The Cramér-Rao Lower Bound in the Phase Retrieval Problem

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Notations and Assumptions
Phase Retrievability and Identifiability

- Hilbert space $H = \mathbb{C}^n$, $\hat{H} = H/T^1$, frame $\mathcal{F} = \{f_1, \cdots, f_m\} \subset \mathbb{C}^n$ and
$$\alpha : \hat{H} \to \mathbb{R}^m \quad , \quad \alpha(x) = (|\langle x, f_k \rangle|)_{1 \leq k \leq m} .$$
- We assume the frame is *phase retrievable*, i.e., α is injective. Hence $(|\langle x, f_k \rangle|)_{1 \leq k \leq m}$ determine uniquely x up to a global phase factor.

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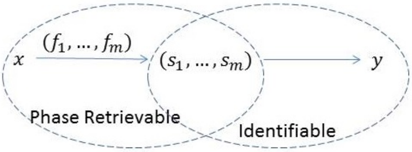
- We assume the frame is *phase retrievable*, i.e., α is injective. Hence $(|\langle x, f_k \rangle|)_{1 \leq k \leq m}$ determine uniquely x up to a global phase factor.
- Measurement process: $y = (y_k)_{1 \leq k \leq m}$. We assume the distribution of y, $p(y; x)$ depends on $\alpha(x)$ only. For instance:
$$y_k = |\langle x, f_k \rangle + \mu_k|^a + \nu_k \ , \ \mu_k \sim \mathbb{CN}(0, \rho^2) \ , \ \nu_k \sim \mathbb{N}(0, \sigma^2)$$
Specifically: $p(y; x) = F(s_1, \cdots, s_m, y)$, where $s_k = |\langle x, f_k \rangle|$.
- We assume *identifiability and regularity*: (1) If $\forall y \in \mathbb{R}^m$, $F(s^{[1]}, y) = F(s^{[2]}, y)$ then $s^{[1]} = s^{[2]}$; and, (2) The Fisher Infomatrix $\mathbb{E}[\frac{\partial log(F)}{\partial s_k} \frac{\partial log(F)}{\partial s_j}]$ is continuous and has constant rank on an open neighborhood of the operating point [Rthbrg71].

Problem Statement

FIM vs. CRLB



Assumptions:

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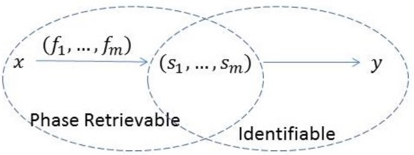

In previous works we derived various Fisher Information Matrix expressions. We have also derived a Cramér-Rao Lower Bound (CRLB) for a specific estimation model. In this paper we analyze a second identification problem and compare the two CRLBs:

Problem

The problem is not how to compute the Fisher Information Matrix (FIM). The problem is how to use FIM, to derive Cramér-Rao Lower Bounds.

Fisher Info Matrix for the AWGN Model

- For the AWGN model:

$$y_k = |\langle x, f_k \rangle|^2 + \nu_k \quad , \quad 1 \leq k \leq m$$

with $\nu_k \sim \mathbb{C}\mathcal{N}(0, \sigma^2)$ i.i.d. the Fisher Information Matrix:

$$\mathbb{I} = \mathbb{E}\left[(\nabla_x \, log \, p(y; x))(\nabla_x \, log \, p(y; x))^*\right]$$

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- $\mathbb{I}^{AWGN,real}(x) = \frac{4}{\sigma^2} \sum_{k=1}^m |\langle x, f_k \rangle|^2 f_k f_k^T = \frac{4}{\sigma^2} \sum_{k=1}^m (f_k f_k^T) x x^T (f_k f_k^T)$

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- $\mathbb{I}^{AWGN,cplx}(x) = \frac{4}{\sigma^2}\sum_{k=1}^m \Phi_k \xi \xi^* \Phi_k$ [Bal13,BCMN13] with $\Phi_k \in \mathbb{R}^{2n \times 2n}$ and $\xi \in \mathbb{R}^{2n}$.

FIM for Non-AWGN

- Consider the Non-AWGN model:

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- The likelihood function:

$$p(y;x) = \frac{1}{\rho^{2m}} exp\left\{-\frac{1}{\rho^2}\left(\sum_{k=1}^{m} y_k + \sum_{k=1}^{m} |\langle x, f_k\rangle|^2\right)\right\} \prod_{k=1}^{m} I_0\left(\frac{2|\langle x, f_k\rangle|\sqrt{y_k}}{\rho^2}\right)$$

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- Realification: $x \mapsto \xi = [real(x) \; imag(x)]^T$ and $|\langle x, f_k \rangle| = \sqrt{\langle \Phi_k \xi, \xi \rangle}$ where Φ_k is a rank-2 replacing $f_k f_k^*$.

