Nonlinear Analysis with Frames

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Problem Formulation
The phase retrieval problem

- Hilbert space $H = \mathbb{C}^n$, $\hat{H} = H/T^1$, frame $\mathcal{F} = \{f_1, \cdots, f_m\} \subset \mathbb{C}^n$ and

$$\alpha : \hat{H} \to \mathbb{R}^m \quad , \quad \alpha(x) = (|\langle x, f_k \rangle|)_{1 \leq k \leq m} .$$

$$\beta : \hat{H} \to \mathbb{R}^m \quad , \quad \beta(x) = \left(|\langle x, f_k \rangle|^2 \right)_{1 \leq k \leq m} .$$

The frame is said *phase retrievable* (or that it gives phase retrieval) if α (or β) is injective.

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  The frame is said *phase retrievable* (or that it gives phase retrieval) if α (or β) is injective.

- The general *phase retrieval problem* a.k.a. *phaseless reconstruction*: Decide when a given frame is phase retrievable, and, if so, find an algorithm to recover x from $y = \alpha(x)$ (or from $y = \beta(x)$) up to a global phase factor.

Problem Formulation

Lipschitz Reconstruction

- Our Problems Today: Assume $\mathcal{F}$ is phase retrievable.
  1. Are the nonliner maps α, β bi-Lipschitz with respect to appropriate metrics?
  2. Do they admit left inverses that are globally Lipschitz?
  3. What are the Lipschitz constants?

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- Additionally, we want to understand the structure of Lipschitz bounds (to be defined shortly).

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Metric Space Structures

Topological Structures

Let $H = \mathbb{C}^n$. The quotient space $\hat{H} = \mathbb{C}^n / T^1$, with classes induced by $x \sim y$ if there is real φ with $x = e^{i\varphi} y$.

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Topologically:

$$\hat{\mathbb{C}}^n = \{0\} \cup \left((0,\infty) \times \mathbb{CP}^{n-1} \right)$$

with

$$\mathring{\hat{\mathbb{C}}}^n = \hat{\mathbb{C}}^n \setminus \{0\} = (0,\infty) \times \mathbb{CP}^{n-1}$$

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a real analytic manifold of real dimension $2n-1$.

Another embedding is into the space of symmetric matrices $Sym(\mathbb{C}^n)$. Specifically let

$$\mathcal{S}^{p,q}(H) = \{ T \in Sym(H) \ , \ T \text{ has at most } p \text{ pos.eigs. and } q \text{ neg.eigs}\}$$

Then:

$$\kappa_\beta : \hat{H} \to \mathcal{S}^{1,0} \quad , \quad \hat{x} \mapsto = xx^* \quad , \quad \text{is an embedding.}$$

Metric Space Structures

The matrix-norm induced metric structure

Fix $1 \leq p \leq \infty$. The *matrix-norm induced distance*

$$d_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ \ d_p(\hat{x}, \hat{y}) = \|xx^* - yy^*\|_p$$

with the p-norm of the singular values. In the case $p = 2$ we obtain

$$d_2(x, y) = \sqrt{\|x\|^4 + \|y\|^4 - 2|\langle x, y \rangle|^2}$$

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Lemma (BZ15)

1. $(d_p)_{1 \leq p \leq \infty}$ *are equivalent metrics and the identity map* $i : (\hat{H}, d_p) \to (\hat{H}, d_q)$, $i(x) = x$ *has Lipschitz constant*

$$Lip^d_{p,q,n} = \max(1, 2^{\frac{1}{q} - \frac{1}{p}}).$$

2. *The metric space* $(\hat{H}, d_p)$ *is isometrically isomorphic to* $\mathcal{S}^{1,0}$ *endowed with the p-norm via* $\kappa_\beta : \hat{H} \to \mathcal{S}^{1,0}$, $x \mapsto \kappa_\beta(x) = xx^*$.

Metric Space Structures

The natural metric structure

Fix $1 \leq p \leq \infty$. The *natural metric*

$$D_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ \ D_p(\hat{x}, \hat{y}) = \min_{\varphi} \| x - e^{i\varphi} y \|_p$$

with the usual p-norm on $\mathbb{C}^n$. In the case $p = 2$ we obtain

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2. *The metric space* $(\hat{H}, D_2)$ *is Lipschitz isomorphic to* $\mathcal{S}^{1,0}$ *endowed with the* 2*-norm via* $\kappa_\alpha : \hat{H} \to \mathcal{S}^{1,0} \quad , \quad x \mapsto \kappa_\alpha(x) = \frac{1}{\|x\|} xx^*$.

