The Iterative and Regularized Least Squares Algorithm for Phase Retrieval

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Phase Retrieval
The phase retrieval problem

Hilbert space $H = \mathbb{C}^n$, $\hat{H} = H/T^1$, frame $\mathcal{F} = \{f_1, \cdots, f_m\} \subset \mathbb{C}^n$ and measurements

$$y_k = |\langle x, f_k \rangle|^2 + \nu_k, \quad 1 \leq k \leq m.$$  

The frame is said *phase retrievable* (or that it gives phase retrieval) if $\hat{x} \mapsto (|\langle x, f_k \rangle|)_{1 \leq k \leq m}$ is injective.

The general *phase retrieval problem* a.k.a. *phaseless reconstruction*: Decide when a given frame is phase retrievable, and, if so, find an algorithm to recover $x$ from $y = (y_k)_k$ up to a global phase factor.

Our problem today: A reconstruction algorithm.
General Purpose Algorithms
Unstructured Frames. Unstructured Data

1. Iterative Algorithms:
   - Gerchberg-Saxton [Gerchberg&all]
   - Wirtinger flow - gradient descent [CLS14]
   - IRLS [B13]

2. Rank 1 Tensor Recovery:
   - PhaseLift; PhaseCut [Candes&all]; [Waldspurger&all]
   - Higher-Order Tensor Recovery [B.]
Structuralized Algorithms
Structured Frames and/or Structured Data

1. Structured Frames:
   - Fourier Frames: $4n-4$ [BH13]; Masking DFT [CLS13]; STFT/Spectograms [B.] [Eldar&all][Hayes&all]; Alternating Projections [GriffinLim][Fannjiang]
   - Polarization: 3-term [ABFM12], masking [BCM]
   - Shift-Invariant Spaces: Bandlimited [Thakur]; Filterbanks/Circulant Matrices [IVW2]; Other spaces [Chen&all]
   - X-Ray Crystallography – over 100 years old, lots of Nobel prizes ...

2. Special Signals:
   - Sparse general case: GESPAR[SBE14];
   - Specialized: sparse [IVW1]; speech [ARF03]

... and others – ”phase retrieval” in title: 2680 papers
The IRLS Algorithm
First Motivation: Graduation Method. Homotopic Continuation

The IRLS algorithm belongs to the class of *Graduation Methods*, or *Homotopic Continuations*.

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Our target is to optimize a complicated (possibly non-convex) optimization criterion $J(x)$, $\arg\min_{x \in D} J(x)$. 
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Our target is to optimize a complicated (possibly non-convex) optimization criterion $J(x)$, $\arg\min_{x \in D} J(x)$.
However we know how to optimize a closely related criterion $J_0(x)$, $\arg\min_{x \in D_0} J_0(x)$.
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However we know how to optimize a closely related criterion $J_0(x)$, $\text{argmin}_{x \in D_0} J_0(x)$.

Then we introduce a monotonic sequence $0 \leq t_n \leq 1$ with $t_0 = 1$ and $t_n \to 0$ and solve iteratively

$$x^{n+1} = \text{argmin}_{x \in D_n} F(t_n, J(x), J_0(x))$$

using $x^n$ as starting point. Here $F$ is a continue function so that $F(1, J(x), J_0(x)) = J_0(x)$ and $F(0, J(x), J_0(x)) = J(x)$. 

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Least Angle Regression (LARS) [EHJT04] designed to solve LASSO, or variants:

\[ \text{argmin}_x \| y - Ax \|_2^2 + \lambda \| x \|_1 \]
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$$\text{argmin}_x \|y - Ax\|_2^2 + \lambda \|x\|_1$$

It is proved the optimizer $x_{opt} = x(\lambda)$ is a continuous and piecewise differentiable function of $\lambda$ (linear, in the case of LASSO).
The IRLS Algorithm

LARS Algorithm

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Method: Start with \(\lambda = \lambda_0 = \frac{2}{\|A^Ty\|_2}\) and the optimal solution is \(x^0 = 0\).