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FIM for Non-AWGN

Theorem (Bal15)

The Fisher Information Matrix for the Non-AWGN model is given by

$$\mathbb{I}(\xi) = \frac{4}{\rho^4}\sum_{k=1}^{m}\left(G_1\left(\frac{\langle\Phi_k\xi,\xi\rangle}{\rho^2}\right)-1\right)\Phi_k\xi\xi^*\Phi_k$$

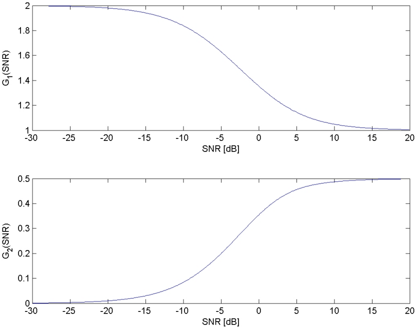
$$= \frac{4}{\rho^2}\sum_{k=1}^{m}G_2\left(\frac{\langle\Phi_k\xi,\xi\rangle}{\rho^2}\right)\frac{1}{\langle\Phi_k\xi,\xi\rangle}\Phi_k\xi\xi^*\Phi_k$$

where

$$G_1(a) = \frac{e^{-a}}{8a^3}\int_0^\infty \frac{I_1^2(t)}{I_0(t)}t^3 e^{-\frac{t^2}{4a}}dt \quad , \quad G_2(a) = a(G_1(a)-1)$$

FIM for Non-AWGN

Asymptotic Regimes



FIM for Non-AWGN

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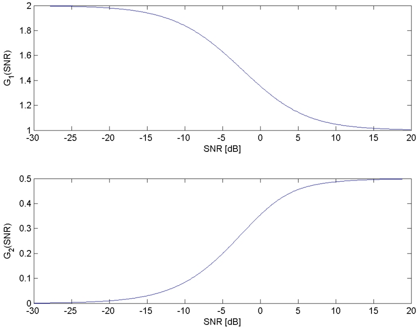

Form 1: Low SNR

$$\mathbb{I}(\xi) = \frac{4}{\rho^4}\sum_{k=1}^m \left(G_1\left(\frac{\langle \Phi_k\xi,\xi\rangle}{\rho^2}\right) - 1\right)\Phi_k\xi\xi^*\Phi_k \approx \frac{4}{\rho^4}\sum_{k=1}^m \Phi_k\xi\xi^*\Phi_k$$

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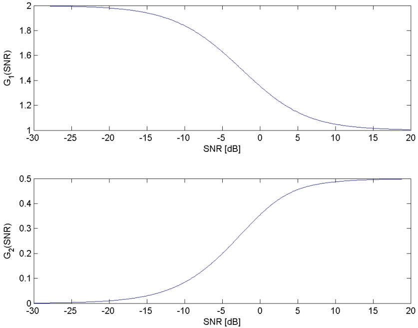

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Form 2: High SNR

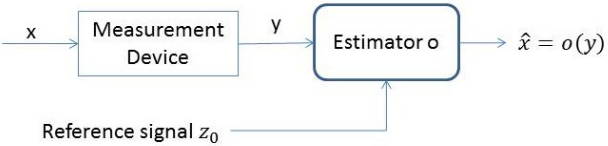
$$\mathbb{I}(\xi) = \frac{4}{\rho^2}\sum_{k=1}^{m} G_2\left(\frac{\langle \Phi_k\xi,\xi\rangle}{\rho^2}\right)\frac{1}{\langle \Phi_k\xi,\xi\rangle}\Phi_k\xi\xi^*\Phi_k \approx \frac{2}{\rho^2}\sum_{k=1}^{m}\frac{1}{\langle \Phi_k\xi,\xi\rangle}\Phi_k\xi\xi^*\Phi_k$$

Setup 1: Reference signal based estimation

In the first setup we fix a reference unit-norm signal $z_0 \in \mathbb{C}^n$. The unknown (to-be-estimated) signal x is assumed to come from set:

$$V_{z_0} = \{x \in \mathbb{C}^n \; : \; imag(\langle x, z_0 \rangle) = 0 \; , \; real(\langle x, z_0 \rangle) > 0 \; \}.$$

The estimator has access to the reference signal z_0:



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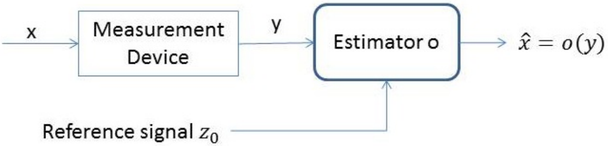


Let $\mathcal{V}_{\zeta_0} = \{\xi \in \mathbb{R}^{2n} \ , \ \langle \xi, \zeta_0 \rangle) \geq 0, \langle \xi, J\zeta_0 \rangle) = 0\}.$, $\mathcal{E}_{\zeta_0} = span_{\mathbb{R}}(\mathbb{V}_{\zeta_0})$ with $\zeta_0 = [real(z_0) \ imag(z_0)]^T$. The estimator $o : \mathbb{R}^m \to \mathcal{E}_{\zeta_0}$ is *unbiased* if $\mathbb{E}[o(y); \xi] = \xi$ for every $x \in V_{z_0}$, with $\xi = [real(x); imag(x)]$.

