In chapter 2 I present a geometric theory of frames, emphasizing certain equivalence relations. Within an equivalence class, a distance between equivalent frames is introduced. This geometric study is also used in the next chapter.

In the next chapter I analyze two stability results; one is an extension of Kadec'$\frac{1}{4}$-stability theorem for nonharmonic Fourier series from Riesz bases to frames. The other result generalizes an observation by Daubechies and Tchamitchian. They showed for Meyer's orthogonal wavelet that the Riesz basis property is preserved under small perturbations of the translation parameter. I have extended this stability property to more general wavelets. This result also shows the time-scale (or time-frequency) density of the discrete set has no connection with the behavior of the wavelet set (as opposed to the Weyl-Heisenberg case when the density plays a fundamental role).

In chapter 4 I study the localization of the wavelet generator. An uncertainty inequality is proved where the lower bound of $\frac{1}{2}$ (as in the case of the classical Heisenberg uncertainty principle) is replaced by $\frac{3}{2}$. This bound comes from the zeroth order vanishing moment property of the wavelet.

In the last chapter I present an application of the Weyl-Heisenberg Riesz bases for their span in a signal processing problem. The problem is to find the best approximation of a stochastic signal by Weyl-Heisenberg expansions. This approximation represents the equivalent of the Karhunen-Loève expansions in the Weyl-Heisenberg coherent set framework. I further analyze different sources of error (distortion) when using a Weyl-Heisenberg set in an encoding-decoding scheme. For sufficiently small quantization interlevel, the total distortion is mainly given by the stochastic approximation computed before. Some numerical simulations accompany this study.

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