Then LARS finds monotonically decreasing \(\lambda\) values where the slope (and support) of \(x(\lambda)\) changes. The algorithm ends at the desired value of \(\lambda = \lambda_\infty\) (see also Hierarchical Decompositions of Tadmor&all).
The IRLS Algorithm

The Iterative Regularized Least-Squares Algorithm attempts to find the global minimum of the non-convex problem

$$\arg\min_x \sum_{k=1}^m |y_k - |x, f_k||^2 + 2\lambda_\infty \|x\|^2$$
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$$\arg\min_x \sum_{k=1}^m |y_k - |\langle x, f_k \rangle|^2|^2 + 2\lambda_\infty \|x\|^2_2$$

using a sequence of iterative least-squares problems:

$$x^{(t+1)} = \arg\min_x \sum_{k=1}^m |y_k - |\langle x, f_k \rangle|^2|^2 + 2\lambda_t \|x\|^2_2 + \mu_t \|x - x^{(t)}\|^2$$
The IRLS Algorithm

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\]

together with a polarization relaxation:

\[
|\langle x, f_k \rangle|^2 \approx \frac{1}{2} (\langle x, f_k \rangle \langle f_k, x^{(t)} \rangle + \langle x^{(t)}, f_k \rangle \langle f_k, x \rangle)
\]
The optimization problem:

\[
\begin{array}{c}
\mathbf{x}^{(t+1)} = \text{argmin}_x \sum_{k=1}^{m} \left| y_k - \frac{1}{2} \left( \langle x, f_k \rangle \langle f_k, x^{(t)} \rangle + \langle x^{(t)}, f_k \rangle \langle f_k, x \rangle \right) \right|^2 + \\
+ \lambda_t \| x \|_2^2 + \mu_t \| x - x^{(t)} \|_2^2 + \lambda_t \| x^{(t)} \|_2^2 \\
= \text{argmin}_x \ J(x, x^{(t)}; \lambda, \mu)
\end{array}
\]
The IRLS Algorithm

Main Optimization

The optimization problem:

\[ x^{(t+1)} = \arg\min_x \sum_{k=1}^{m} \left| y_k - \frac{1}{2}(\langle x, f_k \rangle \langle f_k, x^{(t)} \rangle + \langle x^{(t)}, f_k \rangle \langle f_k, x \rangle) \right|^2 + \\
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\[ = \arg\min_x J(x, x^{(t)}; \lambda, \mu) \]

Note:
The IRLS Algorithm

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= \arg\min_x J(x, x^{(t)}; \lambda, \mu)
\]

Note:

- \( J(x, .; ., .) \) is quadratic in \( x \) \( \Rightarrow \) hence a least-squares problem!
The IRLS Algorithm

Main Optimization

The optimization problem:

$$
x^{(t+1)} = \arg\min_x \sum_{k=1}^m \left| y_k - \frac{1}{2} \left( \langle x, f_k \rangle \langle f_k, x^{(t)} \rangle + \langle x^{(t)}, f_k \rangle \langle f_k, x \rangle \right) \right|^2 +$$

$$+ \lambda_t \| x \|_2^2 + \mu_t \| x - x^{(t)} \|_2^2 + \lambda_t \| x^{(t)} \|_2^2$$

$$= \arg\min_x J(x, x^{(t)}; \lambda, \mu)$$

Note:

- $J(x, .; ., .)$ is quadratic in $x$ $\Rightarrow$ hence a least-squares problem!
- $J(x, x; \lambda, \mu) = \sum_{k=1}^m \left| y_k - \left| \langle x, f_k \rangle \right|^2 \right|^2 + 2\lambda \| x \|_2^2 \Rightarrow$ Fixed points of IRLS are local minima of the original problem.
Another motivation: seek $X = xx^*$ that solves

$$\min_{X \geq 0, \text{rank}(X) = 1} \sum_{k=1}^{m} |y_k - \langle X, f_k f_k^* \rangle_{HS}|^2 + 2\lambda \text{trace}(X).$$
Another motivation: seek $X = xx^*$ that solves

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PhaseLift algorithm removes the condition $\text{rank}(X) = 1$ and shows (for large $\lambda$) this produces the desired result with high probability.
The IRLS Algorithm

Second Motivation: Relaxation of Constraints

Another motivation: seek $X = xx^*$ that solves

$$
\min_{X \geq 0, rank(X) = 1} \sum_{k=1}^{m} \left| y_k - \langle X, f_k f_k^* \rangle_{HS} \right|^2 + 2\lambda \trace(X).
$$

PhaseLift algorithm removes the condition $rank(X) = 1$ and shows (for large $\lambda$) this produces the desired result with high probability. Another way to relax the problem is to search for $X$ in a larger space. The IRLS is essentially equivalent to optimize a convex functional of $X$ on the larger space

$$
S^{1,1} = \{ T = T^* \in \mathbb{C}^{n \times n}, \ T \text{ has at most one positive eigenvalue and at most one negative eigenvalue} \}. 
$$
The IRLS Algorithm
Second Formulation

