Phase Retrieval using Lipschitz Continuous Maps

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Problem Formulation

The phase retrieval problem

- Hilbert space $H = \mathbb{C}^n$, $\hat{H} = H / T^1$, frame $\mathcal{F} = \{f_1, \ldots, f_m\} \subset \mathbb{C}^n$ and

  $\alpha : \hat{H} \to \mathbb{R}^m$, \quad \alpha(x) = (|\langle x, f_k \rangle|)_{1 \leq k \leq m}.$

  $\beta : \hat{H} \to \mathbb{R}^m$, \quad \beta(x) = \left(|\langle x, f_k \rangle|^2\right)_{1 \leq k \leq m}.$

  The frame is said \textit{phase retrievable} (or that it gives phase retrieval) if $\alpha$ (or $\beta$) is injective.
Problem Formulation
The phase retrieval problem

- Hilbert space $H = \mathbb{C}^n$, $\hat{H} = H/T^1$, frame $\mathcal{F} = \{f_1, \ldots, f_m\} \subset \mathbb{C}^n$ and
  \[ \alpha : \hat{H} \to \mathbb{R}^m, \quad \alpha(x) = (|\langle x, f_k \rangle|)_{1 \leq k \leq m}. \]
  \[ \beta : \hat{H} \to \mathbb{R}^m, \quad \beta(x) = \left(|\langle x, f_k \rangle|^2\right)_{1 \leq k \leq m}. \]

  The frame is said **phase retrievable** (or that it gives phase retrieval) if $\alpha$ (or $\beta$) is injective.

- The general **phase retrieval problem** a.k.a. **phaseless reconstruction**: Decide when a given frame is phase retrievable, and, if so, find an algorithm to recover $x$ from $y = \alpha(x)$ (or from $y = \beta(x)$) up to a global phase factor.
Our Problems Today: Assume $\mathcal{F}$ is phase retrievable.

1. Are the nonliner maps $\alpha, \beta$ bi-Lipschitz with respect to appropriate metrics?

2. Do they admit left inverses that are globally Lipschitz?

3. What are the Lipschitz constants?
Our Problems Today: Assume $\mathcal{F}$ is phase retrievable.

1. Are the nonlinear maps $\alpha, \beta$ bi-Lipschitz with respect to appropriate metrics?

2. Do they admit left inverses that are globally Lipschitz?

3. What are the Lipschitz constants?

Additionally, we want to understand the structure of Lipschitz bounds (to be defined shortly).
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Let $H = \mathbb{C}^n$. The quotient space $\hat{H} = \mathbb{C}^n / T^1$, with classes induced by $x \sim y$ if there is real $\varphi$ with $x = e^{i\varphi}y$. 

Topologically: $\hat{\mathbb{C}}^n = \{0\} \cup \left((0, \infty) \times \mathbb{CP}^{n-1}\right)$ with $\hat{\mathbb{C}}^n \setminus \{0\} = (0, \infty) \times \mathbb{CP}^{n-1}$ a real analytic manifold of real dimension $2n - 1$. Another embedding is into the space of symmetric matrices $\text{Sym}(\mathbb{C}^n)$. Specifically let $S_{p, q}(H) = \{T \in \text{Sym}(H), T\text{ has at most } p\text{ pos.eigs and } q\text{ neg.eigs}\}$ Then: $\kappa_\beta : \hat{H} \to S^1_0, \hat{x} \mapsto xx^*$, is an embedding.
Metric Space Structures

Topological Structures

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Then:

$$\kappa : \hat{H} \rightarrow S^1, \hat{x} \mapsto xx^*$$

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Problem Formulation

Metric Space Structures

Main Results

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Metric Space Structures

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$$S^{p,q}(H) = \{ T \in \text{Sym}(H) \mid T \text{ has at most } p \text{ pos.eigs. and } q \text{ neg.eigs} \}$$

