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# DISTURBANCE DECOUPLING PROBLEM with STABILITY

– Graduation Paper –

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# Chapter 1

# Introduction

## 1.1 The Organization of the Thesis

By a *nonlinear dynamical system* we understand a system of the form:

$$\begin{cases} \dot{x} & = f(x) + \sum_{i=1}^{m} g_i(x)u_i + \sum_{i=1}^{r} e_i(x)q_i \stackrel{\text{def}}{=} f(x) + g(x)u + e(x)q \\ y & = [h_1(x), \ldots, h_l(x)]^\top \stackrel{\text{def}}{=} h(x) \end{cases} \tag{1.1}$$

where $x, u, q$ and $y$ denote the states, inputs, disturbances and outputs, respectively, and $g(x)$ and $e(x)$ are matrices of dimension $n \times m$ and $n \times r$ with columns $g_i(x), i = \overline{1,m}$ and $e_j(x), j = \overline{1,r}$ respectively. The vector fields $f$ and $g_i, i = \overline{1,m}$ and $e_j, j = \overline{1,r}$ and the output functions $h_i, i = \overline{1,l}$ are assumed to be smooth, i.e. infinitely many times continuously differentiable. Note that the system 1.1 is affine in the inputs $u$ and the disturbances $q$.

In order to obtain certain properties of the dynamic of the system, we have to design a compensator such that when we close the loop, the dynamics of the new system have the desired properties. Two questions arise:
1) What can we require from the closed-loop system ? (that means what are the properties that we can ask for the new system ?)
2) When we know what we want, how could we find the compensator ? (in other words, we ask for an algorithm).

A general method for solving the problems is the linear designer of the compensator. For this we have to follow three steps:
1) Linearization of the nonlinear process.
2) Choice of a linear algorithm and design the compensator.
3) Simulation of the closed-loop and verification of the performances (when the process is implemented by its nonlinear equations).

But, unfortunately, there are cases when this method does not work. In §1.2 we give a such example and we plead for directly nonlinear algorithms.

In §1.3 we list a sequence of problems whose solutions are well known. Our goal is to solve one of these problems, more exactly the LOCAL DISTURBANCE DECOUPLING PROBLEM with STABILITY. An introduction in this problem is made in §1.4.

In Chap.2 we present two geometric tools: distributions and their dual objects, codistributions. We give also Frobenius' theorem (in §2.1) and the meaning of these notions as part of the theory of nonlinear systems regarding the local decomposition (in §2.2).

The fundamental notion of the solution of disturbance decoupling problem is the controlled invariant distribution. We speak about this in Chap.3. Here we shall give four algorithms to compute the maximal controlled invariant distribution included in $Ker(dh)$. The classical way to solve the linear disturbance decoupling problem with stability is to use the controllability spaces. The nonlinear equivalent is the controllability distribution. In the last section of Chap.3 we discus about it.

Our problem (LDDPS) has two solutions. Both are presented in [vdWe91]. We present these solutions in Chap.4.

The Chap.5 is reserved for the conclusions.

I want to express my deeply felt gratitude to Professor Andrea Bacciotti for the help in the preparation of this thesis. I never had such interesting discussions about the theory of nonlinear systems as here.

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## 1.2 Nonlinear versus Linear Control

We shall prove that for a given nonlinear system it does not exist a linear stabilizer but there is a nonlinear state feedback that stabilizes the closed- loop system.

This example is due to [Ka89] (see also [Bacc92]).

**EXAMPLE 1.1** *Let us consider the single-input system:*

$$\begin{cases} \dot{x}_1 &= u \\ \dot{x}_2 &= x_2 - x_1^3 \end{cases} \tag{1.2}$$

*with the state vector* $x = (x_1, x_2) \in \mathbf{R}^2$.

*A. If we try a linear stabilizer we obtain:*

$$\begin{cases} \dot{x}_1 &= f_1 x_1 + f_2 x_2 \\ \dot{x}_2 &= x_2 - x_1^3 \end{cases}$$

*where* $f_1, f_2 \in \mathbf{R}$.

*The linearization of the system has the form:*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

*and the eigenvalues are* 1 *and* $f_1$ *so the system is obviously unstable.*

*B. We consider the function:*

$$u(x) = -x_1 + x_2 + \frac{4}{3} x_2^{1/3} - x_1^3 \tag{1.3}$$

*and:*

$$V(x) = \frac{x_1^4}{4} - x_1 x_2 + x_2^{4/3}$$

*as a candidate Liapunov function.*

*Along any curve* $x_2 = m^3 x_1^3$ *we obtain:*

$$V(x_1, m^3 x_1^3) = (m^4 - m^3 + \frac{1}{4}) x_1^4 \geq \frac{37}{256} x_1^4$$

*So:* $V(x_1, x_2) > 0$ *for* $(x_1, x_2) \neq (0, 0)$.

*On the other hand:*

$$\dot{V}(x) = -(x_2 - x_1^3)^2 \leq 0$$

*and the set* $N \stackrel{\text{def}}{=} \{(x_1, x_1^3) | x_1 \in \mathbf{R}\} = \{x \in \mathbf{R} | \dot{V}(x) = 0\}$ *does not include any positive orbit. By using the LaSalle's invariance principle we obtain that* (1.3) *is a nonlinear state feedback that stabilizes the closed-loop system.* ◇

We point out that the compensator is not smooth but is a continuous one. This example proves that when we have a nonlinear system it is much better (and sometimes it is the only way) to use nonlinear algorithms to solve the problem.

## 1.3 A List of Nonlinear Problems (and Solutions)

This section is written using three references: [Bacc92], [Is89] and [NiSc90]. The last two books present the geometric theory of nonlinear systems and its applications. We point out that there exist two kind of solutions: local and global (there is, also, a third type of solution: semiglobal – see [Su90] for a negative result). Our thesis consists only in a local study, so that we shall limit to this case. Also, we discus only about the continuous and not discrete systems.

Now a briefly presentation of the main nonlinear problems and solutions of them.

### 1.3.1 Asymptotic Stabilization via State Feedback

Let us consider a nonlinear dynamical system of the form:

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{1.4}$$

where $u \in \mathbf{R}^m$, $y \in \mathbf{R}^l$ and let $x_0$ be an equilibrium point (that means $f(x_0) = 0$ and $g(x_0) = 0$). The problem is to design a state feedback:

$$u = u(x)$$

such that $x_0$ becomes an asymptotic equilibrium point.

First we observe that the output equation does not play any role here. Then we can suppose, without loss of generality, that $x_0 = 0$. The problem is well posed if $f$ has not in $x_0$ an asymptotic equilibrium point because else the trivial solution $u \equiv 0$ is obvious. There are two types of solutions: direct and indirect.

**Direct Approach.** That means that the design of the feedback is induced directly from the form of the system. There are two methods given by:

· Artstein-Sontag Theorem

· Jurdjevic-Quinn method.

Both of them use Liapunov functions (see [Bacc92] for details).

**Indirect Approaches.** That means that first we associate to (1.4) a new system constructed by a certain way and then we design the state feedback using this new system. There are three families of solutions (after [Bacc92]):

*Local approximation.* There are two types:

· linearization method (and its generalization to homogeneous Taylor expansion)

· approximation using a non-standard dilation (this idea is due to M. Kawaski).

*Equivalence of systems.* First, under a certain assumptions, the system is brought in the *normal form* using a state feedback and then, if the zero dynamics is asymptotically stable, one can design very easily the state feedback that stabilizes the nonlinear system (see [Is89]).

*Reduction of dimension.* There are also two types:
· the technique involving center manifold theory;
· decomposition in a cascade connection.
More about these issues can be found in [Bacc92].

### 1.3.2 Asymptotic Output Tracking

We consider the system described by the equation (1.4) and let $y_R(t)$ be the desired output. The problem is to find a feedback control law which is able to impose on the error:

$$e(t) = y(t) - y_R(t)$$

a behavior which asymptotically decays to zero as time tends to infinity.

In [Is89] two ways of approaching this problem are described.

**The first method** uses the *relative degree* - denoted by $r$ - of the plant and the solution is proved to be:

$$u = \frac{1}{L_g L_f^{r-1} h(x)}(-L_f^r + y_R^{(r)} - \sum_{i=1}^{r} c_{i-1}(L_f^{(i-1)} h(x) - y_R^{(i-1)}))$$

where $(c_i)_{i=\overline{0,r-1}}$ are real numbers chosen in such away that the error $(e(t))$ verifies the following differential equation:

$$e^{(r)} + c_{r-1} e^{(r-1)} + \ldots + c_1 \dot{e} + c_0 e = 0$$

**In the second approach** the desired output is assumed to be the output of a certain dynamic system (called the exosystem):

$$\begin{cases} y_R(t) &= -q(w(t)) \\ \dot{w} &= s(w) \end{cases}$$

We meet two problems and also two solutions (see [Is89] pg 350):

*State feedback regulator problem* where is required a state feedback $u = \alpha(x, w)$ such that $\alpha(0,0) = 0$

*Error feedback regulator problem* where it is required a dynamical system with the error as its input and the control $u$ as its output:

$$\begin{cases} \dot{z} &= \eta(z, e) \\ u &= \theta(z) \end{cases}$$

such that $\eta(0,0) = 0, \theta(0) = 0$.

In both cases we ask for the asymptotic stability in the first approximation of the closed-loop system at the equilibrium point $x_0 = 0$ and the asymptotic tracking of the reference.

### 1.3.3 Disturbance Decoupling with Stability

The discussion of this problem is postponed to the next paragraph.

### 1.3.4 Noninteracting Control

We consider again the system described by the equation (1.4). We wish to use feedback in order to reduce the system in such a form that, from an input-output point of view, it looks lika an aggregate of independent single-input single-output channels.

Usually, the problem is formulated when the number of inputs ($m$) coincides with the number of outputs ($l$), but one can extend this to systems having the number of inputs larger than the number of outputs (see [Is89], pg. 264).

The solution of the form:

$$u = \alpha(x) + \beta(x)v$$

with $\beta(x)$ nonsingular and $v$ the new inputs, exists in a neighborhood $\mathcal{U}$ of the fixed point $x_0$ if the system has a finite *vector relative degree* $\{r_1, \ldots, r_m\}$ *at* $x_0$ (that means:

$(i)$ $L_{g_j} L_f^k h_i(x) = 0$ for all $1 \leq j \leq n,\ 1 \leq i \leq m,\ k < r_i - 1$ and $x \in \mathcal{U}$.

$(ii)$ $[L_{g_1} L_f^{r_i - 1} h_i(x_0) \ldots L_{g_m} L_f^{r_i - 1} h_i(x_0)] \neq [0 \ldots 0]$ for all $1 \leq i \leq m$.

$(ii)$ The matrix:

$$A(x) = \begin{bmatrix} L_{g_1} L_f^{r_1 - 1} h_1(x) & \cdots & L_{g_m} L_f^{r_1 - 1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{r_m - 1} h_1(x) & \cdots & L_{g_m} L_f^{r_m - 1} h_1(x) \end{bmatrix}$$

called the *decoupling matrix* is nonsingular at $x_0$; see also §3.24). For details see [Is89] §5.3 or [NiSc90] §8.1, §13.3.

