The Iterative and Regularized Least Squares Algorithm for Phase Retrieval

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Phase Retrieval

The phase retrieval problem

Hilbert space $H = \mathbb{C}^n$, $\hat{H} = H/T^1$, frame $\mathcal{F} = \{f_1, \cdots, f_m\} \subset \mathbb{C}^n$ and measurements

$$y_k = |\langle x, f_k \rangle|^2 + \nu_k \quad , \quad 1 \leq k \leq m.$$

The frame is said *phase retrievable* (or that it gives phase retrieval) if $\hat{x} \mapsto (|\langle x, f_k \rangle|)_{1 \leq k \leq m}$ is injective.

The general *phase retrieval problem* a.k.a. *phaseless reconstruction*: Decide when a given frame is phase retrievable, and, if so, find an algorithm to recover x from $y = (y_k)_k$ up to a global phase factor.

Our problem today: A reconstruction algorithm.

General Purpose Algorithms

Unstructured Frames. Unstructured Data

1. Iterative Algorithms:
   - Gerchberg-Saxton [Gerchberg&all]
   - Wirtinger flow - gradient descent [CLS14]
   - IRLS [B13]
2. Rank 1 Tensor Recovery:
   - PhaseLift; PhaseCut [Candes&all];[Waldspurger&all]
   - Higher-Order Tensor Recovery [B.]

Specialized Algorithms

Structured Frames and/or Structured Data

1. Structured Frames:
   - Fourier Frames: 4n-4 [BH13]; Masking DFT [CLS13]; STFT/Spectograms [B.][Eldar&all][Hayes&all]; Alternating Projections [GriffinLim][Fannjiang]
   - Polarization: 3-term [ABFM12], masking [BCM]
   - Shift-Invariant Spaces: Bandlimited [Thakur]; Filterbanks/Circulant Matrices [IVW2]; Other spaces [Chen&all]
   - X-Ray Crystallography – over 100 years old, lots of Nobel prizes ...
2. Special Signals:
   - Sparse general case: GESPAR[SBE14];
   - Specialized: sparse [IVW1]; speech [ARF03]

... and others – "phase retrieval" in title: 2680 papers

The IRLS Algorithm

First Motivation: Graduation Method. Homotopic Continuation

The IRLS algorithm belongs to the class of *Graduation Methods* , or *Homotopic Continuations*.

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Our target is to optimize a complicated (possibly non-convex) optimization criterion $J(x)$, $argmin_{x \in D} J(x)$.

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However we know how to optimize a closely related criterion $J_0(x)$, $argmin_{x \in D_0} J_0(x)$.

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Then we introduce a monotonic sequence $0 \leq t_n \leq 1$ with $t_0 = 1$ and $t_n \to 0$ and solve iteratively

$$x^{n+1} = argmin_{x \in D_n} F(t_n, J(x), J_0(x))$$

using x^n as starting point. Here F is a continue function so that $F(1, J(x), J_0(x)) = J_0(x)$ and $F(0, J(x), J_0(x)) = J(x)$.

The IRLS Algorithm

LARS Algorithm

Least Angle Regression (LARS) [EHJT04] designed to solve LASSO, or variants:

$$argmin_x \|y - Ax\|_2^2 + \lambda \|x\|_1$$

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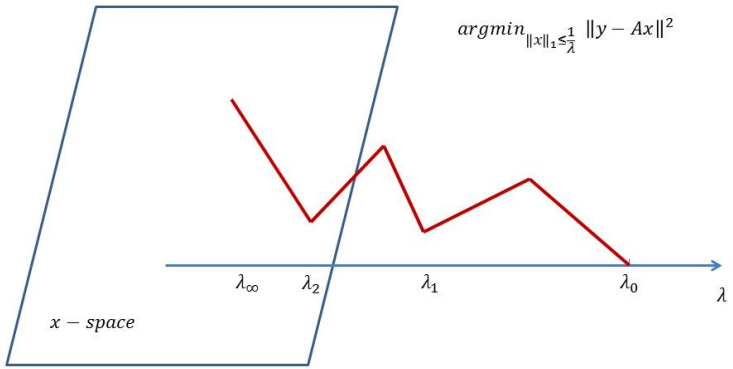
Then LARS finds monotonically decreasing λ values where the slope (and support) of $x(\lambda)$ changes. The algorithm ends at the desired value of $\lambda = \lambda_\infty$ (see also Hierarchical Decompositions of Tadmor&all).