Then:

$$\kappa_\beta : \hat{H} \to S^{1,0}, \quad \hat{x} \mapsto xx^*$$

is an embedding.
Metric Space Structures
The matrix-norm induced metric structure

Fix $1 \leq p \leq \infty$. The *matrix-norm induced distance*

$$d_p : \hat{H} \times \hat{H} \to \mathbb{R}, \quad d_p(\hat{x}, \hat{y}) = \|xx^* - yy^*\|_p$$

with the $p$-norm of the singular values. In the case $p = 2$ we obtain

$$d_2(x, y) = \sqrt{\|x\|^4 + \|y\|^4 - 2|\langle x, y \rangle|^2}$$
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$$d_2(x, y) = \sqrt{\|x\|_4^4 + \|y\|_4^4 - 2|\langle x, y \rangle|^2}$$

**Lemma (BZ15)**

1. $(d_p)_{1 \leq p \leq \infty}$ are equivalent metrics and the identity map $i : (\hat{H}, d_p) \to (\hat{H}, d_q)$, $i(x) = x$ has Lipschitz constant

$$Lip^d_{p,q,n} = \max(1, 2^{\frac{1}{q} - \frac{1}{p}}).$$

2. The metric space $(\hat{H}, d_p)$ is isometrically isomorphic to $S^{1,0}$ endowed with the $p$-norm via $\kappa_{\beta} : \hat{H} \to S^{1,0}$, $x \mapsto \kappa_{\beta}(x) = xx^*$. 

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Metric Space Structures

The natural metric structure

Fix $1 \leq p \leq \infty$. The *natural metric*

$$D_p : \hat{H} \times \hat{H} \to \mathbb{R}, \quad D_p(\hat{x}, \hat{y}) = \min_{\varphi} \|x - e^{i\varphi} y\|_p$$

with the usual $p$-norm on $\mathbb{C}^n$. In the case $p = 2$ we obtain

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Fix $1 \leq p \leq \infty$. The \textit{natural metric}

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with the usual $p$-norm on $\mathbb{C}^n$. In the case $p = 2$ we obtain

$$D_2(\hat{x}, \hat{y}) = \sqrt{\|x\|^2 + \|y\|^2 - 2|\langle x, y \rangle|}$$

\textbf{Lemma (BZ15)}

\begin{enumerate}
  \item $(D_p)_{1 \leq p \leq \infty}$ are equivalent metrics and the identity map
        \begin{align*}
          i : (\hat{H}, D_p) &\to (\hat{H}, D_q), \quad i(x) = x \text{ has Lipschitz constant } \\
          Lip^D_{p,q,n} &\ = \max(1, n^q \frac{1}{p}) .
        \end{align*}
  \item The metric space $(\hat{H}, D_2)$ is Lipschitz isomorphic to $S^{1,0}$ endowed
        with the 2-norm via $\kappa_{\alpha} : \hat{H} \to S^{1,0}$, \quad $x \mapsto \kappa_{\alpha}(x) = \frac{1}{\|x\|} xx^*$.
\end{enumerate}
Two different structures: topologically equivalent, BUT the metrics are NOT equivalent:

**Lemma (BZ15)**

The identity map $i : (\hat{H}, D_p) \rightarrow (\hat{H}, d_p)$, $i(x) = x$ is continuous but it is not Lipschitz continuous. Likewise, the identity map $i : (\hat{H}, d_p) \rightarrow (\hat{H}, D_p)$, $i(x) = x$ is continuous but it is not Lipschitz continuous. Hence the induced topologies on $(\hat{H}, D_p)$ and $(\hat{H}, d_p)$ are the same, but the corresponding metrics are not Lipschitz equivalent.
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Main Results

Lipschitz inversion: $\alpha$

**Theorem (BZ15)**

Assume $\mathcal{F}$ is a phase retrievable frame for $H$. Then:

1. The map $\alpha : (\hat{H}, D_2) \rightarrow (\mathbb{R}^m, \| \cdot \|_2)$ is bi-Lipschitz. Let $\sqrt{A_0}, \sqrt{B_0}$ denote its Lipschitz constants: for every $x, y \in \hat{H}$:

   $$A_0 \min_\varphi \| x - e^{i\varphi} y \|_2^2 \leq \sum_{k=1}^{m} \| \langle x, f_k \rangle - \langle y, f_k \rangle \|_2^2 \leq B_0 \min_\varphi \| x - e^{i\varphi} y \|_2^2.$$

2. There is a Lipschitz map $\omega : (\mathbb{R}^m, \| \cdot \|_2) \rightarrow (\hat{H}, D_2)$ so that: (i) $\omega(\alpha(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $\text{Lip}(\omega) \leq \frac{4+3\sqrt{2}}{\sqrt{A_0}} = \frac{8.24}{\sqrt{A_0}}$. 

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Main Results
Lipschitz inversion: $\beta$

Theorem (BZ15)

Assume $F$ is a phase retrievable frame for $H$. Then:

1. The map $\beta : (\hat{H}, d_1) \rightarrow (\mathbb{R}^m, \| \cdot \|_2)$ is bi-Lipschitz. Let $\sqrt{a_0}, \sqrt{b_0}$ denote its Lipschitz constants: for every $x, y \in \hat{H}$:

$$a_0 \| xx^* - yy^* \|_1^2 \leq \sum_{k=1}^{m} \left| \| \langle x, f_k \rangle \|_2^2 - \| \langle y, f_k \rangle \|_2^2 \right|^2 \leq b_0 \| xx^* - yy^* \|_1^2.$$

2. There is a Lipschitz map $\psi : (\mathbb{R}^m, \| \cdot \|_2) \rightarrow (\hat{H}, d_1)$ so that: (i) $\psi(\beta(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $\text{Lip}(\psi) \leq \frac{4+3\sqrt{2}}{\sqrt{a_0}} = \frac{8.24}{\sqrt{a_0}}$. 

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Main Results

Prior Works

Prior literature:
Main Results

Prior Works

Prior literature:

- **2012**: B.: Cramer-Rao lower bound in the real case; Eldar & Mendelson: map $\alpha$ in the real case

\[ \| \alpha(x) - \alpha(y) \| \geq C \| x - y \| \| x + y \|. \]

- **2013**: Bandeira, Cahill, Mixon, Nelson: improved the estimate of $C$. B.: $\beta$ bi-Lipschitz in real and complex case.

- **2014**: B. & Yang: Find the exact Lipschitz constant for $\alpha$ in the real case - the constants $A_0, B_0$; B. & Z.: constructed a Lipschitz left inverse for $\beta$; B.: lower Lipschitz constant $A_0$ connected to CRLB’s for a non-AWGN model.

- **2015**: B. & Z.: Proved $\alpha$ is bi-Lipschitz in the complex case; constructed a Lipschitz left inverse.
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The proofs involve several steps.
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1. **Part 1: Injectivity $\rightarrow$ bi-Lipschitz:** Upper bounds are not too hard; lower bounds: relatively easy for $\beta$ (the "square" map), but very hard for $\alpha$. 
The proofs involve several steps.

1. **Part 1: Injectivity \( \rightarrow \) bi-Lipschitz:** Upper bounds are not too hard; lower bounds: relatively easy for \( \beta \) (the "square" map), but very hard for \( \alpha \).