## 1.4 Disturbance Decoupling Problem with Stability

We consider again the system described by the equation (1.1). We say that a feedback:

$$u = \alpha(x) + \beta(x)v \quad u, v \in \mathbf{R}^m,\ x \in \mathbf{R}^n \tag{1.5}$$

is a *regular static state feedback* (or a *regular feedback*) if $\beta(x)$ is a nonsingular matrix for all $x$. Under this feedback, the nonlinear system will have the following dynamic:

$$\dot{x} = (f(x) + g(x)\alpha(x)) + g(x)\beta(x)v = F(x) + G(x)v$$

We can meet three types of problems. We define these problems as follows (see [vdWe91]):

**Disturbance Decoupling Problem (DDP)** *What are the conditions that allow us to find a smooth regular static state feedback such that in the closed-loop system the disturbances $q$ do not influence the outputs $y$ ?*

Note that the decoupling requirement must hold for all initial points $x_0$ and all controlled inputs $v$. A version of this problem is the:

**Local Disturbance Decoupling Problem (LDDP)** *"Local" refers to the fact that we search for a feedback defined on a neighborhood $\mathcal{U}$ of a given point such that the disturbance decoupling requirement holds for all initial point in $\mathcal{U}$ and all controlled inputs $v$ as long as the state trajectories remain within $\mathcal{U}$.*

Moreover, we shall want to obtain a stable closed-loop system. We suppose that $f(x_0) = 0$. We state the problem that will be approached in this study:

**Local Disturbance Decoupling Problem with Stability (LDDPS)** *Under what conditions can we find a smooth regular static state feedback* (1.5) *defined locally around $x = 0$ with $\alpha(0) = 0$ such that in the feedback system the disturbances $q$ do not influence the outputs $y$, and $x = 0$ is a locally exponentially stable equilibrium of the modified drift dynamics $\dot{x} = f(x) + g(x)\alpha(x)$ ?*

From a well-known theorem it follows that the linearized system:

$$\dot{z} = [\frac{\partial f}{\partial x}(x_0) + g(x_0)\frac{\partial \alpha}{\partial x}(x_0)]z$$

is asymptotically stable at $z_0 = 0$. Then a necessary condition is that the pair $(\frac{\partial f}{\partial x}(x_0), g(x_0))$ is stabilizable.

# Chapter 2

# Distributions and Codistributions

## 2.1 Integrability. Forms of Frobenius' Theorem

### 2.1.1 Definitions

Let $M$ be a $n$-dimensional paracompact and smooth manifold and $x \in M$ be an arbitrary point. We shall denote by $T_x M$ the tangent space to $M$ at $x$, by $T_x^* M$ the cotangent space to $M$ at $x$, by $\mathcal{F}(M)$ the ring of all smooth real-valued functions, by $TM$ the tangent bundle, by $T^*M$ the cotangent bundle, by $V^\infty(M)$ the $\mathcal{F}(M)$-module of smooth vector fields, by $\Lambda^k(M)$ the $\mathcal{F}(M)$-module of smooth $k$-forms and by $\Lambda(M)$ the exterior algebra of smooth forms. For details of definitions see [Nara73].

A. We call *distribution* on $M$, the mapping:

$$D : x \in M \to D(x) \subset T_x M$$

where $D(x)$ is a vector subspace of the tangent space to $M$ at $x$.

The *dimension* (or *rank*) of the distribution is $\dim D(x)$. Let $\mathcal{L}$ be a $\mathcal{F}(M)$-module of smooth vector fields ($\mathcal{L} \subset V^\infty(M)$). We say that $\mathcal{L}$ *generates* the distribution $L$ (or the distribution $L$ is *generated* by $\mathcal{L}$) if:

$$L(x) = \{v|_x,\ v \in \mathcal{L}\},\ \forall x \in M$$

In this case we say that $L$ is a $\mathcal{C}^\infty$ (or smooth)-distribution. We shall deal only with distributions generated by $\mathcal{F}(M)$-modules of smooth vector fields. We say that a vector field $X$ *belongs to the distribution* $L$ (and we write $X \in L$) if for every $p \in M$, $X|_p \in L(p)$. We denote by $smt(L)$ the set of all vector fields that belong to the distribution: $smt(L) = \{X \in V^{\infty}(M) | X \in L\}$.

We say that the distribution $L$ (or the $\mathcal{F}(M)$-module $\mathcal{L}$) is *involutive* if for every $X, Y \in L$ (or $X, Y \in \mathcal{L}$) we obtain $[X, Y] \in L$ (or $[X, Y] \in \mathcal{L}$). For connections between the two types of involutivity see Appendix A.

Let us consider the distribution $L$ and a point $x_0 \in M$. If there exists a neighborhood of $x_0$ where the distribution has constant dimension, then the point is called an *ordinary point* (or *regular point*), otherwise it is called a *singular point*. If the distribution has singular points then we say that it is a *distributions with singularities*, otherwise we call it a *regular distribution*. In the last case $\mathcal{L} = smt(L)$. From the next chapter we shall deal only with regular distributions.

The distribution $L$ is said to be *punctually integrable at* $x_0 \in M$ if there exists a submanifold $\mathcal{N}_{x_0} \overset{i}{\hookrightarrow} M$ ($i$ being the canonical inclusion) passing through $x_0$ such that:

$$L(x) = T_x \mathcal{N}_{x_0}, \ for \ all \ x \in \mathcal{N}_{x_0}$$

(more precisely, we have: $i_{*,x}(T_x \mathcal{N}_{x_0}) = L(x)$). $\mathcal{N}_{x_0}$ is called an *integral manifold* of the distribution. The distribution is called *locally integrable* if for each point in $M$ there is an integral manifold of the distribution, and it is called *(globally) integrable* if there exists a partition of $M$ in integral manifolds of the distribution.

B. The dual notion of the distribution is codistribution and it is defined as follows. We call *codistribution* on $M$, the mapping:

$$P : x \in M \longrightarrow P(x) \subset T_x^* M$$

where $P(x)$ is a vector subspace of the cotangent space to $M$ at $x$.

By a $\mathcal{C}^{\infty}$*-(Pfaffian) differential system* we shall mean a $\mathcal{F}(M)$- module of smooth 1-forms. We denote it by $\mathcal{P}$. So $\mathcal{P} \subset \Lambda^1(M)$. The codistribution $P$ is called a *smooth* or $\mathcal{C}^{\infty}$*-codistribution* if there is a $\mathcal{C}^{\infty}$-Pfaffian differential system that generates the codistribution.

For every distribution $L$ we can associate in a canonical way a codistribution $Ort(L)$ by setting:

$$Ort(L)(x) = (L(x))^{\perp} \overset{\text{def}}{=} \{\omega_x \in T_x^* M | \omega_x(v_x) = 0, \ for \ all \ \omega_x \in P(x)\}$$

and an orthogonal $\mathcal{F}(M)$-module of smooth forms by:

$$L^{\perp} \overset{\text{def}}{=} \{\omega \in \Lambda^1(M) | \omega|_x(v_x) = 0, \ for \ all \ x \in M, v_x \in L(x)\}$$

Conversely, to every codistribution $P$ we can associate a distribution $Ker(P)$ that is punctually orthogonal with respect to the inner product:

$$(KerP)(x) = (P(x))^{\perp} \overset{\text{def}}{=} \{v_x \in T_x M | \omega_x(v_x) = 0, \ for \ all \ \omega_x \in P(x)\}$$

and a $\mathcal{F}(M)$-module of smooth vector fields:

$$P^{\perp} \stackrel{\text{def}}{=} \{v \in V^{\infty}(M)|\ \omega_x(v|_x) = 0,\ for\ all\ x \in M, \omega_x \in P(x)\}$$

Obviously: $L^{\perp} \subset OrtL$ ; $P^{\perp} = smt(KerP) \subset KerP$.

We shall say that the codistribution $P$ is punctually, locally or globally integrable (at $x_0 \in M$) if $KerP$ is punctually, locally or globally integrable. We observe that if $L$ is a smooth distribution, then $OrtL$ is a smooth codistribution without singularities too. The converse is also true (for $P$ and $KerP$).

For details about codistributions with singularities see Appendix B.

C. For the third form of Frobenius' theorem we shall use a system of partial differential equations of the form:

$$\frac{\partial y(x)}{\partial x_i} = \Gamma^i(x)y(x)\ ,\ 1 \le i \le m \qquad (2.1)$$

where $\Gamma^1, \ldots, \Gamma^m$ are smooth functions defined on an open set $\mathcal{U}$ in $\mathbf{R}^m$:

$$\Gamma^i : \mathcal{U} \longrightarrow \mathbf{R}^{n \times n}$$

$x_1, \ldots, x_m$ denote the coordinates of a point $x$ in $\mathbf{R}^m$, $y_1, \ldots, y_n$ the coordinates of a point $y$ in $\mathbf{R}^n$ and the solution $y$ denotes a function:

$$y : \mathcal{U} \to V$$

where $V$ is an open set in $\mathbf{R}^n$.

### 2.1.2 Statements

For the proofs see [Nara73].

**THEOREM 2.1 (Frobenius' Theorem: First Form)** *Let $L$ be a smooth regular distribution generated by the $\mathcal{F}(M)$-module of smooth vector fields $\mathcal{L}$. Then the following conditions are equivalent:*
*1) $L$ is locally integrable.*
*2) $L$ is globally integrable.*
*3) $L$ is involutive.*
*4) $\mathcal{L}$ is involutive.*
*5) There exists a coordinates system $(y_1, \ldots, y_n)$ such that:*

$$\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_k}\}\ ,\ k = rank\ \mathcal{L}$$

*is a set of generators of $\mathcal{L}$ and:*

$$\mathcal{N}_{x_0} = \{y|\ y_{k+1} = y^0_{k+1}, \ldots, y_n = y^0_n\}$$

*is an integral manifold of the distribution passing through the point $x_0$ of the coordinates $(y^0_i)_{1 \le i \le n}$.* □

For the proof that $1 \Rightarrow 2$ see also the remark from Appendix A about Sussmann's paper.

**THEOREM 2.2 (Frobenius' Theorem: Second Form)** *Let $\omega_{k+1}, \ldots, \omega_n$ be smooth 1-forms which are linearly independent at every point , $\mathcal{P}$ be the $\mathcal{F}(M)$-module spanned by these forms and let $P$ denote the associated regular codistribution. Then the following three conditions are equivalent:*
*1) $P$ is locally or globally integrable.*
*2) For every $\omega \in \mathcal{P}$ there exist $n-k$ smooth 1-forms: $\pi_{k+1}, \ldots, \pi_n \in \Lambda^1(M)$ such that:*

$$d\omega = \sum_{j=k+1}^{n} \pi_j \wedge \omega_j$$

*3) There exist smooth 1-forms $\lambda_{ij} \in \Lambda^1(M)$, $k+1 \leq i, j \leq n$, such that:*

$$d\omega_i = \sum_{j=k+1}^{n} \lambda_{ij} \wedge \omega_j$$

**THEOREM 2.3 (Frobenius' Theorem: Third Form)** *We consider the system from the paragraph C from the previous subsection. Given a point $(x^0, y^0) \in \mathcal{U} \times V$ there exist a neighborhood $\mathcal{U}^0$ of $x_0$ in $\mathcal{U}$ and a unique smooth function:*

$$y : \mathcal{U}^0 \longrightarrow V$$

*which satisfies the equations (2.1) and is such that $y(x_0) = y^0$ if and only if the functions $\Gamma^1, \ldots, \Gamma^m$ satisfy the conditions:*

$$\frac{\partial \Gamma^i}{\partial x_k} - \frac{\partial \Gamma^k}{\partial x_i} + \Gamma^i \Gamma^k - \Gamma^k \Gamma^i = 0 \ , \ 1 \leq i, k \leq m \tag{2.2}$$

*for all $x \in \mathcal{U}$.* □

For details and proof of this theorem see [Is89], Theorem 2.3, pp. 312–319 or [Nara73], Theorem 2.11.4, pp. 120–121.