The IRLS Algorithm

IRLS Algorithm

The Iterative Regularized Least-Squares Algorithm attempts to fnd the global minimum of the non-convex problem

$$argmin_x \sum_{k=1}^{m} |y_k - |\langle x, f_k \rangle|^2|^2 + 2\lambda_\infty \|x\|_2^2$$

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using a sequence of iterative least-squares problems:

$$x^{(t+1)} = argmin_x \sum_{k=1}^{m} |y_k - |\langle x, f_k \rangle|^2|^2 + 2\lambda_t \|x\|_2^2 + \mu_t \|x - x^{(t)}\|^2$$

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together with a polarization relaxation:

$$|\langle x, f_k \rangle|^2 \approx \frac{1}{2}(\langle x, f_k \rangle \langle f_k, x^{(t)} \rangle + \langle x^{(t)}, f_k \rangle \langle f_k, x \rangle)$$

The IRLS Algorithm

Main Optimization

The optimization problem:

$$
\begin{aligned}
x^{(t+1)} &= argmin_x \sum_{k=1}^{m} \left| y_k - \frac{1}{2}(\langle x, f_k\rangle\langle f_k, x^{(t)}\rangle + \langle x^{(t)}, f_k\rangle\langle f_k, x\rangle) \right|^2 + \\
&\quad + \lambda_t \|x\|_2^2 + \mu_t \|x - x^{(t)}\|_2^2 + \lambda_t \|x^{(t)}\|_2^2 \\
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Note:

- $J(x, .; ., .)$ is quadratic in $x \Rightarrow$ hence a least-squares problem!
- $J(x, x; \lambda, \mu) = \sum_{k=1}^{m} |y_k - |\langle x, f_k\rangle|^2|^2 + 2\lambda\|x\|_2^2 \Rightarrow$ Fixed points of IRLS are local minima of the original problem.

The IRLS Algorithm

Second Motivation: Relaxation of Constraints

Another motivation: seek $X = xx^*$ that solves

$$\min_{X \geq 0, rank(X)=1} \sum_{k=1}^{m} |y_k - \langle X, f_k f_k^* \rangle_{HS}|^2 + 2\lambda trace(X).$$

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PhaseLift algorithm removes the condition $rank(X) = 1$ and shows (for large λ) this produces the desired result with high probability.

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PhaseLift algorithm removes the condition $rank(X) = 1$ and shows (for large λ) this produces the desired result with high probability.
Another way to relax the problem is to search for X in a larger space. The IRLS is essentially equivalent to optimize a convex functional of X on the larger space

$$\mathcal{S}^{1,1} = \{T = T^* \in \mathbb{C}^{n \times n}, \ T \text{ has at most one positive eigenvalue and at most one negative eigenvalue}\}.$$

The IRLS Algorithm

Second Formulation

Consider the following three convex criteria:

$$
\begin{aligned}
J_1(X;\lambda,\mu) &= \sum_{k=1}^{m} |y_k - \langle X, f_k f_k^* \rangle_{HS}|^2 + 2(\lambda+\mu)\|X\|_1 - 2\mu trace(X) \\
J_2(X;\lambda,\mu) &= \sum_{k=1}^{m} |y_k - \langle X, f_k f_k^* \rangle_{HS}|^2 + 2\lambda eig_{max}(X) - (2\lambda + 4\mu) eig_{min}(X) \\
J_3(X;\lambda,\mu) &= \sum_{k=1}^{m} |y_k - \langle X, f_k f_k^* \rangle_{HS}|^2 + 2\lambda\|X\|_1 - 4\mu eig_{min}(X)
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which coincide on $\mathcal{S}^{1,1}$.