Setup 1: Positive correlation with a reference signal
The CRL Bound

Let $\Pi_\eta = 1 - \frac{1}{\|\eta\|^2} J\eta\eta^T J^T$ and $L = I - \frac{1}{\langle \xi, \zeta_0 \rangle} J\zeta_0 \xi^T J^T$, with J the symplectic form matrix $[0, -I; I, 0]$.

Theorem

Assume the measurement model $y = (y_k)_{1 \le k \le m}$ where the likelihood function $p(y; x) = F(|\langle x, f_1 \rangle|, \cdots, |\langle x, f_m \rangle|, y)$ is identifiable and regular. Then the covariance of any unbiased estimtor $\omega : \mathbb{R}^m \to \mathcal{E}_{\zeta_0}$ is bounded below by

$$Cov[\omega(y); \xi] \ge (\Pi_{z_0} \mathbb{I}(\xi) \Pi_{z_0})^\dagger = L^T (\mathbb{I}(\xi))^\dagger L.$$

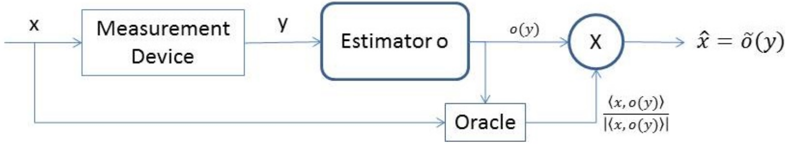
In particular: $\mathbb{E}[\|\omega(y) - \xi\|^2; \xi] \ge trace\left\{(\Pi_{z_0} \mathbb{I}(\xi) \Pi_{z_0})^\dagger\right\} = trace(\mathbb{I}(\xi))^\dagger + \frac{\|\xi\|^2}{|\langle \xi, \zeta_0 \rangle|^2} \langle (\mathbb{I}(\xi))^\dagger J\zeta_0, J\zeta_0 \rangle.$

Remark: First inequality was derived in 2015 paper; the second equality is new.

Setup 2: Oracle-based signal estimation

Consider now a different setup, where $x \in \mathbb{C}^n$ is unconstrained and the estimation is performed in two stages: (i) the first stage returns a "class" estimate through $o : \mathbb{R}^m \to \mathbb{C}^n$; (ii) in the second stage, an oracle provides the optimal global phase $\frac{\langle x,o(y)\rangle}{|\langle x,o(y)\rangle|}$. Thus, the overall estimator:

$$\tilde{o} : \mathbb{R}^m \to \mathbb{C}^n \ , \ \tilde{o}(y) = o(y)\frac{\langle x, o(y)\rangle}{|\langle x, o(y)\rangle|}.$$



The estimator is *unbiased* if $\mathbb{E}[\tilde{o}(y); x] = x$ for every $x \in \mathbb{C}^n$.

Setup 2: Oracle-based signal estimation
The CRL Bound

Theorem

Assume the measurement model $y = (y_k)_{1\leq k\leq m}$ *where the likelihood function* $p(y;x) = F(|\langle x, f_1\rangle|, \cdots, |\langle x, f_m\rangle|, y)$ *is identifiable and regular. Let* $\tilde{o} : \mathbb{R}^m \to \mathbb{C}^n$ *be an unbiased estimator in Setup 2 (Oracle-based estimator). Denote by* $\omega(y) = [real(o(y)); imag(o(y))]$ *and* $om\tilde{e}ga(y) = [real(\tilde{o}(y)); imag(\tilde{o}(y))]$. *Then for any* $\xi = [real(x); imag(x)] \neq 0$,

$$Cov[\tilde{\omega}(y);\xi] \geq (I-\Delta)(\mathbb{I}(\xi))^{\dagger}(I-\Delta)$$

where $\Delta =$

$$\mathbb{E}\left[\frac{(\langle\omega, J\xi\rangle)^2}{((\langle\omega,\xi\rangle)^2+(\langle\omega,J\xi\rangle)^2)^{3/2}}\omega\omega^T + \frac{\langle\omega,\xi\rangle\langle\omega,J\xi\rangle}{((\langle\omega,\xi\rangle)^2+(\langle\omega,J\xi\rangle)^2)^{3/2}}(J\omega\omega^T+\omega\omega^TJ^T) + \frac{(\langle\omega,\xi\rangle)^2}{((\langle\omega,\xi\rangle)^2+(\langle\omega,J\xi\rangle)^2)^{3/2}}J\omega\omega^TJ^T\right]$$

and satisfies $\Delta = \Delta^T \geq I - \Pi_\xi \geq 0$, $\Delta J\xi = J\xi$ *and* $\Delta\xi = 0$.

Conclusions and Open Questions

We obtained Cramér-Rao (type) Lower Bounds for two setups:

1. Positive Correlation with a reference signal: CRLB has a simple form.
2. Oracle-based global phase: CRLB seems very complicated, and estimator dependent. (Remark: Estimator dependency is known for other classes of estimators)

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Open Question: Which of the two CRL bounds is smaller?

Intuitively, Oracle-based estimator seems to have more information than the reference signal based estimator. But is this true/quantifiable?
Easy case: $CRLB_{Setup\ 1} \to \infty$ as $\xi \perp \zeta_0$.

Thank you! Merci!

Questions?

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