We shall give now a fourth form of Frobenius' theorem that can be found in [JaRe80] (see [vdWe91]). A set of distributions $\{\Delta_1, \ldots, \Delta_r\}$ is called *nested* if $\Delta_1 \subset \Delta_2 \subset \cdots \Delta_r$. A collection $\Delta_1 \subset \Delta_2 \subset \cdots \Delta_r$ of nested nonsingular smooth distributions on $M$ is completely integrable if at each $x \in M$ there exists a local chart $(\mathcal{U}, \varphi)$ such that, for $i = 1, \ldots, r$

$$\Delta_i(y) = span\{\frac{\partial}{\partial z_1}|_y, \ldots, \frac{\partial}{\partial z_{d_i}}|_y\} \ for \ all \ y \in \mathcal{U} \ (d_i = dim(\Delta_i)).$$

**THEOREM 2.4 (Frobenius' Theorem: Fourth Form)** *A collection $\Delta_1 \subset \Delta_2 \subset \cdots \Delta_r$ of nested nonsingular distributions is completely integrable if and only if each distribution $\Delta_i$, $i = 1, \ldots, r$ is involutive.*

□

## 2.2 Invariant Distributions and Codistributions. Local Decompositions

### 2.2.1 Invariant Distributions and Codistributions

Let $f$ be a smooth vector field and let $\Delta$ be a smooth distribution. We call $\Delta$ an *invariant distribution* under the vector field $f$ if for any vector field $X \in \Delta$ we have $[f, X] \in \Delta$. We set:

$$[f, \Delta] \stackrel{\text{def}}{=} span_{\mathcal{F}(M)}\{[f, \tau], \tau \in \Delta\}$$

So: $\Delta$ is invariant under the vector field $f \iff [f, \Delta] \subset \Delta$. The meaning of this notion is given by the following lemma (for proof see [Is89], Lemma 6.3):

**LEMMA 2.5** *Let $\Delta$ be a nonsingular involutive distribution of dimension $d$ and suppose that $\Delta$ is invariant under the vector field $f$. Then for each point $x^0$ there exist a neighborhood $\mathcal{U}^0$ of $x^0$ and a coordinates transformation $z = \Phi(x)$ defined on $\mathcal{U}^0$, such that $f$ is represented in new coordinates by:*

$$\bar{f}(z) = \begin{bmatrix} f_1(z_1, \ldots, z_d, z_{d+1}, \ldots, z_n) \\ \cdots \\ f_d(z_1, \ldots, z_d, z_{d+1}, \ldots, z_n) \\ f_{d+1}(z_{d+1}, \ldots, z_n) \\ \cdots \\ f_n(z_{d+1}, \ldots, z_n) \end{bmatrix}$$

□

A codistribution $\Omega$ is said to be *invariant* under the vector field $f$ if the derivative $L_f\omega \in \Omega$, for all $\omega \in \Omega$ (i.e. $L_f\Omega \subset \Omega$, with an analogous notation). Using the well-known formula:

$$(L_f\omega)(\tau) = f(\omega(\tau)) - \omega([f, \tau])$$

one can prove the following lemma:

**LEMMA 2.6** *If a smooth distribution $\Delta$ is invariant under the vector field $f$, then the codistribution generated by $\Delta^\perp$ is also invariant under $f$. If a smooth codistribution $\Omega$ is invariant under the vector field $f$, then the distribution generated by $\Omega^\perp$ is also invariant under $f$.* □

Let $\Delta$ be a smooth distribution, $\Omega$ be a smooth codistribution and $f_1, \ldots, f_n$ be smooth vector fields. We shall denote by $< f_1, \ldots, f_n | \Delta >$ the smallest distribution (with respect to the inclusion) that includes $\Delta$ and is invariant under the action of the vector fields $f_1, \ldots, f_n$. We shall denote by $< f_1, \ldots, f_n | \Omega >$ the minimal element of the family of the codistributions that include $\Omega$ and are invariant under the action of the vector fields $f_1, \ldots, f_n$. We point out that the smoothness guarantees the existence of both structures.

### 2.2.2 Local Decompositions

Let us consider again the nonlinear control system described by the equation 1.4.

**PROPOSITION 2.7** *Let $\Delta$ be a nonsingular involutive distribution of dimension $d$ and assume that $\Delta$ is invariant under the vector fields $f, g_1, \ldots, g_m$. Moreover, suppose that the distribution $span\{g_1, \ldots, g_m\}$ is contained in $\Delta$. Then, for each point $x^0$ it is possible to find a neighborhood $\mathcal{U}^0$ of $x^0$ and a local coordinates transformation $z = \Phi(x)$ defined on $\mathcal{U}^0$ such that, in the new coordinates, the control system is represented by equations of the form:*

$$\left\{ \begin{array}{lcl} \dot{\xi}_1 & = & f_1(\xi_1, \xi_2) + \sum_{i=1}^{m} g_{1i}(\xi_1, \xi_2) u_i \\ \dot{\xi}_2 & = & f_2(\xi_2) \\ y & = & h(\xi_1, \xi_2) \end{array} \right.$$

*where $\xi_1 = (z_1, \ldots, z_d)$ and $\xi_2 = (z_{d+1}, \ldots, z_n)$.* $\square$

**Remarks**

1. Let $P =< f, g_1, \ldots, g_m | span\{g_1, \ldots, g_m\} >$. One can prove that if $P$ is a regular distribution then $P$ is involutive and then we can take $\Delta = P$ (see Lemma 8.7 from [Is89]).

2. This proposition allows us to obtain the input-state behavior. Suppose that the inputs $u_i$ are piecewise constant functions of time. Set $x(0) = x^0$ and let $x^0(t) = exp\ tf.x^0$ be the point of $\mathcal{U}^0$ reached at time $t$ when no input is imposed. Then the set of the point reachable at time $t$ is a subset of the slice:

$$\{x \in \mathcal{U}^0 | \xi_2(x) = \xi_2(x^0(t))\}$$

3. For details and proof see Proposition 7.1, pp. 53–54 from [Is89]

**PROPOSITION 2.8** *Let $\Delta$ be a nonsingular involutive distribution of dimension $d$ and assume that $\Delta$ is invariant under the vector fields $f, g_1, \ldots, g_m$. Moreover, suppose that the codistribution $span\{dh_1, \ldots, dh_p\}$ is contained in the codistribution $\Delta^\perp$. Then, for each point $x^0$ it is possible to find a neighborhood $\mathcal{U}^0$ of $x^0$ and a local coordinates transformation $z = \Phi(x)$ defined on $\mathcal{U}^0$ such that, in the new coordinates, the control system is represented by equations on the form:*

$$\left\{ \begin{array}{lcl} \dot{\zeta}_1 & = & f_1(\zeta_1, \zeta_2) + \sum_{i=1}^{m} g_{1i}(\zeta_1, \zeta_2) u_i \\ \dot{\zeta}_2 & = & f_2(\zeta_2) + \sum_{i=1}^{m} g_{2i}(\zeta_2) u_i \\ y & = & h(\zeta_2) \end{array} \right.$$

*where $\zeta_1 = (z_1, \ldots, z_d)$ and $\zeta_2 = (z_{d+1}, \ldots, z_n)$.*

**Remarks**

1. Let $Q =< f, g_1, \ldots, g_m | span\{dh_1, \ldots, dh_p\} >^\perp$. One can prove that if $Q$ is a regular distribution then $Q$ is involutive and then we can take $\Delta = Q$ (see Lemma 9.6 from [Is89]).

2. This proposition allows us to obtain the state-output interaction. For every pair $x_0^a$ and $x_0^b$ of initial states such that $\zeta_2(x_0^a) = \zeta_2(x_0^b)$ and for arbitrary constant picewise constant control $(u)$ we obtain that $x_u^a(t)$ and $x_u^b(t)$, the state functions under the action of the control $u$, are always on the same slice:

$$\{x \in \mathcal{U}^0 | \zeta_2(x) = \zeta_2(x_u^a(t))\}$$

and they produce the same outputs. We say that they are *indistinguishable* . As a matter of fact, all the initial states on the same slice are indistinguishable (the slices are defined by setting $\zeta_2$ with a constant value).

3. For details see Proposition 7.2, pp. 54–56 from [Is89].

# Chapter 3

# Controlled Invariant Distributions

## 3.1 General Results

### 3.1.1 Definitions

Let us consider the nonlinear system given by 1.4 and a regular feedback control law of the form given by 1.5:

$$\begin{cases} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i \\ y &= h(x) \end{cases}$$

$$u = \alpha(x) + \beta(x) v \ ; \quad \beta(x) \text{ nonsingular}$$

When we close the loop we obtain the system:

$$\begin{cases} \dot{x} &= \tilde{f}(x) + \sum_{i=1}^{m} \tilde{g}_i(x) v_i \\ y &= h(x) \end{cases}$$

where: $\tilde{f}(x) = f(x) + \sum_{i=1}^{m} g_i(x)\alpha_i(x) \equiv f(x) + g(x)\alpha(x)$ and:
$\tilde{g}_i(x) = \sum_{j=1}^{m} g_j(x)\beta_{ji}(x)$. So: $\tilde{g}(x) = g(x)\beta(x)$.

A distribution $\Delta$ is said to be *controlled invariant on* $\mathcal{U}$ if there exists a regular feedback pair $(\alpha, \beta)$ defined on $\mathcal{U}$ with the property that $\Delta$ is invariant under the vector fields: $\tilde{f}, \tilde{g}_1, \ldots, \tilde{g_m}$ (i.e. $[\tilde{f}, \Delta] \subset \Delta$; $[\tilde{g}_i, \Delta] \subset \Delta, \ 1 \leq i \leq m$). A distribution is said to be *locally controlled invariant* if for each $x \in \mathcal{U}$ there exists a neighborhood $\mathcal{U}^0$ of $x$ with the property that $\Delta$ is controlled invariant on $\mathcal{U}^0$.

We can rewrite the condition of controlled invariance as follows:

$$\Delta = < \tilde{f}, \tilde{g}_1, \ldots, \tilde{g}_m | \Delta > \tag{3.1}$$

### 3.1.2 The Main Results

We set: $\mathcal{G} = span_{\mathcal{F}(M)}\{g_1, \ldots, g_m\}$ and denote by $G$ the associated distribution.

**THEOREM 3.1** *Let $\Delta$ be an involutive distribution. Suppose $\Delta, G$ and $\Delta + G$ are nonsingular on $\mathcal{U}$. Then $\Delta$ is locally controlled invariant if and only if:*
$[f, \Delta] \subset \Delta + G$
$[g_i, \Delta] \subset \Delta + G$, *for* $1 \le i \le m$.