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which coincide on $\mathcal{S}^{1,1}$. Consider the optimization problem

$$
(Jopt, X) = \min_{X \in \mathcal{S}^{1,1}} J_k(X;\lambda,\mu) \quad , \quad 1 \le k \le 3
$$

The IRLS Algorithm

Second Formulation -2

The following are true:

1. Optimization in $\mathcal{S}^{1,1}$:

$$\min_{X \in \mathcal{S}^{1,1}} J_k(X; \lambda, \mu) = \min_{u,v \in \mathbb{C}^n} J(u, v; \lambda, \mu)$$

If $\hat{X}$ and $(\hat{u}, \hat{v})$ denote optimizers so that $imag(\langle \hat{u}, \hat{v} \rangle) = 0$, then $\hat{X} = \frac{1}{2}(\hat{u}\hat{v}^* + \hat{v}\hat{u}^*)$.

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2. Optimization in $\mathcal{S}^{1,0}$:

$$\min_{X \in \mathcal{S}^{1,0}} J_k(X; \lambda, \mu) = \min_{x \in \mathbb{C}^n} J(x, x; \lambda, \mu)$$

   If $\hat{X}$ and $\hat{x}$ denote optimizers, then $\hat{X} = \hat{x}\hat{x}^*$. $\mathcal{S}^{1,0} = \{xx^*\}$.

The IRLS Algorithm

Initialization

For $\lambda \geq eig_{max}(R(y))$, where $R(y) = \sum_{k=1}^{m} y_k f_k f_k^*$, $J(x;\lambda) = \sum_{k=1}^{m} |y_k - |\langle x, f_k \rangle|^2|^2 + 2\lambda \|x\|_2^2$ is convex. The unique global minimum is $x^0 = 0$.

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Initialization Procedure:

- Solve the principal eigenpair (e, eig_{max}) of matrix $R(y)$ using e.g. the power method;
- Set

$$\lambda_0 = (1-\varepsilon)eig_{max} \ , \ x^0 = \sqrt{\frac{(1-\varepsilon)eig_{max}}{\sum_{k=1}^{m} |\langle e, f_k\rangle|^4}}\, e.$$

  Here $\varepsilon > 0$ is a parameter that depends on the frame set as well as the spectral gap of $R(y)$.
- Set $\mu_0 = \lambda_0$ and $t = 0$.

The IRLS Algorithm

Iterations

Repeat the following steps until stopping:

- Optimization: Solve the least-square problem:

$$
\begin{aligned}
x^{(t+1)} &= argmin_x \sum_{k=1}^{m} \left| y_k - \frac{1}{2}(\langle x, f_k\rangle\langle f_k, x^{(t)}\rangle + \langle x^{(t)}, f_k\rangle\langle f_k, x\rangle)\right|^2 + \\
&\quad +\lambda_t \|x\|_2^2 + \mu_t \|x - x^{(t)}\|_2^2 + \lambda_t \|x^{(t)}\|_2^2 \\
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\end{aligned}
$$

- Update: $\lambda_{t+1} = \gamma\lambda_t$, $\mu_{t+1} = \max(\gamma\mu_t, \mu^{\min})$, $t = t + 1$. Here γ is the learning rate, and μ^{min} is related to performance.

The IRLS Algorithm

Performance

Let $y_k = |\langle x, f_k \rangle|^2 + \nu_k$. Assume the algorithm is stopped at some T so that

$$J(x^{(T)}, x^{(T-1)}; \lambda, \mu) \leq J(x, x; \lambda, \mu).$$

Denote $\hat{X} = \frac{1}{2}(x^{(T)} x^{(T-1)*} + x^{(T-1)} x^{(T)*})$ and $\hat{x}\hat{x}^* = P_+(\hat{X})$.

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Denote $\hat{X} = \frac{1}{2}(x^{(T)}x^{(T-1)*} + x^{(T-1)}x^{(T)*})$ and $\hat{x}\hat{x}^* = P_+(\hat{X})$.
Then the following hold true:

1. Matrix norm error:

$$\|\hat{X} - xx^*\|_1 \leq \frac{\lambda}{C_0} + \sqrt{C_0}\|\nu\|$$

2. Natural distance:

$$D(\hat{x}, x)^2 = \|\hat{X} - xx^*\|_1 + |eig_{min}(\hat{X})| \leq \frac{\lambda}{C_0} + \sqrt{C_0}\|\nu\| + \frac{\|\nu\|^2}{4\mu} + \frac{\lambda\|x\|^2}{2\mu}$$

where C_0 is a frame dependent constant (lower Lipschitz constant in $\mathcal{S}^{1,1}$).