Metric Space Structures
Distinct Structures

Two different structures: topologically equivalent, BUT the metrics are NOT equivalent:

Lemma (BZ15)

The identity map $i:(\hat{H},D_p)\rightarrow(\hat{H},d_p)$, $i(x)=x$ is continuous but it is not Lipschitz continuous. Likewise, the identity map $i:(\hat{H},d_p)\rightarrow(\hat{H},D_p)$, $i(x)=x$ is continuous but it is not Lipschitz continuous. Hence the induced topologies on $(\hat{H},D_p)$ and $(\hat{H},d_p)$ are the same, but the corresponding metrics are not Lipschitz equivalent.

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Main Results

Lipschitz inversion: α

Theorem (BZ15)

Assume $\mathcal{F}$ is a phase retrievable frame for H. Then:

1. *The map $\alpha : (\hat{H}, D_2) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz. Let $\sqrt{A_0}, \sqrt{B_0}$ denote its Lipschitz constants: for every $x, y \in \hat{H}$:*

$$A_0 \min_{\varphi} \|x - e^{i\varphi} y\|_2^2 \leq \sum_{k=1}^{m} ||\langle x, f_k\rangle| - |\langle y, f_k\rangle||^2 \leq B_0 \min_{\varphi} \|x - e^{i\varphi} y\|_2^2 .$$

2. *There is a Lipschitz map $\omega : (\mathbb{R}^m, \|\cdot\|_2) \to (\hat{H}, D_2)$ so that: (i) $\omega(\alpha(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $Lip(\omega) \leq \frac{4+3\sqrt{2}}{\sqrt{A_0}} = \frac{8.24}{\sqrt{A_0}}$.*

Main Results

Lipschitz inversion: β

Theorem (BZ15)

Assume $\mathcal{F}$ is a phase retrievable frame for H. Then:

1. *The map $\beta : (\hat{H}, d_1) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz. Let $\sqrt{a_0}, \sqrt{b_0}$ denote its Lipschitz constants: for every $x, y \in \hat{H}$:*

$$a_0 \|xx^* - yy^*\|_1^2 \leq \sum_{k=1}^{m} \left| |\langle x, f_k \rangle|^2 - |\langle y, f_k \rangle|^2 \right|^2 \leq b_0 \|xx^* - yy^*\|_1^2.$$

2. *There is a Lipschitz map $\psi : (\mathbb{R}^m, \|\cdot\|_2) \to (\hat{H}, d_1)$ so that: (i) $\psi(\beta(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $Lip(\psi) \leq \frac{4+3\sqrt{2}}{\sqrt{a_0}} = \frac{8.24}{\sqrt{a_0}}$.*

Main Results
Prior Works

Prior literature:

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- 2012: **B.**: Cramer-Rao lower bound in the real case; **Eldar&Mendelson** : map α in the real case

$$\|\alpha(x)-\alpha(y)\| \geq C\|x-y\|\|x+y\|.$$

- 2013: **Bandeira,Cahill,Mixon,Nelson**: improved the estimate of C. **B.**: β bi-Lipschitz in real and complex case.
- 2014: **B.&Yang**: Find the exact Lipschitz constant for α in the real case - the constants A_0, B_0; **B.&Z.**:constructed a Lipschitz left inverse for β; **B.**: lower Lipschitz constant A_0 connected to CRLB's for a non-AWGN model.
- 2015: **B.&Z.**: Proved α is bi-Lipschitz in the complex case; constructed a Lipschitz left inverse.

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Overview

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Overview

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1. Part 1: Injectivity $\longrightarrow$ bi-Lipschitz: Upper bounds are not too hard; lower bounds: relatively easy for β (the "square" map), but very hard for α.
2. Part 2: Left inverse construction is done in three steps:
   1. The left inverse is first extended to $\mathbb{R}^m$ into $Sym(H)$ using Kirszbraun's theorem;
   2. Then we show that $\mathcal{S}^{1,0}(H)$ is a Lipschitz retract in $Sym(H)$;
   3. The proof is concluded by composing the two maps.