**Sketch of Proof** *(for a complete proof see [Is89], Lemma 2.1, pp. 311–319)*
*"$\Rightarrow$" Let $\tau \in \Delta$. We have:*

$$[\tilde{f}, \tau] = [f + g\alpha, \tau] = [f, \tau] + \sum_{j=1}^{m}[g_j, \tau]\alpha_j - \sum_{j=1}^{m}(L_\tau \alpha_j)g_j \in \Delta$$

$$[\tilde{g}_i, \tau] = [\sum_{j=1}^{m} g_j\beta_{ji}, \tau] = \sum_{j=1}^{m}[g_j, \tau]\beta_{ji} - \sum_{j=1}^{m}(L_\tau \beta_{ji})g_j \in \Delta$$

*Then we conclude* $[f, \tau] \in \Delta + G$, $[g_j, \tau] \in \Delta + G$

*"$\Leftarrow$" Now we use the fact that $\Delta, G$ and $\Delta + G$ are nonsingular. Let $d = dim\,\Delta$ and $p = dim\,G - dim\,\Delta \cap G$.*

*We change the set of generators of $G$ using a nonsingular $m \times m$ matrix $T$:*

$$\hat{g}_i = \sum_{j=1}^{m} t_{ji} g_j$$

*in order to obtain the following relations:*

$$span\{\hat{g}_{p+1}, \ldots, \hat{g}_m\} \subset \Delta \cap G$$

$$\Delta + G = \Delta \oplus span\{\hat{g}_1, \ldots, \hat{g}_p\}$$

*Let $\{\tau_1, \ldots, \tau_d\}$ be a set of vector fields which locally span $\Delta$ around $x^0$. From the given relations, setting $\hat{g}_0 = f$, we obtain:*

$$[\hat{g}_i, \tau_k] = \sum_{j=1}^{p} c_{ji}^k \hat{g}_j + \delta_i^k \;\; ; \;\; 0 \le i \le m \;\; ; \;\; 1 \le k \le d$$

*where: $\delta_i^k \in \Delta$ are unique vector fields.*

*Using Frobenius' Theorem (Third Form) - Theorem(2.3) - we obtain the existence of a $m \times m$ matrix $\hat{B}$ and a $m \times 1$ vector $\hat{a}$ such that:*

$$-L_{\tau_k}\hat{b}_{hi} + \sum_{j=1}^{m} c_{hj}^k \hat{b}_{ji} = 0$$

$$-L_{\tau_k}\hat{a}_h + \sum_{j=1}^{m} c_{hj}^k \hat{a}_j + c_{h0}^k = 0$$

*Then we set $\beta = T\hat{B}$ and $\alpha = T\hat{a}$ and we obtain a regular feedback that proves that $\Delta$ is a controlled invariant distribution. Q.E.D.* □

The notion of controlled invariant distribution is of particular interest in the problem of using feedback for the purpose of bringing the system in a decoupled form. To be more exact we state the following result (whose proof is obvious):

**THEOREM 3.2** *Let us consider the nonlinear dynamical system given by the equation (1.1). Let $\Delta$ be an involutive and nonsingular controlled invariant distribution included in:*

$$Ker(dh) = \bigcap_{j=1}^{m} Ker(dh_j)$$

*If $span\{e_i(x);\ 1 \leq i \leq r\} \subset \Delta$ then in a neighborhood of each point we can choose a regular feedback $(\alpha, \beta)$ and a coordinates system such that the closed-loop system is represented by equations of the form:*

$$\begin{cases} \dot{x_1} &= \tilde{f}_1(x_1, x_2) + \tilde{g}_1(x_1, x_2)v + \tilde{e}(x_1, x_2)w \\ \dot{x_2} &= \tilde{f}_2(x_2) + \tilde{g}_2(x_2)v \\ y &= h(x_2) \end{cases}$$

□

A few remarks are necesarry.

**Remarks**

1) If $\Delta$ is involutive, from $e_i \in \Delta$ we obtain that $\Delta$ is $e_i$-invariant (that means invariant under the action of the vector field $e_i$).

2) In order to solve a class of problem as larger as possible, we look for the maximal controlled invariant distribution included in $Ker(dh)$. Let $\Delta$ denote a controlled invariant distribution included in $Ker(dh)$. We recall its properties:

A) There exists a regular feedback pair $(\alpha, \beta)$ such that for the closed-loop $\Delta$ is an invariant distribution.

B) $\Delta \subset Ker(dh)$

Since a sum of two controlled invariant distributions is also a controlled invariant distribution (it results from Theorem(3.1)) the family of all smooth controlled invariant distributions included in $Ker(dh)$ (that we shall denote by $\mathcal{J}(f, g, Ker(dh))$) has a maximal element, namely the sum of all the members of the family. The distribution which we are interested in (i.e. the maximal controlled invariant distribution) must be also involutive. In the next section we shall give an algorithm that leads us ( under a few conditions of regularity) to the maximal involutive regular controlled invariant distribution.

3) The dynamics of the nonlinear system restricted to the leaf of $\Delta$ passing through $(x_{10}, x_{20})$ is given by:

$$\dot{x}_1 = \tilde{f}_1(x_1, x_{20}) + \tilde{g}_1(x_1, x_{20})v + \tilde{e}(x_1, x_{20})w$$

4) In the next chapter we shall use another variant of this theorem given by the following result:

**THEOREM 3.3** *Let us consider the nonlinear system given by* 1.4 *and two nested nonsingular and involutive controlled invariant distributions included in* $Ker(dh)$*:* $\Delta_1 \subset \Delta_2 \subset Ker(dh)$*. If there exists a pair of feedback* $(\alpha, \beta)$ *that renders invariant both distributions then there exist a coordinates system such that the closed-loop system is represented by:*

$$\begin{array}{rcllll} \dot{z}_1 &=& \tilde{f}_1(z_1, z_2, z_3) & +\tilde{g}_{11}(z_1, z_2, z_3)v^1 & +\tilde{g}_{12}(z_1, z_2, z_3)v^2 & +\tilde{g}_{13}(z_1, z_2, z_3)v^3 \\ \dot{z}_2 &=& \tilde{f}_2(z_2, z_3) & & +\tilde{g}_{22}(z_2, z_3)v^2 & +\tilde{g}_{23}(z_2, z_3)v^3 \\ \dot{z}_3 &=& \tilde{f}_3(z_3) & & & +\tilde{g}_{33}(z_3)v^3 \\ y &=& h(z_3) & & & \end{array} \tag{3.2}$$

*with* $\Delta_1 = span\{\frac{\partial}{\partial z_1}\}$, $\Delta_2 = span\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\}$*;* $z_1 = (x_1, x_2, \ldots, x_{d_1})$, $z_2 = (x_{d_1+1}, \ldots, x_{d_2})$, $d_1 = dim\Delta_1, d_2 = dim\Delta_2$*;* $\tilde{g}_{11} = (g_1, g_2, \ldots, g_{k_1}) \in \Delta_1 \cap G$,
$\tilde{g}_2 = (g_{k_1+1}, \ldots, g_{k_2}) \in \Delta_2 \backslash \Delta_2) \cap G$ *and* $v = (v_1, v_2, v_3)$, $v_1 = (u_1, \ldots, u_{k_1})$, $v_2 = (u_{k_1+1}, \ldots, u_{k_2})$, $v_3 = (u_{k_2+1}, \ldots, u_n)$ □

**Remarks**

1) Even it is a very simple result, the understanding of the solution of the LDDPS (and also of the local noninteractive control with stability) is conditioned by the understanding of this theorem.

2) We point out that it is crucial that both controlled invariant distributions have the same feedback that renders them invariant.

## 3.2 Algorithms

### 3.2.1 The $D^*$-Algorithm

The following algorithm is the nonlinear analogous of the linear algorithm for computing the maximal controlled invariant subspace.

**ALGORITHM 1 ($D^*$-Algorithm)**
*Step 0:* $D^0 := TM$
*Step k:*

$$D_k := Ker(dh) \cap \{X \in V^\infty(M) | [f, X] \in D_{k-1} + G, [g_i, X] \in D_{k-1} + G, 1 \leq i \leq m\}$$

With the help of the above algorithm we can obtain the maximal controlled invariant distribution included in $Ker(dh)$ as is stated in the following proposition:

**PROPOSITION 3.4 ([NiSc90])** *If for all $k \geq 0$ the distribution $D_k$ and $D_k \cap G$ as well as the distribution $G$ have constant dimension on $M$ then:*
*(i)* $D^0 \supset D^1 \supset \cdots \supset D^k \supset D^{k+1} \supset \cdots$
*(ii)* $D^k$ *is involutive for* $K \geq 0$
*(iii)* $\Delta^* = D^n$. □

For proof see Proposition 7.16 from [NiSc90], pp.223–224.

This algorithm has the disadvantage that there is not an efficient method to compute the set of vector fields as in Step $k$.

### 3.2.2 The Controlled Invariant Distribution Algorithm

We present now the dual form of the previous algorithm.

**ALGORITHM 2 (The Controlled Invariant Distribution Algorithm)**

*Step 0:* $\Omega_0 := span\{\, dh \,\}$
*Step k:* $\Omega_k := \Omega_{k-1} + L_f(\Omega_{k-1} \cap G^\perp) + \sum_{i=1}^{m} L_{g_i}(\Omega_{k-1} \cap G^\perp)$

We shall denote by $\Delta^*$ the maximal controlled invariant distribution included in $Ker(dh)$ (that is the maximal element of $\mathcal{J}(f, g, Ker(dh))$).

**LEMMA 3.5 ([Is89])** *Suppose there exists an integer $k^*$ such that $\Omega_{k^*+1} = \Omega_{k^*}$. Then $\Omega_k = \Omega_{k^*}$ for all $k > k^*$. If $\Omega_{k^*} \cap G^\perp$ and $\Omega_{k^*}^\perp$ are smooth, then $\Omega_{k^*}^\perp = \Delta^*$.* □

For proof see Lemma 3.2,§6.3,pp. 322–323 of Isidori's book.
We set:

$$D^* = (\Omega_0 + \Omega_1 + \cdots + \Omega_k + \cdots)^\perp$$

and we say that $D^*$ is *finitely computable* if there exists an integer $k^*$ such that $\Omega_{k^*} = \Omega_{k^*+1}$. In this case $D^* = (\Omega_{k^*})^\perp$.

**LEMMA 3.6** (**[Is89]**) *Suppose $D^*$ is finitely computable and $G, D^*, D^* + G$ are nonsingular. Then $D^*$ is involutive and $D^* = \Delta^*$.* □

A case where the above assumptions are verified is provided by the following lemma:

**LEMMA 3.7** (**[vdWe91]**) *Assume that the codistributions $G^\perp, \Omega_k$ and $\Omega_k \cap G^\perp$ have constant dimension for all $k \geq 0$. Then $\Delta^* = \Omega_n^\perp = Ker\Omega_n$. Moreover, $\Delta^*, G$ and $\Delta^* + G$ are nonsingular.* □

For proof see [NiSc90], Proposition 7.18,pp.225.

A point $x_0$ is called a *regular point of controlled invariant distribution algorithm* if in a neighborhood of $x_0$ the distribution $G$ (or codistribution $G^\perp$) and the codistributions $\Omega_k$ and $\Omega_k \cap G^\perp$ for all $k \geq 0$ are nonsingular. Then the previous lemma can be stated as follows:

**LEMMA 3.8** (**[Is89]**) *If $x_0$ is a regular point of the controlled invariant distribution algorithm then $\Delta^* = \Omega_n^\perp$ and it is involutive.* □

### 3.2.3 The Structure Algorithm

We shall use the matrix notations: an 1-form will be identified with a row vector and a set of 1-forms with the rows of a matrix. We shall also identify a differential system with the codistribution generated or with a set of generators and a $\mathcal{F}(M)$-module of smooth vector fields with its distribution associated or with a set of generators.

We shall compute $\Omega_{k+1}$ using the nonlinear structure algorithm. First we suppose that $\Omega_k$ is spanned by the exact 1-forms:

$$dc_1, \ldots, dc_{\rho_k}$$

Let $dc$ denote the matrix having $dc_i$'s as rows.