Numerical Simulations

Setup

The algorithm requires $O(m)$ memory. Simulations with $m = Rn$ (complex case) with $n = 1000$ and $R \in \{4, 6, 8, 12\}$. Frame vectors corresponding to masked (windowed) DFT:

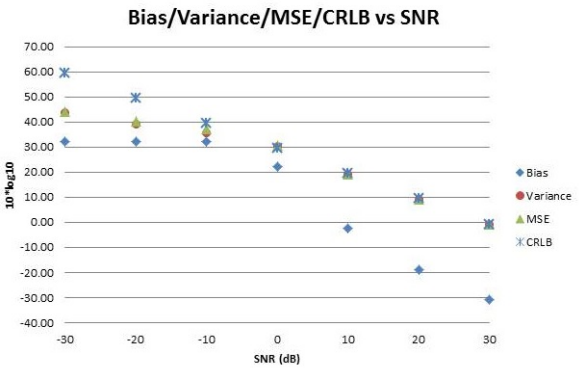
$$f_{jn+k} = \frac{1}{\sqrt{8n}} \left(w_l^j e^{2\pi i k(l-1)/n} \right)_{0 \le l \le n-1}, \quad 1 \le j \le R, 1 \le k \le n$$

$$\begin{bmatrix} f_1 & f_2 & \cdots & f_m \end{bmatrix} = \begin{bmatrix} Diag(w^1) & \cdots & Diag(w^R) \end{bmatrix} \begin{bmatrix} DFT_n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & DFT_n \end{bmatrix}$$

Parameters: $\varepsilon = 0.1$, $\gamma = 0.95$, $\mu^{min} = \frac{\mu^0}{10}$. Power method tolerance: 10^{-8} Conjugate gradient tolerance: 10^{-14}.

Numerical Simulations

MSE Plots



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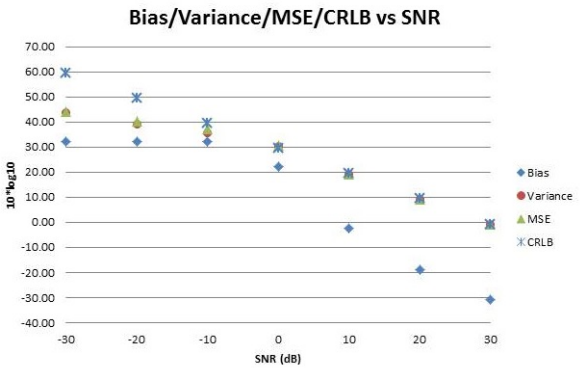