Proofs

Part 1: Bi-Lipschitzianity for β

Key Remark (B.Bodmann,Casazza,Edidin - 2007): The nonlinear map β is the restrictrion of the linear map

$$\mathbb{A} : Sym(H) \to \mathbb{R}^m \quad , \quad \mathbb{A}(T) = (\langle Tf_k, f_k\rangle)_{1\leq k\leq m}$$

Specifically: $\beta(x) = \mathbb{A}(xx^*)$.

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$$\begin{aligned} \|\beta(x) - \beta(y)\| = \|\mathbb{A}(xx^*) - \mathbb{A}(yy^*)\| &= \|\mathbb{A}(xx^* - yy^*)\| \\ &= \|xx^* - yy^*\| \|\mathbb{A}\left(\frac{xx^* - yy^*}{\|xx^* - yy^*\|}\right)\| \end{aligned}$$

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$$a_0 = \min_{T \in \mathcal{S}^{1,1}, \|T\|_1 = 1} \|\mathbb{A}(T)\| > 0 \quad , \quad b_0 = \max_{T \in \mathcal{S}^{1,1}, \|T\|_1 = 1} \|\mathbb{A}(T)\|$$

Proofs

Part 2: Extension of the inverse for β

Assume $\beta : (\hat{H}, d_1) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz:

$$a_0 d_1(x,y)^2 \leq \|\beta(x) - \beta(y)\|^2 \leq b_0 d_1(x,y)^2$$

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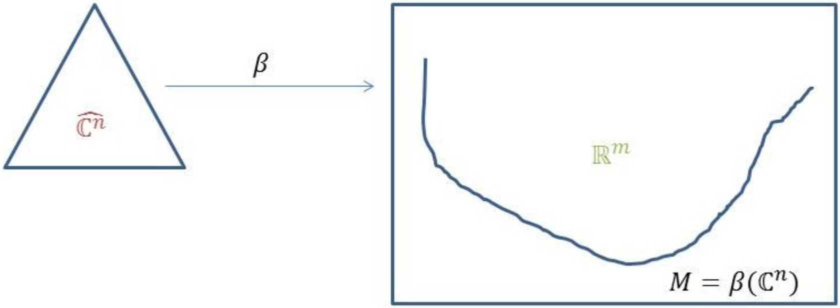
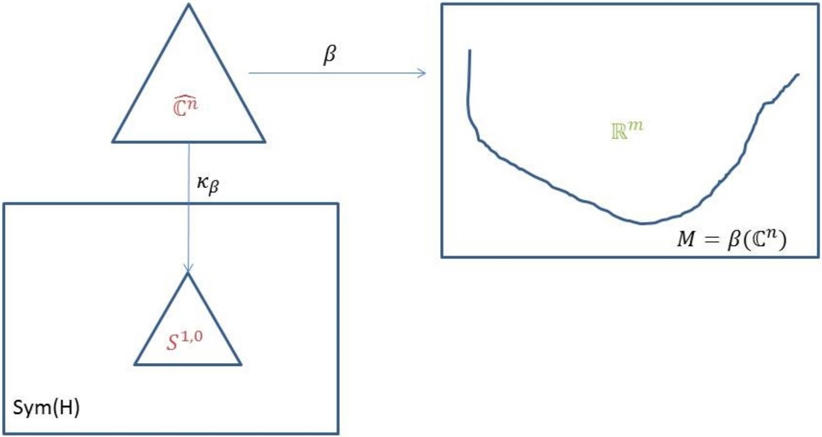
Let $M = \beta(\hat{H}) \subset \mathbb{R}^m$.



Proofs

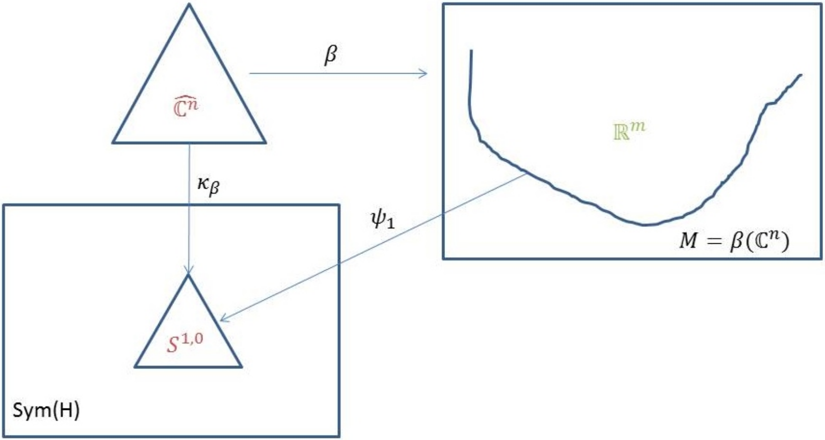
Part 2: Extension of the inverse for β

First identify $\hat{H}$ with $\mathcal{S}^{1,0}(H)$.