The Evaluation of $\Omega_k \cap G^\perp$

Let $N$ denote the following $\rho_k \times m$ matrix of functions:

$$N_{ij} = dc_i(g_j) = L_{g_j} c_i$$

We see that the elements of $N$ span $dc(G)$. We put $r_k = rank\, N$ and we suppose that the first $r$ rows of $N$ are independent. Then we partition $N$ and $dc$ according with the above agreement:

$$N = \begin{bmatrix} \bar{N} \\ \cdots \\ \tilde{N} \end{bmatrix} \begin{matrix} \}r_k \\ \\ \}\rho_k - r_k \end{matrix} \qquad dc = \begin{bmatrix} d\bar{c} \\ \cdots \\ d\tilde{c} \end{bmatrix} \tag{3.3}$$

We obtain that there exists a $(\rho_k - r_k) \times r_k$ matrix $M$ such that:

$$\tilde{N} + M\bar{N} = 0$$

And then:

$$\Omega_k \cap G^{\perp} = span\{d\tilde{c}_i + \sum_{j=1}^{r_k} M_{ij} d\bar{c}_j \mid r_k + 1 \leq i \leq \rho_k\}$$

where: $\tilde{c}_i = c_i$, $\bar{c}_j = c_{r_k+j}$ and the first subscript of $M$ runs from $r_k + 1$ to $\rho_k$.

The Evaluation of $L_f(\Omega_k \cap G^{\perp})$ and $L_{g_j}(\Omega_k \cap G^{\perp})$

We have now:

$$L_v(d\tilde{c}_i + \sum_{j=1}^{r_k} M_{ij} d\bar{c}_j) = d(L_v \tilde{c}_i + \sum_{j=1}^{r_k} M_{ij} L_v \bar{c}_j) + (H dc)_i - \sum_{j=1}^{r_k} d\bar{c}_j(v) dM_{ij} \quad (3.4)$$

where $H = L_v M$. For $v = f$ we denote $H^0 = L_f M$ and for $v = g_j$ we put $H^j = L_{g_j} M$, $1 \leq j \leq m$. Also we denote by $M_i$ the transpose of the $i$'s row of the matrix $M$:

$$M_i = \begin{bmatrix} M_{i1} \\ \vdots \\ M_{ir_k} \end{bmatrix}$$

Using the relation (3.4) with the particular notation just introduced , we obtain for $L_f(\Omega_k \cap G^{\perp})$ a set of generators written in the matrix notation as follows:

$$d[(d\tilde{c} + M d\bar{c})(f)] + H^0 d\bar{c} - \begin{bmatrix} [d\bar{c}(f)]^T dM_{r_k+1} \\ \cdots \\ [d\bar{c}(f)]^T dM_{\rho_k} \end{bmatrix}$$

or:

$$d[\tilde{M} dc(f)] + H^0 d\bar{c} - \begin{bmatrix} [d\bar{c}(f)]^T dM_{r_k+1} \\ \cdots \\ [d\bar{c}(f)]^T dM_{\rho_k} \end{bmatrix}$$

where: $\tilde{M} = [M \ I_{\rho_k - r_k}]$. Analogously, for $L_{g_j}(\Omega_k \cap G^{\perp})$ we obtain as a set of generators the matrix:

$$H^j d\bar{c} - \begin{bmatrix} [d\bar{c}(g_j)]^T dM_{r_k+1} \\ \cdots \\ [d\bar{c}(g_j)]^T dM_{\rho_k} \end{bmatrix}$$

The Evaluation of $\Omega_{k+1}$

Let $NewC$ denote the set of generators of $\Omega_{k+1}$ computed using the relation defining of $\Omega_{k+1}$:

$$\Omega_{k+1} = \Omega_k + L_f(\Omega_k \cap G^{\perp}) + \sum_{j=1}^{m} L_{g_j}(\Omega_k \cap G^{\perp})$$

Then:

$$NewC = \begin{bmatrix} set\_of\_generators\_of\_\Omega_k \\ set\_of\_generators\_of\_L_f(\Omega_k \cap G^{\perp}) \\ set\_of\_generators\_of\_L_{g_1}(\Omega_k \cap G^{\perp}) \\ \cdots \\ set\_of\_generators\_of\_L_{g_m}(\Omega_k \cap G^{\perp}) \end{bmatrix}$$

We may use elementary operations on the rows of the matrix and we obtain:

$$NewC = \begin{bmatrix} dc = \begin{bmatrix} d\bar{c} \\ d\tilde{c} \end{bmatrix} \\ d[\tilde{M} dc(f)] + H^0 d\bar{c} - \begin{bmatrix} [d\bar{c}(f)]^T dM_{r_k+1} \\ \cdots \\ [d\bar{c}(f)]^T dM_{\rho_k} \end{bmatrix} \\ H^1 d\bar{c} - \begin{bmatrix} [d\bar{c}(g_1)]^T dM_{r_k+1} \\ \cdots \\ [d\bar{c}(g_1)]^T dM_{\rho_k} \end{bmatrix} \\ \cdots \\ H^m d\bar{c} - \begin{bmatrix} [d\bar{c}(g_m)]^T dM_{r_k+1} \\ \cdots \\ [d\bar{c}(g_m)]^T dM_{\rho_k} \end{bmatrix} \end{bmatrix} \equiv$$

$$\equiv \begin{bmatrix} dc \\ d[\tilde{M} dc(f)] - \begin{bmatrix} [d\bar{c}(f)]^T dM_{r_k+1} \\ \cdots \\ [d\bar{c}(f)]^T dM_{\rho_k} \end{bmatrix} \\ [d\bar{c}(G)]^T dM_{r_k+1} \\ \cdots \\ [d\bar{c}(G)]^T dM_{\rho_k} \end{bmatrix} = \begin{bmatrix} dc \\ d[\tilde{M} dc(f)] - \begin{bmatrix} [d\bar{c}(f)]^T dM_{r_k+1} \\ \cdots \\ [d\bar{c}(f)]^T dM_{\rho_k} \end{bmatrix} \\ \bar{N}^T dM_{r_k+1} \\ \cdots \\ \bar{N}^T dM_{\rho_k} \end{bmatrix}$$

Since $\bar{N}$ has full row rank we can still write:

$$NewC \equiv \begin{bmatrix} dc \\ d[\tilde{M} dc(f)] - \begin{bmatrix} [d\bar{c}(f)]^T dM_{r_k+1} \\ \cdots \\ [d\bar{c}(f)]^T dM_{\rho_k} \end{bmatrix} \\ dM_{r_k+1} \\ \cdots \\ dM_{\rho_k} \end{bmatrix} \equiv \begin{bmatrix} dc \\ d[\tilde{M} dc(f)] \\ dM_{r_k+1} \\ \cdots \\ dM_{\rho_k} \end{bmatrix}$$

Let $d$ denote a column-vector constructed from the elements of the matrix $M$ (the length of $d$ is $\rho_k(\rho_k - r_k)$). Since $dc(f) = L_f c$ the previous formula can be written in the following form:

$$NewC \equiv d \begin{bmatrix} c \\ \tilde{M} \cdot L_f c \\ d \end{bmatrix}$$

We can observe that the set of generators of $\Omega_{k+1}$ is also made from exact 1-forms. Since $\Omega_0 = dh$ is obviously generated by exact forms, we obtain that every $\Omega_k$ is involutive and also, if they are nonsingular distributions, integrable.

In order to obtain a simple form for computing $M$ and $\tilde{M}dc(f)$ we use the following algorithm:

**ALGORITHM 3 (The Structure Algorithm)**

*Step 0:* $c = h$ ; $\rho_0 = l$
*Step k:*

1. $\Gamma_{k+1} = \begin{bmatrix} L_f c & L_{g_1} c \cdots L_{g_m} c \\ c & 0 \end{bmatrix} \begin{matrix} \}\rho_k \\ \}\rho_k \end{matrix}$

2. $T_{k+1}\Gamma_{k+1} = \begin{bmatrix} c'_{k+1} & A_{k+1} \\ c_k & 0 \end{bmatrix} \begin{matrix} \}r_k \\ \phantom{x} \end{matrix}$

*where $T_{k+1}$ is a nonsingular matrix and $A_{k+1}$ is a full row rank matrix with $m$ columns and $r_k$ rows.*
3. *We partition*

$$T_{k+1} = \begin{bmatrix} * \\ \cdots \\ \tilde{M} \\ \cdots \\ * \end{bmatrix} \begin{matrix} \}r_k \\ \\ \\ \\ \}\rho_k \end{matrix}$$

*and we denote by $d$ the column-vector obtained from the elements of $\tilde{M}$ excepting the constant functions.*

4. $c := \begin{bmatrix} \hat{c} \\ d \end{bmatrix}$

5. $\rho_{k+1} = dim\, c \leq 2\rho_k - r_k + 2\rho_k(\rho_k - r_k)$

**PROPOSITION 3.9** *If $x_0$ is a regular point for the controlled invariant distribution algorithm then the structure algorithm ends in the most $n$ steps and the codistribution spanned by $dc_n$ equals $(\Delta^*)^\perp$. Moreover, the regular state feedback that proves this is given by:*

$$\begin{cases} A_n\alpha + c'_n = 0 \\ A_n\beta = [I_{r_n} \;\; 0] \end{cases}$$

**Proof**

*We have already proved that if $x_0$ is a regular point for the controlled invariant distribution algorithm then $\Omega_n$ is spanned by a set of exact 1-forms and from Lemma(3.8) it follows that $(\Delta^*)^\perp = span\{dc_i\}$.*

*For the second part we see that the given relations are equivalent with:*

$$\begin{cases} (L_g\bar{c})\alpha + L_f\bar{(c)} = 0 \\ (L_g\bar{c})\beta = [I_{r_n} \ 0] \end{cases}$$

*where c is the column-vector of functions obtained at the nth step and $\bar{c}$,$\tilde{c}$ are the partition of this vector according with the relations* 3.3. *If $\tilde{f} = f + g\alpha$ and $\tilde{g} = g\beta$ denote the modified dynamic then:*

$$L_{\tilde{f}}(dc) = d(L_{\tilde{f}}c) = d\begin{bmatrix} L_f\bar{c} + (L_g\bar{c})\alpha \\ L_f\tilde{c} + (L_g\tilde{c})\alpha \end{bmatrix} =$$

$$= d\begin{bmatrix} 0 \\ L_f\tilde{c} - M(L_g\bar{c})\alpha \end{bmatrix} = d\begin{bmatrix} 0 \\ L_f\tilde{c} + M \cdot L_f\bar{c} \end{bmatrix} = d\begin{bmatrix} 0 \\ \tilde{M} \cdot L_f c \end{bmatrix} \in \Omega_n$$

*And:*

$$L_{\tilde{g}}(dc) = d(L_{\tilde{g}}c) = d\begin{bmatrix} L_g\bar{c} \\ L_g\tilde{c} \end{bmatrix}\beta) = d(\begin{bmatrix} 0 \\ \tilde{N} \end{bmatrix}\beta) =$$

$$= d\begin{bmatrix} 0 \\ -M(L_g\bar{c})\beta \end{bmatrix} = -d\begin{bmatrix} 0 \\ one\_column\_of\_M \end{bmatrix} \in \Omega_n$$

*Q.E.D.* □

**Remark** In the second step of the algorithm, the matrix $T_{k+1}$ can be obtained as a product of a suitable reflectors in order to obtain a Gauss echelon-form for $A_{k+1}$.

### 3.2.4 The Ker-Algorithm

With stronger assumptions than in the previous algorithm we can obtain a new form of this. First a definition:

Consider the smooth nonlinear system 1.4. The *(vector) relative degree* $(r_1(x), \ldots, r_l(x))$ is the vector of the smallest integer such that:

$$\begin{cases} L_{g_j}L_f^k h_i(x) = 0 & , \text{ for all } j = 1, \ldots, m, \ k < r_i(x) - 1 \\ L_{g_j}L_f^{r_i(x)-1}h_i(x) \neq 0 & , \text{ for some } j \end{cases}$$

and ,moreover, the decoupling matrix, defined by $(A(x))_{ij} = L_{g_j}L_f^{r_i(x)-1}h_i(x)$ has full row rank at $x$.