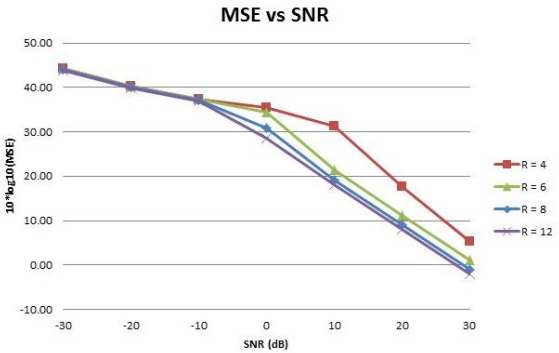

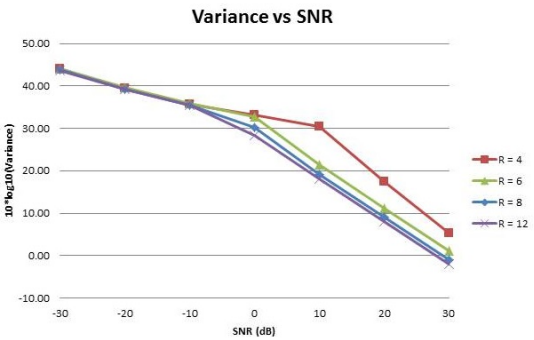

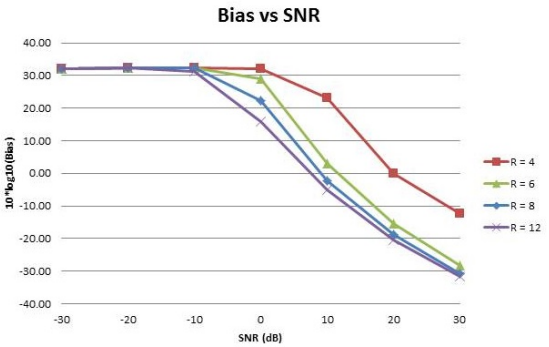

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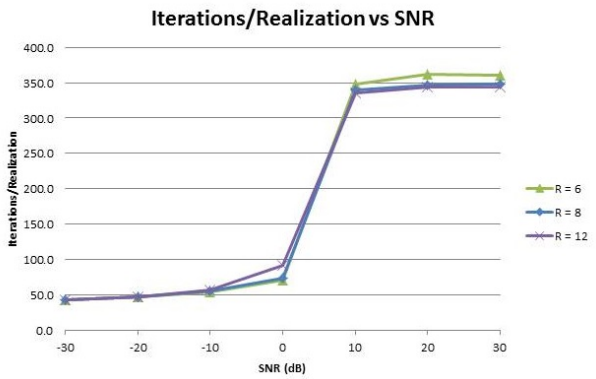

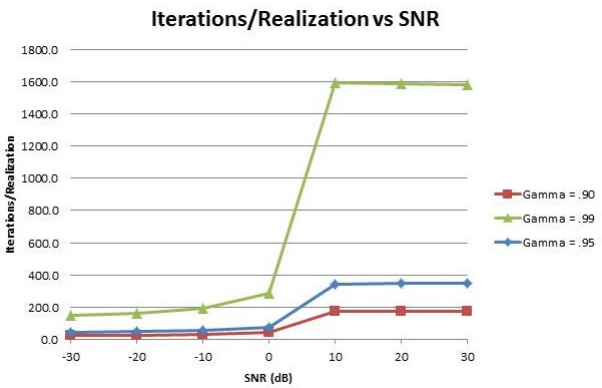

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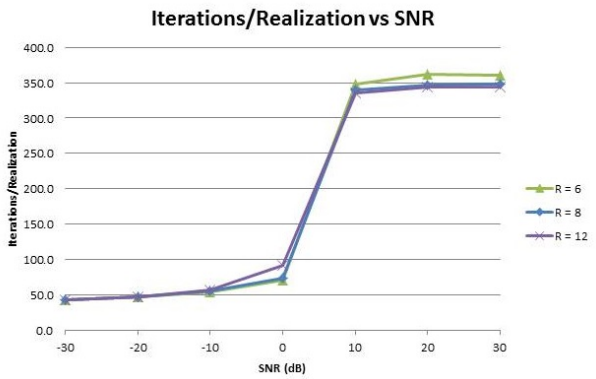
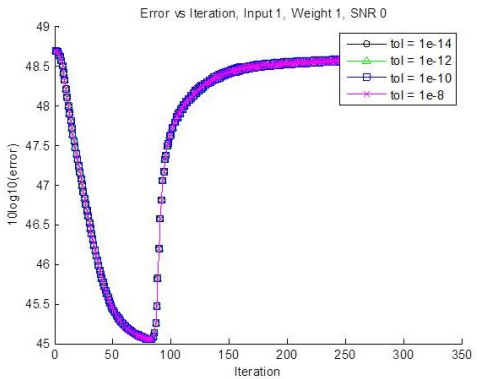
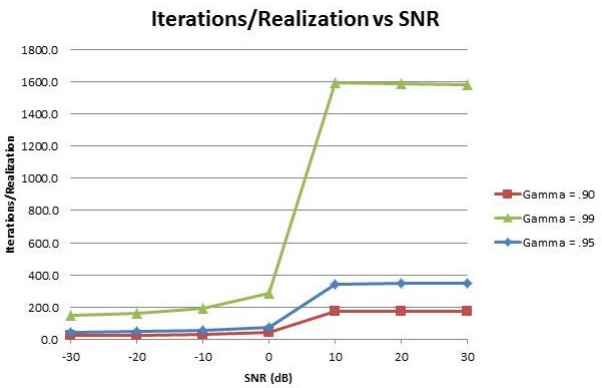
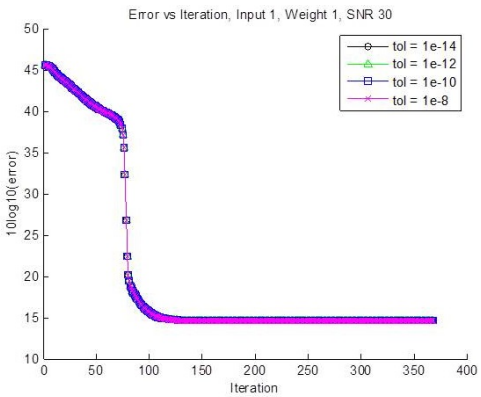