2. **Part 2: Left inverse construction is done in three steps:**
   1. The left inverse is first extended to \( \mathbb{R}^m \) into \( \text{Sym}(H) \) using Kirszbraun’s theorem;
   2. Then we show that \( S^{1,0}(H) \) is a Lipschitz retract in \( \text{Sym}(H) \);
   3. The proof is concluded by composing the two maps.
Part 1: Bi-Lipschitzianity for $\beta$

Key Remark (B. Bodmann, Casazza, Edidin - 2007): The nonlinear map $\beta$ is the restriction of the linear map

$$\mathbb{A}: \text{Sym}(H) \to \mathbb{R}^m, \quad \mathbb{A}(T) = (\langle T f_k, f_k \rangle)_{1 \leq k \leq m}$$

Specifically: $\beta(x) = \mathbb{A}(xx^*)$. 
Proofs

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Specifically: $\beta(x) = A(xx^*)$.

$$\|\beta(x) - \beta(y)\| = \|A(xx^*) - A(yy^*)\| = \|A(xx^* - yy^*)\|$$

$$= \|xx^* - yy^*\| \|A\left(\frac{xx^* - yy^*}{\|xx^* - yy^*\|}\right)\|$$
Proofs

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$$= \|xx^* - yy^*\| \|A \left( \frac{xx^* - yy^*}{\|xx^* - yy^*\|} \right) \|$$

$$a_0 = \min_{T \in S^{1,1}, \|T\|_1 = 1} \|A(T)\| > 0 \ , \ b_0 = \max_{T \in S^{1,1}, \|T\|_1 = 1} \|A(T)\|$$
Proofs

Part 2: Extension of the inverse for $\beta$

Assume $\beta : (\hat{H}, d_1) \rightarrow (\mathbb{R}^m, \| \cdot \|_2)$ is bi-Lipschitz:

$$a_0 d_1(x, y)^2 \leq \| \beta(x) - \beta(y) \|^2 \leq b_0 d_1(x, y)^2$$
Part 2: Extension of the inverse for $\beta$

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Let $M = \beta(\hat{H}) \subset \mathbb{R}^m$. 

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Proofs
Part 2: Extension of the inverse for $\beta$

First identify $\hat{H}$ with $S^{1,0}(H)$. 

\begin{align*}
\hat{C}^n & \xrightarrow{\hat{\kappa}_\beta} S^{1,0} \\
S^{1,0} & \xrightarrow{\beta} \mathbb{R}^m \\
\text{Sym}(H) & \xrightarrow{\kappa_\beta} \hat{C}^n
\end{align*}

\[ M = \beta(\mathbb{C}^n) \]
Then construct the local left inverse $\psi_1 : M \to \hat{H}$ with $\text{Lip}(\psi_1) = \frac{1}{\sqrt{a_0}}$. 

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Proofs

Part 2: Extension of the inverse for $\beta$

Use Kirsbraun’s theorem to extend isometrically $\psi_2 : \mathbb{R}^m \rightarrow \text{Sym}(H)$. 

\[ M = \beta(\mathbb{C}^n) \]
Construct a Lipschitz "projection" $\pi : \text{Sym}(H) \to S^{1,0}(H)$.
Proofs
Part 2: Extension of the inverse for $\beta$

Compose the two maps to get $\psi : \mathbb{R}^m \rightarrow S^{1,0}$, $\psi = \pi \circ \psi_2$. 
Proofs

Part 2: $S^{1,0}(H)$ as Lipschitz retract in $\text{Sym}(H)$

How to obtain $\pi : \text{Sym}(H) \rightarrow S^{1,0}(H)$?
Lemma

Consider the spectral decomposition of the self-adjoint operator $A$ in $\text{Sym}(H)$, $A = \sum_{k=1}^{d} \lambda_{m(k)} P_k$. Then the map

$$\pi : \text{Sym}(H) \to S^{1,0}(H), \quad \pi(A) = (\lambda_1 - \lambda_2)P_1$$

satisfies the following two properties:

1. for $1 \leq p \leq \infty$, it is Lipschitz continuous from $(\text{Sym}(H), \| \cdot \|_p)$ to $(S^{1,0}(H), \| \cdot \|_p)$ with Lipschitz constant less than or equal to $3 + 2^{1+\frac{1}{p}}$;

2. $\pi(A) = A$ for all $A \in S^{1,0}(H)$. 

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Part 2: $S^{1,0}(H)$ as Lipschitz retract in $\text{Sym}(H)$

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2. $\pi(A) = A$ for all $A \in S^{1,0}(H)$.