**PROPOSITION 3.10** *Consider the smooth nonlinear system* 1.4. *If it has a finite and constant relative degree on a neighborhood of the point $x_0$, namely $(r_1, \ldots, r_l)$, then the maximal controlled invariant distribution included in $Ker(dh)$ is given by:*

$$\Delta^* = \bigcap_{i=1}^{l}\bigcap_{k=0}^{r_i-1} Ker\, dL_f^k h_i \qquad (3.5)$$

*Moreover, a regular state feedback solving the LDDP follows from the equations:*

$$A(x)\alpha(x) + b(x) = 0$$
$$A(x)\beta(x) = [I_l \;\; 0]$$

*where* $b(x)$ *is defined by:* $(b(x))_i = L_f^{r_i} h_i(x),\ i = 1, \ldots, l$ □

**Remark** The relation 3.5 can be rewritten in the following form:

$$(\Delta^*)^{\perp} = span\{dL_f^k h_i;\ 1 \leq i \leq l,\ 0 \leq k \leq r_i - 1\}$$

For details of proof see [Is89], Lemma 3.13,pp.334–335.

## 3.3 Controllability Distributions

### 3.3.1 Definitions

A distribution $\Delta$ is said to be a *controllability distribution on* $\mathcal{U}r$ if it is involutive and there exist a feedback pair $(\alpha, \beta)$ defined on $\mathcal{U}$ and a subset $I$ of the index set $\{1, \dots, m\}$ with the property that $\Delta \cap \ G = span\{\tilde{g}_i, i \in I\}$, and $\Delta$ is the smallest distribution which is invariant under the vector fields $\tilde{f}, \tilde{g}_1, \dots, \tilde{g}_m$ and contains $\tilde{g}_i$ for all $i \in I$.

A distribution is said to be a *a local controllability distribution* if for each $x_0 \in \mathcal{U}$ there exists a neighborhood $\mathcal{U}^0$ of $x_0$ with the property that $\Delta$ is a controllability distribution on $\mathcal{U}^0$.

It is clear that, by definition, a (local) controllability distribution is (locally) controlled invariant. Therefore it is interesting to search extra conditions for a complete characterization of a local controllability distribution. First we prove a result as 3.1.

**LEMMA 3.11** *Let $\Delta$ be an involutive distribution. Then $\Delta$ is a (local) controllability distribution if and only if it is a (local) controlled invariant distribution and:*

$$\Delta = < \tilde{f}, \tilde{g}_1, \dots, \tilde{g}_m | \Delta \cap G > \tag{3.6}$$

**Proof**

*" $\Rightarrow$" It is obvious.*

*" $\Leftarrow$" Let*

$$\Delta \cap G = span\{\bar{g}_i\}_{i \in I}$$

*where $\bar{g}_i = \sum_{j=1}^m \tilde{g}_j h_{ji}$ and $Card\, I = rank(\Delta \cap G) = r$.*

*Then we choose $(\bar{\alpha}, \bar{\beta})$ of the form:*

$$\bar{\alpha} = \alpha$$
$$\bar{\beta} = \beta \cdot \bar{H}$$

*where $\bar{H}$ is a $m \times m$ matrix obtained from $(h_{ji})_{1 \le j \le m; 1 \le i \le r}$ by adding $m - r$ columns from the canonical basis in order to obtain a nonsingular matrix. Let us suppose $\bar{H}$ of the form:*

$$\bar{H} = \left[ \begin{array}{c|c} (h_{ij}) & \begin{array}{c} 0 \\ \hline I_{m-r} \end{array} \end{array} \right]$$

*Then the modified dynamics in the new pair of feedback $(\bar{\alpha}, \bar{\beta})$ is given by: $f, \bar{g}_1, \dots, \bar{g}_r, g_{r+1}, \dots, g_m$.*

*Since $\bar{g}_1, \dots, \bar{g}_r \in \Delta$ it is very easy to prove that $\Delta$ is also $\bar{g}_1, \dots, \bar{g}_r$-invariant. Now the proof is complete. Q.E.D.* □

**Remark** Note that $\Pi^*$ is a possible nonlinear analogue for $\mathcal{R}^*$, the largest controllability subspace in the kernel of the output mapping. It is well-known (see [Wo79], Proposition 5.2, pp. 104) that for linear systems the dynamics restricted to $\mathcal{R}^*$ are controllable (so, in particular, stabilizable). For nonlinear systems there is no direct relation between controllability distributions and stabilizability. In fact, in the following example it is shown that the dynamics of a system restricted to the leaf of a controllability distribution through an equilibrium point of the drift vector field ($f$) need not to be stabilizable.

**EXAMPLE 3.12 (see [vdWe91])** *Consider the system (1.4) with* $n = 5, m = 2, l = 1$ *and:*

$$f(x) = x_4 \frac{\partial}{\partial x_4} \ , \ g_1(x) = \frac{\partial}{\partial x_2} \ , \ g_2(x) = x_2 \frac{\partial}{\partial x_1} + (1 + x_1) \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}$$

$$h(x) = x_5$$

*A direct computation shows that:*

$$\Delta^* = span\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\}$$

*(the relative degree is 1). Then, using one of the two algorithms that will follow in next subsections or direct from definition, we obtain:*

$$\Pi^* = span\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4}\}$$

*The leaf of* $\Pi^*$ *that passes through the origin is given by:*

$$N = \{x \in \mathbf{R}^5 | x_3 = 0, x_5 = 0\}$$

*The dynamics of the system restricted to this leaf are given by:*

$$\begin{cases} \dot{x}_1 = 0 \\ \dot{x}_2 = u_1 \\ \dot{x}_4 = x_4 \end{cases}$$

*Clearly, these dynamics are unstabilizabe !* $\diamond$

### 3.3.2 The Controllability Distribution Algorithm

Let $\Delta$ be a fixed distribution.

**ALGORITHM 4 (Controllability Distribution Algorithm)**
*Step 0:* $S_0 = \Delta \cap G$
*Step k:*

$$S_k := \Delta \cap (L_f S_{k-1} + \sum_{j=1}^{m} L_{g_j} S_{k-1} + G) \tag{3.7}$$

**LEMMA 3.13 (see [Is89],pp.** 338) *The sequence* 3.7 *is nondecreasing. If there exists an integer* $k^*$ *such that* $S_{k^*} = S_{k^*+1}$ *then* $S_k = S_{k^*}$ *for all* $k > k^*$. □

We set:

$$S(\Delta) = (S_0 + S_1 + \cdots + S_k + \cdots)^{\perp}$$

If the algorithm 4 ends in a finite number of steps (that means there exists $K^*$ as in Lemma(3.13)) then we say that $S(\Delta)$ is *finitely computable* and then $S(\Delta) = S_{k^*}$.

An "intrinsic" characterization of a local controllability distribution is given by the following theorem (see [Is89],pp. 340–341, for proof)

**THEOREM 3.14** *Let* $\Delta$ *be an involutive distribution. Suppose* $\Delta, G, \Delta + G$ *are nonsingular and that* $S(\Delta)$ *is finitely computable. Then* $\Delta$ *is a local controllability distribution if and only if:*

$$\begin{array}{l} [f, \Delta] \subset \Delta + G \\ [g_i, \Delta] \subset \Delta + G, \ 1 \leq i \leq m \\ S(\Delta) = \Delta \end{array}$$

□

### 3.3.3 The $\Pi^*$-Algorithm

Let us suppose the nonlinear system given by 1.4. As in the case of controlled invariant distributions we look for the largest local controllability distribution included in $Ker(dh)$. Since every (local) controllability distribution is also a (local) controlled invariant distribution, the problem is equivalent to look for the largest local controllability distribution included in $\Delta^*$ that is the maximal local controlled invariant distribution included in $Ker(dh)$.

We have the following lemma (from [Is89]):

**LEMMA 3.15** *Suppose* $\Delta^*, G$ *and* $G+\Delta^*$ *are nonsingular and* $S(\Delta^*)$ *is finitely computable and nonsingular. Then* $S(\Delta^*)$ *is the largest local controllability distribution included in* $Ker(dh)$. □

If we use instead of $(f, g_j)$ the modified vector fields under the action of a pair of feedback $(\alpha, \beta)$ which renders $\Delta^*$ invariant, we obtain a new form of the controllability distribution algorithm.

**ALGORITHM 5 (The $\Pi^*$-Algorithm)** *Step 0:*

1. *Compute* $\Delta^*$, *the maximal controlled invariant distribution included in* $Ker(dh)$ *(which is involutive).*
2. *Establish* $(\alpha, \beta)$ *a pair of feedback which renders* $\Delta^*$ *invariant.*
3. *Modify* $\beta$, *if it is necessary, in order to obtain:*

$$\Delta \cap G = span\{\bar{g}_i, i \in I\}$$

4. *Set* $S_0 = span\{\bar{g}_i, i \in I\}$

*Step k:*

$$S_k = S_{k-1} + L_{\bar{f}} S_{k-1} + \sum_{i=1}^{m} L_{\bar{g}_i} S_{k-1} \qquad \square$$

If the assumptions of Lemma (3.15) are fulfilled then $S(\Delta^*) = \Pi^*$. Moreover, if every $S_k$ is nonsingular then $\Pi^* = S_n$

# Chapter 4

# The Solutions of the LDDPS

## 4.1 General Presentation

The both solutions that we shall present in this chapter are borrowed from [vdWe91]. We suppose that the equilibrium point is $x_0 = 0$ (the origin) and we require only $f(x_0) = 0$.

The first solution uses the stabilizability distributions. This notion will be introduce in the next section, but we can give now the basic idea. Let us consider the decomposed form given by Theorem (3.3) (relation (3.2)) with $\Delta_2 = \Delta^*$. Suppose that the following conditions are accomplished:
1) $span\{e_i\} \subset \Delta_1$
2) There exists a regular feedback of the form:

$$v_1 = \alpha_1(z_1, z_2, z_3) + \beta_1(z_1, z_2, z_3)w_1$$

with $\alpha_1(0) = 0$ such that the linearized dynamics restricted to the leaf of $\Delta_1$ passing through the equilibrium point $x_0 = 0$:

$$\dot{\xi}_1 = (\frac{\partial \tilde{f}_1}{\partial z_1} + \tilde{g}_{11}\frac{\partial \alpha_1}{\partial z_1}(0)\xi_1$$

is asymptotically stable.
3) There exists a regular feedback of the form:

$$\begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \alpha_2(z_2, z_3) \\ \alpha_3(z_2, z_3) \end{bmatrix} + \beta_2(z_2, z_3)\begin{bmatrix} w_2 \\ w_3 \end{bmatrix}$$

with $\alpha_2(0) = 0, \alpha_3(0) = 0$ such that the linearized dynamics restricted to the leaf of the distribution $\mathbf{R}^n/\Delta_1$ passing through the equilibrium point $x_0 = 0$:

$$\begin{bmatrix} \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \left. \begin{bmatrix} \frac{\partial \tilde{f}_2}{\partial z_2} + \tilde{g}_{22}\frac{\partial \alpha_2}{\partial z_2} + \tilde{g}_{23}\frac{\partial \alpha_3}{\partial z_2} & \frac{\partial \tilde{f}_2}{\partial z_3} + \tilde{g}_{22}\frac{\partial \alpha_2}{\partial z_3} + \tilde{g}_{23}\frac{\partial \alpha_3}{\partial z_3} \\ \tilde{g}_{33}\frac{\partial \alpha_3}{\partial z_2} & \frac{\partial \tilde{f}_3}{\partial z_3} + \tilde{g}_{33}\frac{\partial \alpha_3}{\partial z_3} \end{bmatrix} \right|_{(0)} \begin{bmatrix} \xi_2 \\ \xi_3 \end{bmatrix}$$

is asymptotically stable.

