An L1 Matrix Factorization

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Table of Contents:

Problem Formulation

Function Space Formulation

Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a positive semi-definite trace-class compact operator written in integral form

$$Tf(x) = \int_{-\infty}^{\infty} K(x,y)f(y)dy.$$

Assume $K \in M^1(\mathbb{R}^2)$ belongs to the modulation space M^1 (a.k.a. the Feichtinger algebra, or the Segal algebra for TF ops).
Let $(f_k)_{k\geq 0}$ be a set of eigenvectors, $Tf_k = \|f_k\|_2^2 f_k$. Thus $T = \sum_k f_k f_k^*$ and $\sum_k \|f_k\|_2^2 = tr(T) < \infty$.
Fact: It is known [HeilLars04/08] that $f_k \in M^1$ for each k.

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Problem 1 [Feichtinger2004]: Does $\sum_{k\geq 0} \|f_k\|_{M^1}^2 < \infty$?

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Fact: It is known [HeilLars04/08] that $f_k \in M^1$ for each k.
Problem 1 [Feichtinger2004]: Does $\sum_{k\geq 0} \|f_k\|_{M^1}^2 < \infty$?
Problem 2 [HeilLars04]: If the answer is negative to Problem 1, is there a decomposition $T = \sum_k g_k g_k^*$, not necessarily spectral, so that $\sum_{k\geq 0} \|g_k\|_{M^1}^2 < \infty$?

Problem Formulation

Interlude: Modulation space M^1

The Feichtinger space M^1 is defined as follows. Let $g : \mathbb{R} \to \mathbb{R}$, $g(x) = e^{-\pi x^2}$ be the Gaussian window. Let

$$f \in \mathbb{S}' \mapsto V_g f(t,w) = \int_{-\infty}^{\infty} e^{-2\pi i w x} f(x) g(x-t) dx$$

be the windowed Fourier transform of f with respect to g. Then

$$M^1(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) \ , \ \|f\|_{M^1} := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |V_g f(t,w)| dt\, dw < \infty \right\}.$$

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Fact: [FeichtGrochWaln92] The Wilson ONB is an unconditional basis in M^1. Let $(w_n)_{n\geq 0}$ denote this Wilson basis. Then we can identify M^1 with $l^1(\mathbb{N})$ space, with equivalent norms:

$$M^1(\mathbb{R})=\{f=\sum_{n\geq 0}c_nw_n\ ,\ \|f\|_{M^1}\sim\sum_{n\geq 0}|c_n|\}.$$

Problem Formulation

Matrix Reformulation

Consider an infinite matrix $A = (A_{m,n})_{m,n\geq 0}$ so that

$$\|A\|_{\wedge} := \sum_{m,n\geq 0} |A_{m,n}| < \infty.$$

This implies that A acts on $l^2(\mathbb{N})$ as a trace-class compact operator.
Assume additionally $A = A^* \geq 0$.
Let $(e_k)_{k\geq 0}$ denote an orthogonal set of eigenvectors normalized so that $A = \sum_{k\geq 0} e_k e_k^*$. It is easy to check that $e_k \in l^1(\mathbb{N})$, for each k.
Equivalent problems reformulation ([HeilLars04]):

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Equivalent problems reformulation ([HeilLars04]):
Problem 1: Does it hold $\sum_{k \geq 0} \|e_k\|_1^2 < \infty$?
Problem 2: If negative to problem 1, is there a factorization $A = \sum_{k \geq 0} f_k f_k^*$ so that $\sum_{k \geq 0} \|f_k\|_1^2 < \infty$?

Tensor Products

Consider $A \in \mathbb{C}^{n \times n}$. We seek "optimal" decompositions of A into a sum of rank-1 operators: $A = \sum_k u_k v_k^*$.
In this talk we assume A to be positive semi-definite: $A = A^* \geq 0$.