The analysis requires a deeper understanding of local behavior.
Proofs

Part 1: Bi-Lipschitzianity of $\alpha$

The analysis requires a deeper understanding of local behavior.

1. The *global lower and upper Lipschitz bounds*:

$$A_0 = \inf_{x,y \in \hat{H}} \frac{\|\alpha(x) - \alpha(y)\|_2^2}{D_2(x, y)^2}, \quad B_0 = \sup_{x,y \in \hat{H}} \frac{\|\alpha(x) - \alpha(y)\|_2^2}{D_2(x, y)^2}$$
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2. The *type I local lower and upper Lipschitz bounds* at $z \in \hat{H}$:

$$A(z) = \lim_{r \to 0} \inf_{x, y \in \hat{H}} \frac{\|\alpha(x) - \alpha(y)\|^2}{D_2(x, y)^2} \quad \text{s.t.} \quad D_2(x, z) < r, \quad D_2(y, z) < r$$

$$B(z) = \lim_{r \to 0} \sup_{x, y \in \hat{H}} \frac{\|\alpha(x) - \alpha(y)\|^2}{D_2(x, y)^2} \quad \text{s.t.} \quad D_2(x, z) < r, \quad D_2(y, z) < r$$
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2. The type I local lower and upper Lipschitz bounds at $z \in \hat{H}$:

$$A(z) = \lim_{r \to 0} \inf_{x, y \in \hat{H} \atop D_2(x, z) < r, D_2(y, z) < r} \frac{\|\alpha(x) - \alpha(y)\|_2^2}{D_2(x, y)^2}, \quad B(z) = \lim_{r \to 0} \sup_{x, y \in \hat{H} \atop D_2(x, z) < r, D_2(y, z) < r} \frac{\|\alpha(x) - \alpha(y)\|_2^2}{D_2(x, y)^2}$$

3. The type II local lower and upper Lipschitz bounds at $z \in \hat{H}$:

$$\tilde{A}(z) = \lim_{r \to 0} \inf_{x \in \hat{H} \atop D_2(x, z) < r} \frac{\|\alpha(x) - \alpha(z)\|_2^2}{D_2(x, z)^2}, \quad \tilde{B}(z) = \lim_{r \to 0} \sup_{x \in \hat{H} \atop D_2(x, z) < r} \frac{\|\alpha(x) - \alpha(z)\|_2^2}{D_2(x, y)^2}$$
Proofs

Part 1: Bi-Lipschitzianity of $\alpha$

We need to analyze the real structure of $\hat{H}$. 
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Let $\varphi_1, \cdots, \varphi_m, \zeta \in \mathbb{R}^{2n}$, $\Phi_1, \cdots, \Phi_m \in \text{Sym}(\mathbb{R}^{2n})$, $J \in \mathbb{R}^{2n \times 2n}$ defined by:

$$\Phi_k = \varphi_k \varphi_k^T + J \varphi_k \varphi_k^T J^T, \varphi_k = \begin{bmatrix} \text{real}(f_k) \\ \text{imag}(f_k) \end{bmatrix}, J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \zeta = \begin{bmatrix} \text{real}(z) \\ \text{imag}(z) \end{bmatrix}$$

Key relations: $\langle z, f_k \rangle = \langle \zeta, \varphi_k \rangle + i \langle \zeta, J \varphi_k \rangle$, $|\langle z, f_k \rangle| = \sqrt{\langle \Phi_k \zeta, \zeta \rangle}$.
Proofs

Part 1: Bi-Lipschitzianity of $\alpha$

We need to analyze the real structure of $\hat{H}$.