Then applying the both feedbacks we obtain the solution of the LDDPS. The problem is now to obtain a convenable splitting of the system as in Theorem (3.3). In order to solve a large class of problems as possible, we look for the maximal distribution $\Delta_1$ (that will be denoted by $\Delta_s^*$) with the properties given by the hypothesis of Theorem (3.3) and the point 2) from the above discussion. This way is followed in [WeNi89]. Unfortunately, the existence result on $\Delta_s^*$ does not give a method to construct this distribution in practice. Motivated by this remark, Leo van der Wegen in 1989 (see [vdWe89]) proposes another approach to solve the LDPPS.

Let us consider again Theorem (3.3) but this time with $\Delta_1 = \Pi^*$. Suppose the following conditions are fulfilled:

1) $span\{e_i\} \subset \Delta_2$

2) There exists a regular feedback of the form:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \alpha_1(z_1, z_2, z_3) \\ \alpha_2(z_1, z_2, z_3) \end{bmatrix} + \beta_1(z_1, z_2, z_3) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

with $\alpha_1(0) = 0, \alpha_2(0) = 0$ such that the dynamics of the closed- loop system restricted to the leaf of the distribution $\Delta_2$ passing through the origin is asymptotically stable.

3) There exists a regular feedback of the form:

$$v_3 = \alpha_3(z_3) + \beta_3(z_3)w_3$$

with $\alpha_3(0) = 0$, that renders asymptotically stable the dynamics of the linearized closed-loop system restricted to the leaf of the distribution $\mathbf{R}^n/\Delta_2$, passing through the origin.

Again the both feedbacks solve the LDDPS. Unlike the first solution, here we look for the smallest distribution $\Delta_2$ (that will be denoted by $(\Delta^p)_*$) with the above properties. We point out that this distribution depends essentially of the disturbance vector fields $(e_i)$ and from this reason it is a "more oriented problem" solution. Fortunately, there exists an algorithm to compute $(\Delta^p)_*$.

## 4.2 First Solution

### 4.2.1 Stabilizability Distributions

We consider the nonlinear system given by (1.4) having at $x_0 = 0$ an equilibrium point (that means $f(0) = 0$). A distribution $\Delta$ is called a *stabilizability distribution* if :

1) $\Delta$ is a nonsingular involutive controlled invariant distribution;

2) the dynamics of the linearized closed-loop system restricted to the leaf of $\Delta$ passing through $x_0 = 0$ is asymptotically stable.

Since the definition of a stabilizability distribution is independent of the disturbance $q$ in (1.1), we take $q \equiv 0$ in the rest of the subsection. We look for the maximal stabilizability distribution included in $Ker(dh)$ (which will be denoted by $\Delta_s^*$) that is rendered invariant by the same feedback as the distribution $\Delta^*$. We point out that is very important that $\Delta_s^*$ has the same "friend" as $\Delta^*$ ("friend" means the pair of feedback that renders invariant the distribution). The following example shows that if $\Delta_1 \subset \Delta^*$ is a controlled invariant distribution, it does not imply that $\Delta_1$ has the same friend as $\Delta^*$:

**EXAMPLE 4.1** *Let us consider the nonlinear system* (1.4) *with:*
$n = 4, m = 2, l = 1$ *and:*

$$f(x) = \begin{bmatrix} -2x_1 \\ -x_2 + x_4 \\ x_3 - x_2 x_3 \\ 3x_4 \end{bmatrix}, \; g_1(x) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \; g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} h(x) = x_4 \quad (4.1)$$

*In this case* $L_{g_1} = 1$ *and* $L_{g_2} = -1$, *hence* $r_1 = r_2 = 1$ *and:*

$$\Delta^* = span\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\}$$

*It is obviously that* $\Delta^*$ *is invariant under* $f$ *and* $g_1, g_2$ *but:*

$$\Delta_1 = span\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\} \subset \Delta^*$$

*is not:*

$$[f, \frac{\partial}{\partial x_2}] = \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} = \frac{x_3}{2} g_1 + \frac{x_3}{2} g_2 + 2\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1} \in \Delta_1 + G$$

*and it is controlled invariant (by Theorem* (3.1)*). Moreover, the dynamics of the linearized closed-loop system restricted to the leaf passing through the origin is asymptotically stable. Let:*

$$u_1 = \frac{x_2 x_3}{2} + w_1 \quad ; \quad u_2 = \frac{x_2 x_3}{2} + w_2$$

*then:*

$$\tilde{f} = \begin{bmatrix} \frac{x_2 x_3}{2} - 2x_1 \\ -\frac{x_2 x_3}{2} - x_2 + x_4 \\ x_3 \\ 3x_4 \end{bmatrix} \; ; \; \tilde{g}_1 = g_1 \; ; \; \tilde{g}_2 = g_2$$

*and:*

$$[\tilde{f}, \tfrac{\partial}{\partial x_1}] = 2\tfrac{\partial}{\partial x_1} \in \Delta_1$$
$$[\tilde{f}, \tfrac{\partial}{\partial x_2}] = -\tfrac{x_3}{2}\tfrac{\partial}{\partial x_1} + (\tfrac{x_3}{2} + 1)\tfrac{\partial}{\partial x_2} \in \Delta$$

*Hence* $\Delta_1$ *is* $(\tilde{f}, \tilde{g})$*-invariant. The restricted system is obtained by setting* $x_3 = 0$ *and* $x_4 = 0$*:*

$$\begin{cases} \dot{x}_1 = -2x_1 \\ \dot{x}_2 = -x_2 \end{cases}$$

*It is already in linear form and , obviously, it is asymptotically stable. Hence* $\Delta_1$ *is a stabilizability distribution.*
**Remark** *We have:*

$$\Delta^* \cap G = span\{\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3}\}$$

$$So: \quad \Pi^* =< f, g_1, g_2 | span\{\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3}\} >= \Delta^*.$$

*Moreover,* $\Pi^*$ *is a stabilizability distribution because, if we setting:*

$$u_1 = -x_3 + w_1 \quad ; \quad u_2 = -x_3 + w_2$$

*in* (4.1)*, we obtain:*

$$\bar{f} = \begin{bmatrix} -2x_1 - x_3 \\ -x_2 - x_3 + x_4 \\ -x_3 - x_2 x_3 \\ 3x_4 \end{bmatrix} \quad , \quad \bar{g}_1 = g_1 \quad ; \quad \bar{g}_2 = g_2$$

*that also renders invariant* $\Pi^* = \Delta^*$*, and the linearized system evolving on the leaf of* $\Pi^*$ *that passes through the origin:*

$$\dot{\xi} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \xi + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} w$$

*is , obviously, asymptotically stable.* $\diamond$

Now we try to find conditions that ensure us of the existence and unicity of the maximal stabilizability distribution. Since in the linear case the controllability distribution is a stabilizability distribution (see [Wo79]) it is reasonable to ask for the same thing in the nonlinear case. We have already seen that, in the nonlinear case, not every controllability distribution is also a stabilizability distribution (see Example (3.12) ). Then it is necessary to assume two conditions:

**A1.** $\Pi^*$ and $G$ are nonsingular on a neighborhood $\mathcal{U}$ of the origin and $dim\, G = m$.

**A2.** The linearization of the dynamics (1.4) restricted to the leaf $L_0$ of $\Pi^*$ through $x_0 = 0$ is stabilizable.

In order to obtain a decomposed form as given by Theorem (3.3), we are led to assume the following condition:

**A3.** $\Delta^*$ and $\Delta^* + G$ are nonsingular on $\mathcal{U}$.

It is known that if $(\alpha, \beta)$ is a friend of $\Delta^*$, it is also of $\Pi^*$ (see [Is89]). Then we may apply Theorem (3.3) with $\Delta_1 = \Pi^*$,$\Delta_2 = \Delta^*$ and such that the linearized system restricted to the leaf $L_0$ (of $\Pi^*$ that passes through the origin) is already asymptotically stable. Using also A1 and A3 we obtain:

$$\begin{array}{rlll} \dot{x}_1 = & \hat{f}^1(x_1, x_2, x_3) & +\hat{g}^{11}(x_1, x_2, x_3)u^1 & +\hat{g}^{12}(x_1, x_2, x_3)u^2 \\ \dot{x}_2 = & \hat{f}^2(x_2, x_3) & & +\hat{g}^{22}(x_2, x_3)u^2 \\ \dot{x}_3 = & \hat{f}^3(x_3) & & +\hat{g}^{32}(x_3)u^3 \end{array} \tag{4.2}$$

with $\Pi^* = span\{\frac{\partial}{\partial x_1}\}$, $\Delta^* = span\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$ , $\hat{g}^{11} = (g_1, \ldots, g_s) \in G \cap \Pi^* = G \cap \Delta^*$. We point out that: $G \cap (\Delta^* \backslash \Pi^*) = \emptyset$ and this is the reason for what we do not have three terms in $\hat{g}$. We have supposed that $\hat{f}^1$ is already asymptotically stable (that means: $\sigma[\frac{\partial \hat{f}^1(x_1,0,0)}{\partial x_1}|_{x_1=0}] \subset \mathbf{C}^-$). Let $W$ denote the set of all stabilizable distributions that are $(\hat{f}, \hat{g})$-invariant and include $\Pi^*$. This set is not empty because $\Pi^* \subset W$. Since $(\hat{g}^{12}, \hat{g}^{22}, \hat{g}^{32})^T$ has full column rank and $(\hat{g}^{32})$ has full row rank we conclude that the class of feedbacks that render invariant $\Delta^*$ (and also $\Pi^*$) is:

$$\begin{array}{rrl} u^1 = & \alpha_1(x_1, x_2, x_3) & +\beta_1 w^1 \\ u^2 = & \alpha_2(x_3) & +\beta_2 w^2 \end{array} \tag{4.3}$$

with $\alpha_1(0) = 0$, $\alpha_2(0) = 0$. Now we can prove the following lemma:

**LEMMA 4.2** *Let $\Delta_1$ be a stabilizability distribution that includes $\Pi^*$. Then $\Delta_1$ is $(\hat{f}, \hat{g})$-invariant.*

**Proof**

*We have: $\Pi^* \subset \Delta_1 \subset \Delta^*$. Since $\Delta_1$ is controlled invariant:*

$$[\hat{f}, \Delta_1] \subset \Delta_1 + Im(\hat{g}^{11}) + Im(\hat{g}^2)$$

$$[\hat{g}, \Delta_1] \subset \Delta_1 + Im(\hat{g}^{11}) + Im(\hat{g}^2)$$

*But $[\hat{f}, \Delta_1], [\hat{g}, \Delta_1]$ has no components on $\frac{\partial}{\partial x_3}$ and $Im(\hat{g}^{11}) \subset \Pi^* \subset \Delta_1$. Q.E.D.*

□

By using this lemma we may consider the sum of all members of $W$ and the involutive closure of it:

$$\Delta_{\hat{f},\hat{g}} = inv\_close \sum_{\Delta_i \in W} \Delta_i$$

as the candidate to the maximal stabilizability distribution that includes $\Pi^*$. The problem is if it is also a stable distribution, that means the restriction of the linearized dynamics to $Q_0$ (that is its leaf passing through the origin) is asymptotically stable. (Note that if there exists a feedback that stabilizes the linearized dynamics restricted to $Q_0$, then this dynamics was also born stable - see A2, the construction of $\hat{f}$ and $\hat{g}$ and the form of equation (4.3)). We are able to prove the following result:

**LEMMA 4.3** *Assume that $\Delta_{\hat{f},\hat{g}}$ is nonsingular. Then $\Delta_{\hat{f},\hat{g}}$ is a stabilizability distribution. Moreover, in every pair of coordinates as in equation 4.2 $\Delta_{\hat{f},\hat{g}}$ is a stable distribution. (So $\Delta_{\hat{f},\hat{g}}$ does not depend on the choice of the feedback.)*
**Proof**

*Let $M_0$ denote the maximal stable manifold of the vector field $\hat{f}$ (stable manifold means an invariant manifold under the action of the vector field $\hat{f}$ and every trajectory initialized on it tends to the equilibrium point; we know that there exists a unique maximal stable manifold - see Hartmann's theorem). Since every $\Delta_i \in W$ is also a stable distribution (see the above note), it follows then $\Delta_i|_{M_0} \subset TM$. Then $\Delta_{\hat{f},\hat{g}}|_{M_0} \subset TM_0$. Since it is nonsingular and involutive we obtain that its integral manifold that passes through the origin is included in $M_0$.* □

Justified of this lemma, it is reasonable to require the following condition:
**A4.** The distribution $\Delta_{\hat{f},\hat{g}}$ is nonsingular.
Now the following result is a consequence of the foregoing:

**COROLLARY 4.4** *Assume that A1,A2,A3 and A4 hold. Then there exists a unique maximal stabilizability distribution that contains $\Pi^*$ and is included in $Ker(dh)$.* □

We shall denote it by $\Delta_s^*$. In fact $\Delta_s^* = \Delta_{\hat{f},\hat{g}}$.