Proofs
Part 2: Extension of the inverse for β

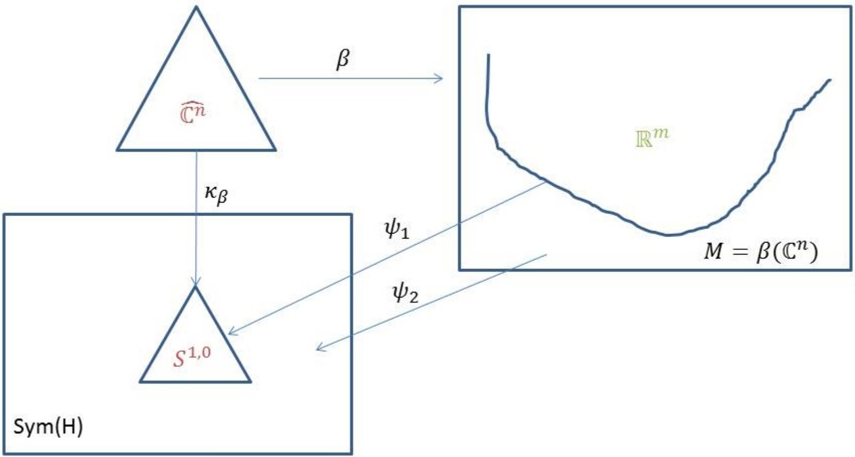
Then construct the local left inverse $\psi_1 : M \to \hat{H}$ with $Lip(\psi_1) = \frac{1}{\sqrt{a_0}}$.



Proofs

Part 2: Extension of the inverse for β

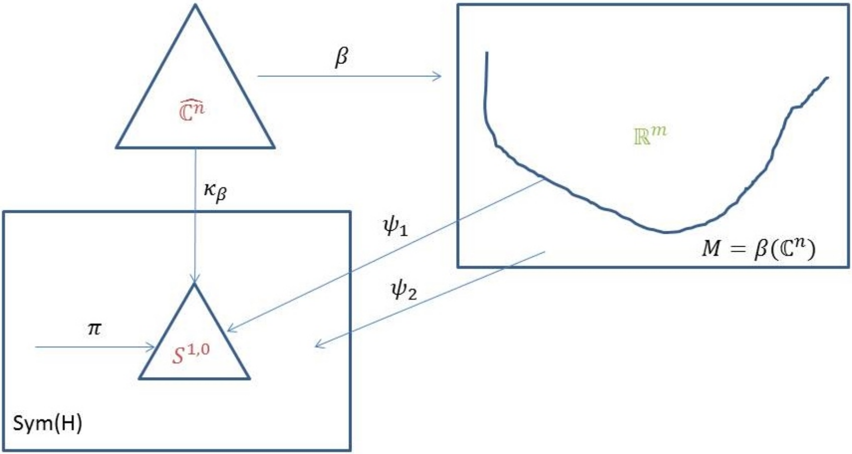
Use Kisrbraun's theorem to extend isometrically $\psi_2 : \mathbb{R}^m \to Sym(H)$.