Criterion 1:

$$J(A) = \inf_{A=\sum_{k=1}^m f_k f_k^*} \sum_{k=1}^{m} \|f_k\|_1^2.$$

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Criterion 3:

$$J_\wedge(A) = \inf_{A=\sum_{k=1}^m f_k g_k^*} \sum_{k=1}^m \|f_k\|_1 \|g_k\|_1$$

What we know

$$J_{\wedge}(A) = \min_{A=\sum_{k=1}^{m} f_k g_k^*} \sum_{k=1}^{m} \|f_k\|_1 \|g_k\|_1$$

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2. $J_\wedge, J_0$ extend to norms on $Sym(\mathbb{C}^n)$.

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1. $J_\wedge, J_0, J$ are positive, homogeneous, and convex on $Sym^+(\mathbb{C}^n)$.
2. $J_\wedge, J_0$ extend to norms on $Sym(\mathbb{C}^n)$.
3. The following hold true:

$$\sum_{i,j}|A_{i,j}| =: \|A\|_\wedge = J_\wedge(A) \le J_0(A) \le 2\|A\|_\wedge \quad , \quad \forall A \in Sym(\mathbb{C}^n).$$

$$\|A\|_\wedge = J_\wedge(A) \le J_0(A) \le J(A) \le n\|A\|_\wedge \quad , \quad \forall A \in Sym^+(\mathbb{C}^n).$$

Central Example

Consider the identity matrix I_n and two possible decompositions:

$$I_n = \sum_{k=1}^{n} \delta_k \delta_k^* = \sum_{k=0}^{n-1} e_{n,k} e_{n,k}^*$$

where $\{\delta_k\}_k$ is the canonical ONB, and $\{e_{n,k}\}_k$ is the Fourier ONB:

$$e_{n,k} = \frac{1}{\sqrt{n}} \left[\begin{array}{cccc} 1 & e^{-2\pi i k/n} & \cdots & e^{-2\pi i k(n-1)/n} \end{array} \right]^T .$$

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Note:

$$\sum_{k=1}^{n} \|\delta_k\|_1^2 = n = \|I_n\|_{\wedge} = J(I_n) \rightarrow \text{"good decomposition"}$$

$$\sum_{k=0}^{n-1} \|e_{n,k}\|_1^2 = n^2 = nJ(I_n) \rightarrow \text{"bad decomposition"}$$

The CounterExample
Block Diagonal Form

We construct an example that answers negatively problem 1, but positively problem 2.
The form: $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n \oplus \cdots$,

$$T = \begin{bmatrix} T_1 & & & & \\ & T_2 & & & \\ & & \ddots & & \\ & & & T_n & \\ & & & & \ddots \end{bmatrix}$$

The CounterExample

Each block T_n is diagonalized by the Fourier ONB, and has positive simple eigenvalues:

$$T_n = \frac{1}{n^3} \sum_{k=0}^{n-1} \left(1 + \frac{k}{n^p}\right) e_{n,k} e_{n,k}^*.$$

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Each block T_n is diagonalized by the Fourier ONB, and has positive simple eigenvalues:

$$T_n = \frac{1}{n^3}\sum_{k=0}^{n-1}\left(1+\frac{k}{n^p}\right)e_{n,k}e_{n,k}^*.$$

Thus:

$$T = \bigoplus_{n\geq 1}\sum_{k=0}^{n-1}\frac{1}{n^3}\left(1+\frac{k}{n^p}\right)e_{n,k}e_{n,k}^*.$$

Problem 1
Negative Answer

The eigendecomposition of T is

$$T=\sum_{n\geq 1}\sum_{k=0}^{n-1} f_{n,k}f_{n,k}^{*} \quad , \quad f_{n,k}=\frac{1}{\sqrt{n^3}}\sqrt{1+\frac{k}{n^p}}e_{n,k}.$$

Then

$$\sum_{n\geq 1}\sum_{k=0}^{n-1}\|f_{n,k}\|_1^2=\sum_{n\geq 1}\sum_{k=0}^{n-1}\frac{1}{n^3}(1+\frac{k}{n^p})n\geq \sum_{n\geq 1}\frac{1}{n}=\infty$$

Hence the answer to problem 1 is negative: There is an operator $S:f\mapsto Sf(x)=\int K(x,y)f(y)dy$ with $K\in M^1(\mathbb{R}^2)$ and $S=S^*\geq 0$, so that its spectral decomposition $S=\sum_{k\geq 1}\langle\cdot,f_k\rangle f_k$ satisfies $\sum_k \|f_k\|_{M^1}^2=\infty$.