### 4.2.2 The First Solution of the LDDPS

Now the solution of the local disturbance decoupling problem with stability (LDDPS) follows straightforward. For convenience we choose new coordinates $x = (z_1, z_2)$ such that:

$$\Delta_s^* = span\{\frac{\partial}{\partial z_1}\}$$

Instead of equations (4.2), this yields

$$\begin{aligned} \dot{z}_1 &= \breve{f}_1(z_1, z_2) + \breve{g}_1(z_1, z_2)v + \breve{e}_1(z_1, z_2)q \\ \dot{z}_2 &= \breve{f}_2(z_2) + \breve{g}_2(z_2)v + \breve{e}_2(z_1, z_2)q \\ y &= \breve{h}(z_2) \end{aligned} \tag{4.4}$$

where we have considered also the disturbance term. We make now two assumptions:

**A5.** The dynamics of the linearized system restricted to the leaf of $\mathbf{R}^n/\Delta_s^*$ passing through the origin is stabilizable.

**A6.** $span\{e_i\} \subset \Delta_s^*$.

Then $\breve{e}_2 = 0$ and, from A5, we can choose a linear feedback:

$$v = Gz_2 + w \tag{4.5}$$

such that: $\sigma(\frac{\partial \breve{f}_2}{\partial z_2}|_0 + \breve{g}_2(0)G) \subset \mathbf{C}^-$. So the LDDPS is solved. We are able to state the following theorem:

**THEOREM 4.5** *Consider the system* (1.1)*. Assume that A1 up to A5 hold. Then the LDDPS for* (1.1) *is solvable if and only if A6 holds.* □

For proof of the only assertion (if it is necessary) see [WeNi89], Theorem 2.1.

For end this section we sum up the discussion:

1. First choose a feedback such that the system is brought into the form given by (4.2) with $\hat{f}_1(x_1, 0, 0)$ asymptotically stable.
2. Determine, if it possible, $\Delta_s^*$.
3. Change, if it needs, the coordinates in order to obtain the form (4.4).
4. Compute the feedback (4.5) and verify A6.

## 4.3 The Second Solution

### 4.3.1 The $(\Delta^p)_*$-Algorithm

This second solution offers an algorithm to solve the problem. The idea of this solution is very simple. We look for the smallest controlled invariant distribution included in $Ker(dh)$ that contains $span\{e_i\}$. Since we try to have an unique distribution, we have to require that this distribution contains also $\Pi^*$. Actually, the problem is to find the smallest controlled invariant distribution included in $Ker(dh)$ that contains $span\{e_i\}$ and $\Pi^*$.
First we have to assume that:

**B1.** $span\{e_i\} \subset \Delta^*$ on a neighborhood $\mathcal{U}$ of $x_0 = 0$.
in order to be able to solve at least the LDDP. And that:

**B2.** $dim\, G = m$ and $\Delta^*$ is constant dimensional on $\mathcal{U}$.

Choose a regular static state feedback:

$$u = \alpha(x) + \beta(x)v \ , \ \alpha(0) = 0 \ , \ \beta(x) \ invertible \ on \ \mathcal{U}$$

such that for the feedback modified system we have that $\Delta^*$ is invariant under $\tilde{f} = f + g\alpha$ and $\tilde{g}_i = (g\beta)_i$, $i = 1, \ldots, m$.

We carry on the following algorithm (see [vdWe91], Algorithm 4.2.1):

**ALGORITHM 6 (The $(\Delta^p)_*$-Algorithm)**
*Step 0:* $\Delta_0 = \Pi^* + span\{e_i\}$
*Step k:* $\Delta_{k+1} = \Delta_k + [\tilde{f}, \Delta_k] + \sum_{i=1}^{m} [\tilde{g}_i, \Delta_k], \ k = 1, 2, \ldots$

Let $\Delta'$ denote the sum of all distributions $\Delta_k$:

$$\Delta' = \Delta_0 + \Delta_1 + \cdots + \Delta_k + \cdots$$

Note that if all distributions $\Delta_k$ are constant dimensional, then the algorithm converges in at most $n$ steps. Let $\Delta^p_{\tilde{f},\tilde{g}}$ denote the involutive closure of $\Delta'$. Assume that:

**B3.** $\Delta^p_{\tilde{f},\tilde{g}}$ is constant dimensional on $\mathcal{U}$.

We are able to prove that $\Delta^p_{\tilde{f},\tilde{g}}$ though a priori depends on the choice of the feedback $(\alpha, \beta)$, in fact it does not depend on this feedback. The proof is very close of that from Lemma (4.3) . We state, only, a theorem that sum up some results:

**THEOREM 4.6** (*see [vdWe91], Theorem 4.2.2, pp 61, for proof) Consider the system* 1.1. *Assume that B1,B2 and B3 hold. Then $\Delta^p_{\tilde{f},\tilde{g}}$ is independent of the choice of the feedback $(\alpha, \beta)$ that renders invariant $\Delta^*$. Moreover, $\Delta^p_{\tilde{f},\tilde{g}}$ is the smallest constant dimensional locally controlled invariant distribution in the kernel of the output mapping that contains $\Pi^*$ and the disturbance vector fields* $\{e_i\}$.

□

In that it follows we shall denote this distribution by $(\Delta^p)_*$. Then $(\Delta^p)_* = \Delta^p_{\tilde{f},\tilde{g}}$.

### 4.3.2 The Second Solution of the LDDPS

The solution of the LDDPS in terms of $(\Delta^p)_*$ is now straightforward:

**THEOREM 4.7** *Consider the smooth system* (1.1)*. Assume that B1,B2 and B3 hold and that:*

**B4.** *The dynamics of the system in closed-loop, restricted to the leaf of* $(\Delta^p)_*$ *through* $x_0 = 0$ *can be exponentially stabilized:*

**B5.** *The linearization of the closed-loop system dynamics restricted to* $\mathbf{R}^n/(\Delta^p)_*$ *is stabilizable.*

*Then the local disturbance decoupling problem with stability for* (1.1) *is solvable. On the other hand, if the LDDPS for* (1.1) *is solvable by making a regular distribution* $\Delta$ *invariant, then the dynamics of the system restricted to the leaf of the* $\Delta$ *through* $x_0 = 0$ *can be stabilized exponentially and the linearization of the dynamics restricted to* $\mathbf{R}^n/\Delta$ *is stabilizable.* □

A few remark are necessary.

**Remarks**

1) The distributions $\Delta$ and $(\Delta^p)_*$ in the above theorem are stabilizability distributions.

2) The conditions B5 and A5 (from the previous section) are implied by:

**C1.** The linearization of the (1.1) around $x_0 = 0$ is stabilizable.

In fact, there is a result for the linear system by the following form:

**PROPOSITION 4.8** *Suppose a linear dynamics given by:*

$$\dot{x} = Ax + Bu \quad , \quad x \in \mathbf{R}^n$$

*with:*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$A_{11}$ *stable and* $(A, B)$ *stabilizable. Then the pair* $(A_{22}, B_{22})$ *is also stabilizable.*

□

Now, using this result the proof of the above assertion is obviously.

# Chapter 5

# Conclusions

Several design problems in linear systems theory have been treated fruitfully by the geometric approach. Motivated by the success of the geometric theory , researchers in nonlinear systems theory tried to translate several geometric concepts to a nonlinear context, using differential geometric tools. This led, for instance, to the definition of (local) controlled invariance and the solution of a local version of the Disturbance Decoupling Problem.

In this thesis the Local Disturbance Decoupling Problem with Stability for nonlinear systems is considered. This problem consists in finding conditions under which there exists a locally defined regular static state feedback that decouples the outputs from the disturbances and exponentially stabilizes the equilibrium of the modified drift dynamics of the feedback systems. For systems for which the linearization of the dynamics around an equilibrium is stabilizable, two methods are proposed to solve this problem.

In the first method the stabilizability distributions for nonlinear systems are introduced and it is shown that under certain regularity assumptions the maximal stabilizability distribution $\Delta_s^*$ in the kernel of the output mapping exists and that the LDDPS is solvable if and only if the disturbance vector fields are contained in $\Delta_s^*$. This distribution forms a nonlinear analogue of the maximal stabilizability subspace $\mathcal{V}_s^*$ for linear systems. But, while in the linear case the dimension of $\mathcal{V}_s^*$ is always equal with the dimension of the maximal stable manifold of the dynamics restricted to the largest locally controlled invariant distribution in the kernel of the output mapping, in the nonlinear case the dimension of $\Delta_s^*$ should be strictly less than the dimension of the stable invariant manifold. This is a nonlinear phenomenon.

A second more " oriented problem" method for the smallest locally controlled invariant distribution $(\Delta^p)_*$ in the kernel of the output mapping containing the disturbance vector fields as well as the largest local controllability distribution in $\Delta^*$. If the linearization of the dynamics of the system restricted to the leaf of $(\Delta^p)_*$ through the equilibrium is stabilizable, then the LDDPS is solvable.

Since stabilizability of the linearization of the nonlinear system around an equilibrium point is a necessary condition for solvability of the LDDPS, one may wonder if solvability of the Disturbance Decoupling Problem with Stability (DDPS) for the linearization is sufficient for solvability of the LDDPS for the nonlinear system. The answer is negative as can easily be seen from an example (for instance, Example 5.2.1, pp 74, from [vdWe91]).

The solutions presented here are locally and in nonsingular cases. The open problems that remains to be studied are:

1) What does it happen if the nonsingularity conditions are not fulfilled ?

2) What are the supplementary conditions that solve the global problem ?

About the first problem we suggest that a point of departure should be the two Appendices of this thesis.

The second problem has not yet any complete solution either for the LDDPS or for any other nonlinear problem. Recently, some articles have been published on (semi)global stabilization of nonlinear control systems (see [Su90]) which show some limitations to this globalization. Our opinion is that the solution of the global problem can be achieved only using algebraic tools. To be more exact, the differential geometric tools have not the "force" to solve the singularities. By this reason it is possible that if one find an algebraic solution of the first problem, this should be used also for the second problem. By this way the two problems can be unified by a unique solution.

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