Let $\varphi_1, \ldots, \varphi_m, \zeta \in \mathbb{R}^{2n}$, $\Phi_1, \ldots, \Phi_m \in \text{Sym}(\mathbb{R}^{2n})$, $J \in \mathbb{R}^{2n \times 2n}$ defined by:

$$\Phi_k = \varphi_k \varphi_k^T + J \varphi_k \varphi_k^T J^T, \varphi_k = \begin{bmatrix} \text{real}(f_k) \\ \text{imag}(f_k) \end{bmatrix}, J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \zeta = \begin{bmatrix} \text{real}(z) \\ \text{imag}(z) \end{bmatrix}.$$ 

Key relations: $\langle z, f_k \rangle = \langle \zeta, \varphi_k \rangle + i \langle \zeta, J \varphi_k \rangle$, $|\langle z, f_k \rangle| = \sqrt{\langle \Phi_k \zeta, \zeta \rangle}$.

Consider the following objects:

$$\mathcal{R} : \mathbb{R}^{2n} \rightarrow \text{Sym}(\mathbb{R}^{2n}), \quad \mathcal{R}(\xi) = \sum_{k=1}^{m} \Phi_k \xi \xi^T \Phi_k, \xi \in \mathbb{R}^{2n}$$

$$\mathcal{S} : \mathbb{R}^{2n} \rightarrow \text{Sym}(\mathbb{R}^{2n}), \quad \mathcal{S}(\xi) = \sum_{k: \Phi_k \xi \neq 0} \frac{1}{\langle \Phi_k \xi, \xi \rangle} \Phi_k \xi \xi^T \Phi_k, \xi \in \mathbb{R}^{2n}$$
Theorem (BZ15)

Assume $\mathcal{F}$ is phase retrievable for $H = \mathbb{C}^n$ and $A, B$ are its optimal frame bounds. Then:

1. For every $0 \neq z \in \mathbb{C}^n$, $A(z) = \lambda_{2n-1} (S(\zeta))$ (the next to the smallest eigenvalue);
2. $A_0 = A(0) > 0$;
3. For every $z \in \mathbb{C}^n$, $\tilde{A}(z) = \lambda_{2n-1} \left( S(\zeta) + \sum_{k: \langle z, f_k \rangle = 0} \Phi_k \right)$ (the next to the smallest eigenvalue);
4. $\tilde{A}(0) = A$, the optimal lower frame bound;
5. For every $z \in \mathbb{C}^n$, $B(z) = \tilde{B}(z) = \lambda_1 \left( S(\zeta) + \sum_{k: \langle z, f_k \rangle = 0} \Phi_k \right)$ (the largest eigenvalue);
6. $B_0 = B(0) = \tilde{B}(0) = B$, the optimal upper frame bound;
Proofs

Lipschitz bounds for $\beta$

**Theorem (cont’d)**

7. For every $0 \neq z \in \mathbb{C}^n$, $a(z) = \tilde{a}(z) = \lambda_{2n-1}(R(\zeta))/\|z\|^2$ (the next to the smallest eigenvalue);

8. For every $0 \neq z \in \mathbb{C}^n$, $b(z) = \tilde{b}(z) = \lambda_1(R(\zeta))/\|z\|^2$ (the largest eigenvalue);

9. $a_0 = \min_{\|\xi\|=1} \lambda_{2n-1}(R(\xi))$ is also the largest constant to that $R(\xi) \succeq a_0(\|\xi\|^2 I - J\xi\xi^T J^T)$;

10. $b(0) = \tilde{b}(0) = b_0 = \max_{\|\xi\|=1} \lambda_1(R(\xi))$ is also the $4^{th}$ power of the frame analysis operator norm $T : (\mathbb{C}^n, \| \cdot \|_2) \to (\mathbb{R}^m, \| \cdot \|_4)$: $b_0 = \| T \|^4_{B(l^2,l^4)} = \max_{\|x\|_2=1} \sum_{k=1}^m \left| \langle x, f_k \rangle \right|^4$;

11. $\tilde{a}(0)$ is given by $\tilde{a}(0) = \min_{\|z\|=1} \sum_{k=1}^m \left| \langle z, f_k \rangle \right|^4$. 

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Thank you!

Questions?
References