Problem 2
Positive Answer

We show now that same operator T we constructed earlier admits a decomposition $T = \sum_m g_m g_m^*$ so that $\sum_m \|g_m\|_1^2 < \infty$.
Notice:

$$T_n = \frac{1}{n^3}\sum_{k=0}^{n-1}\left(1+\frac{k}{n^p}\right)e_{n,k}e_{n,k}^* = \frac{1}{n^3}\sum_{k=0}^{n-1}\delta_k\delta_k^* + \frac{1}{n^{3+p}}\sum_{k=0}^{n-1}ke_{n,k}e_{n,k}^*$$

Thus the induced decomposition

$$T_n = \sum_{k=0}^{n-1} g_{1,n,k}g_{1,n,k}^* + \sum_{k=0}^{n-1} g_{2,n,k}g_{2,n,k}^*$$

satisfies

$$\sum_{k=0}^{n-1}\|g_{1,n,k}\|_1^2 + \|g_{2,n,k}\|_1^2 = \frac{1}{n^2} + \frac{1}{n^{2+p}}\frac{n(n-1)}{2} \leq \frac{1}{n^2} + \frac{1}{n^p}$$

Problem 2
Positive Answer - cont'd

Thus:

$$T = \bigoplus_{n\geq 1} \sum_{k=0}^{n-1} g_{1,n,k} g_{1,n,k}^{*} + g_{2,n,k} g_{2,n,k}^{*}$$

satisfies

$$\sum_{n\geq 1} \sum_{k=0}^{n-1} \|g_{1,n,k}\|_1^2 + \|g_{2,n,k}\|_1^2 \leq \sum_{n\geq 1} \frac{1}{n^2} + \frac{1}{n^p} < \infty$$

Problem 2

Positive Answer - cont'd

Thus:

$$T = \bigoplus_{n\geq 1} \sum_{k=0}^{n-1} g_{1,n,k} g_{1,n,k}^* + g_{2,n,k} g_{2,n,k}^*$$

satisfies

$$\sum_{n\geq 1} \sum_{k=0}^{n-1} \|g_{1,n,k}\|_1^2 + \|g_{2,n,k}\|_1^2 \leq \sum_{n\geq 1} \frac{1}{n^2} + \frac{1}{n^p} < \infty$$

Hence the answer to the second problem is affirmative: There is an operator $S = S^* \geq 0$, $f \mapsto Sf(x) = \int K(x,y)f(y)dy$ with $K \in M^1(\mathbb{R}^2)$ that admits a decomposition $S = \sum_{k\geq 1} \langle \cdot, g_k \rangle g_k$ that satisfies $\sum_k \|g_k\|_{M^1}^2 < \infty$, but whose spectral decomposition does not satisfy the same localization condition.

Open Problem

A remaining open problem:Is there a universal constant $C_0 > 1$ so that for any $n \geq 1$ and every positive semidefinite $A \in \mathbb{C}^{n\times n}$,

$$J(A) = \min_{A=\sum_{k=1}^{m} f_k f_k^*} \|f_k\|_1^2 \leq C_0 \sum_{i,j=1}^{n} |A_{i,j}| \quad ?$$

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Why we care?

If the answer is positive, it follows that, given a trace-class positive semidefinite operator $T : f \mapsto Tf(x) = \int K(x,y)f(y)dy$ the following two statements are equivalent:

1. $K \in M^1(\mathbb{R}^2)$.
2. There are functions $g_k \in M^1(\mathbb{R})$ so that

$$T = \sum_{k\geq 0} \langle \cdot, g_k \rangle g_k$$

and $\sum_{k\geq 0} \|g_k\|_{M^1}^2 < \infty$.

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