Proofs

Part 2: Extension of the inverse for β

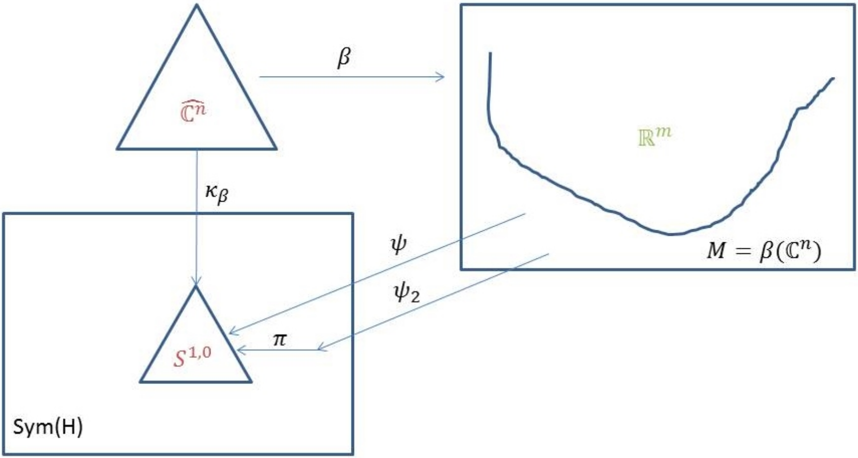
Construct a Lipschitz "projection" $\pi : Sym(H) \to \mathcal{S}^{1,0}(H)$.



Proofs

Part 2: Extension of the inverse for β

Compose the two maps to get $\psi : \mathbb{R}^m \to \mathcal{S}^{1,0}$, $\psi = \pi \circ \psi_2$.



Proofs

Part 2: $\mathcal{S}^{1,0}(H)$ as Lipschitz retract in $Sym(H)$

How to obtain $\pi : Sym(H) \rightarrow \mathcal{S}^{1,0}(H)$?

Proofs

Part 2: $\mathcal{S}^{1,0}(H)$ as Lipschitz retract in $Sym(H)$

Lemma

Consider the spectral decomposition of the self-adjoint operator A in $Sym(H)$, $A = \sum_{k=1}^{d} \lambda_{m(k)} P_k$. Then the map

$$\pi : Sym(H) \to \mathcal{S}^{1,0}(H) \quad , \quad \pi(A) = (\lambda_1 - \lambda_2) P_1$$

satisfies the following two properties:

1. *for $1 \leq p \leq \infty$, it is Lipschitz continuous from $(Sym(H), \|\cdot\|_p)$ to $(\mathcal{S}^{1,0}(H), \|\cdot\|_p)$ with Lipschitz constant less than or equal to $3 + 2^{1+\frac{1}{p}}$;*
2. *$\pi(A) = A$ for all $A \in \mathcal{S}^{1,0}(H)$.*

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Part 2: $\mathcal{S}^{1,0}(H)$ as Lipschitz retract in $Sym(H)$

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2. *$\pi(A) = A$ for all $A \in \mathcal{S}^{1,0}(H)$.*

Proof uses Weyl's inequality and spectral formula on a complex integration contour by Zwald & Blanchard (2006).

Proofs

Part 1: Bi-Lipschitzianity of α

The analysis requires a deeper understanding of local behavior.

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The analysis requires a deeper understanding of local behavior.

1. The *global lower* and *upper Lipschitz bounds*:

$$A_0 = \inf_{x,y\in\hat{H}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2} \ , \ B_0 = \sup_{x,y\in\hat{H}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2}$$

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2. The *type I local lower* and *upper Lipschitz bounds* at $z \in \hat{H}$:

$$A(z) = \lim_{r\to 0} \inf_{\substack{x,y\in\hat{H} \\ D_2(x,z)<r \\ D_2(y,z)<r}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2} \; , \; B(z) = \lim_{r\to 0} \sup_{\substack{x,y\in\hat{H} \\ D_2(x,z)<r \\ D_2(y,z)<r}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2}$$

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3. The *type II local lower* and *upper Lipschitz bounds* at $z\in\hat{H}$:

$$\tilde{A}(z) = \lim_{r\to 0} \inf_{\substack{x\in\hat{H}\\ D_2(x,z)<r}} \frac{\|\alpha(x)-\alpha(z)\|_2^2}{D_2(x,z)^2} \ , \ \tilde{B}(z) = \lim_{r\to 0} \sup_{\substack{x\in\hat{H}\\ D_2(x,z)<r}} \frac{\|\alpha(x)-\alpha(z)\|_2^2}{D_2(x,y)^2}$$

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Part 1: Bi-Lipschitzianity of α

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Let $\varphi_1, \cdots, \varphi_m, \zeta \in \mathbb{R}^{2n}$, $\Phi_1, \cdots, \Phi_m \in Sym(\mathbb{R}^{2n})$, $J \in \mathbb{R}^{2n \times 2n}$ defined by:

$$\Phi_k = \varphi_k \varphi_k^T + J\varphi_k \varphi_k^T J^T , \varphi_k = \begin{bmatrix} real(f_k) \\ imag(f_k) \end{bmatrix} , J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} , \zeta = \begin{bmatrix} real(z) \\ imag(z) \end{bmatrix}$$

Key relations: $\langle z, f_k \rangle = \langle \zeta, \varphi_k \rangle + i \langle \zeta, J\varphi_k \rangle$, $|\langle z, f_k \rangle| = \sqrt{\langle \Phi_k \zeta, \zeta \rangle}$.