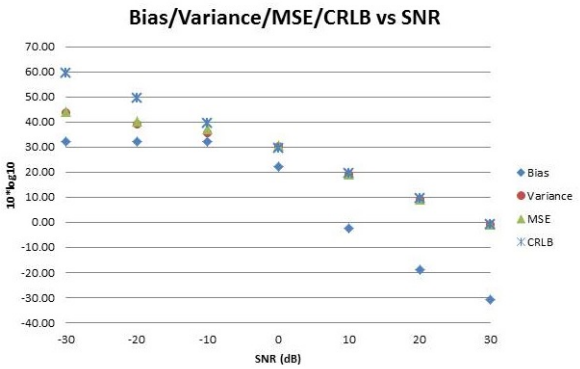






Numerical Simulations

Performance - 2



SNR	$10 \cdot log_{10}(Bias)$	$10 \cdot log_{10}(Variance)$	$10 \cdot log_{10}(MSE)$	CRLB
-30	32.13	43.78	44.07	59.39
-20	32.39	39.29	40.09	49.39
-10	32.27	35.56	37.23	39.39
0	22.17	30.24	30.87	29.39
10	-2.21	19.16	19.19	19.39
20	-18.88	9.05	9.05	9.39
30	-30.63	-0.96	-0.96	-0.61

Numerical Simulations

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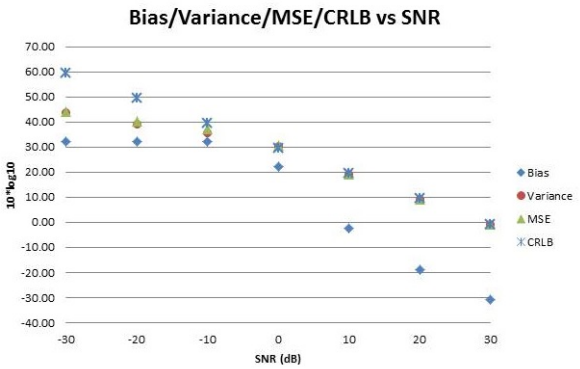

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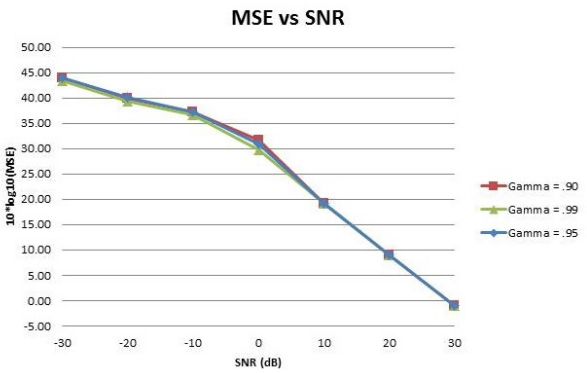
10	-2.21	19.16	19.19	19.39
20	-18.88	9.05	9.05	9.39
30	-30.63	-0.96	-0.96	-0.61



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