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Key relations: $\langle z, f_k\rangle = \langle \zeta, \varphi_k\rangle + i\langle \zeta, J\varphi_k\rangle$, $|\langle z, f_k\rangle| = \sqrt{\langle \Phi_k\zeta, \zeta\rangle}$.
Consider the following objects:

$$\mathcal{R} : \mathbb{R}^{2n} \to Sym(\mathbb{R}^{2n}) \quad , \quad \mathcal{R}(\xi) = \sum_{k=1}^{m} \Phi_k\xi\xi^T\Phi_k \ , \ \xi \in \mathbb{R}^{2n}$$

$$\mathcal{S} : \mathbb{R}^{2n} \to Sym(\mathbb{R}^{2n}) \quad , \quad \mathcal{S}(\xi) = \sum_{k:\Phi_k\xi\neq 0} \frac{1}{\langle \Phi_k\xi, \xi\rangle}\Phi_k\xi\xi^T\Phi_k \ , \ \xi \in \mathbb{R}^{2n}$$

Proofs

Lipschitz bounds for α

Theorem (BZ15)

Assume $\mathcal{F}$ is phase retrievable for $H = \mathbb{C}^n$ and A, B are its optimal frame bounds. Then:

1. *For every $0 \neq z \in \mathbb{C}^n$, $A(z) = \lambda_{2n-1}(\mathcal{S}(\zeta))$ (the next to the smallest eigenvalue);*
2. $A_0 = A(0) > 0$;
3. *For every $z \in \mathbb{C}^n$, $\tilde{A}(z) = \lambda_{2n-1}\left(\mathcal{S}(\zeta) + \sum_{k:\langle z,f_k\rangle=0} \Phi_k\right)$ (the next to the smallest eigenvalue);*
4. *$\tilde{A}(0) = A$, the optimal lower frame bound;*
5. *For every $z \in \mathbb{C}^n$, $B(z) = \tilde{B}(z) = \lambda_1\left(\mathcal{S}(\zeta) + \sum_{k:\langle z,f_k\rangle=0} \Phi_k\right)$ (the largest eigenvalue);*
6. *$B_0 = B(0) = \tilde{B}(0) = B$, the optimal upper frame bound;*

Proofs

Lipschitz bounds for β

Theorem (cont'd)

7. *For every* $0 \neq z \in \mathbb{C}^n$, $a(z) = \tilde{a}(z) = \lambda_{2n-1}(\mathcal{R}(\zeta))/\|z\|^2$ *(the next to the smallest eigenvalue);*
8. *For every* $0 \neq z \in \mathbb{C}^n$, $b(z) = \tilde{b}(z) = \lambda_1(\mathcal{R}(\zeta))/\|z\|^2$ *(the largest eigenvalue);*
9. $a_0 = \min_{\|\xi\|=1} \lambda_{2n-1}(\mathcal{R}(\xi))$ *is also the largest constant to that* $\mathcal{R}(\xi) \geq a_0(\|\xi\|^2 I - J\xi\xi^T J^T)$;
10. $b(0) = \tilde{b}(0) = b_0 = \max_{\|\xi\|=1} \lambda_1(\mathcal{R}(\xi))$ *is also the* 4^{th} *power of the frame analysis operator norm* $T : (\mathbb{C}^n, \|\cdot\|_2) \to (\mathbb{R}^m, \|\cdot\|_4)$: $b_0 = \|T\|^4_{B(l^2,l^4)} = \max_{\|x\|_2=1} \sum_{k=1}^m |\langle x, f_k\rangle|^4$;
11. $\tilde{a}(0)$ *is given by* $\tilde{a}(0) = \min_{\|z\|=1} \sum_{k=1}^m |\langle z, f_k\rangle|^4$.

Thank you!

Questions?

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