Consider the following three convex criteria:

\[ J_1(X; \lambda, \mu) = \sum_{k=1}^{m} |y_k - \langle X, f_k f_k^* \rangle_{HS}|^2 + 2(\lambda + \mu)\|X\|_1 - 2\mu \text{trace}(X) \]

\[ J_2(X; \lambda, \mu) = \sum_{k=1}^{m} |y_k - \langle X, f_k f_k^* \rangle_{HS}|^2 + 2\lambda \text{eig}_{\text{max}}(X) - (2\lambda + 4\mu)\text{eig}_{\text{min}}(X) \]

\[ J_3(X; \lambda, \mu) = \sum_{k=1}^{m} |y_k - \langle X, f_k f_k^* \rangle_{HS}|^2 + 2\lambda \|X\|_1 - 4\mu \text{eig}_{\text{min}}(X) \]

which coincide on \( S^{1,1} \).
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which coincide on $S^{1,1}$. Consider the optimization problem

\[ (J_{\text{opt}}, X) = \min_{X \in S^{1,1}} J_k(X; \lambda, \mu), \quad 1 \leq k \leq 3 \]
The IRLS Algorithm
Second Formulation -2

The following are true:

1. **Optimization in $S^{1,1}$:**

   $\min_{X \in S^{1,1}} J_k(X; \lambda, \mu) = \min_{u, v \in \mathbb{C}^n} J(u, v; \lambda, \mu)$

   If $\hat{X}$ and $(\hat{u}, \hat{v})$ denote optimizers so that $\text{imag}(\langle \hat{u}, \hat{v} \rangle) = 0$, then

   $\hat{X} = \frac{1}{2}(\hat{u}\hat{v}^* + \hat{v}\hat{u}^*)$. 
The IRLS Algorithm
Second Formulation -2

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2. Optimization in $S^{1,0}$:

$$\min_{X \in S^{1,0}} J_k(X; \lambda, \mu) = \min_{x \in \mathbb{C}^n} J(x, x; \lambda, \mu)$$

If $\hat{X}$ and $\hat{x}$ denote optimizers, then

$$\hat{X} = \hat{x}\hat{x}^*. \quad S^{1,0} = \{xx^*\}. $$
For $\lambda \geq eig_{max}(R(y))$, where $R(y) = \sum_{k=1}^{m} y_k f_k f_k^*$, 
$J(x; \lambda) = \sum_{k=1}^{m} |y_k - |\langle x, f_k \rangle| 2^2 + 2\lambda\|x\|_2^2$ is convex. The unique global minimum is $x^0 = 0$. 

Initialization Procedure: 
1. Solve the principal eigenpair $(e, eig_{max})$ of matrix $R(y)$ using e.g. the power method; 
2. Set $\lambda_0 = (1 - \epsilon) eig_{max}$, $x_0 = \sqrt{(1 - \epsilon) eig_{max}} \sum_{k=1}^{m} |\langle e, f_k \rangle|^4 e$. 

Here $\epsilon > 0$ is a parameter that depends on the frame set as well as the spectral gap of $R(y)$. 

Set $\mu_0 = \lambda_0$ and $t = 0$. 

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The IRLS Algorithm

Initialization

For $\lambda \geq eig_{max}(R(y))$, where $R(y) = \sum_{k=1}^{m} y_k f_k f_k^*$,

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Here $\varepsilon > 0$ is a parameter that depends on the frame set as well as the spectral gap of $R(y)$.

- Set $\mu_0 = \lambda_0$ and $t = 0$. 
The IRLS Algorithm

Iterations

Repeat the following steps until stopping:

- **Optimization**: Solve the least-square problem:

\[
\begin{align*}
x^{(t+1)} &= \arg\min_x \sum_{k=1}^{m} \left| y_k - \frac{1}{2} \left( \langle x, f_k \rangle \langle f_k, x^{(t)} \rangle + \langle x^{(t)}, f_k \rangle \langle f_k, x \rangle \right) \right|^2 + \\
&\quad + \lambda_t \|x\|_2^2 + \mu_t \|x - x^{(t)}\|_2^2 + \lambda_t \|x^{(t)}\|_2^2 \\
&= \arg\min_x J(x, x^{(t)}; \lambda, \mu)
\end{align*}
\]

- **Update**: \( \lambda_{t+1} = \gamma \lambda_t, \mu_{t+1} = \max(\gamma \mu_t, \mu_{\text{min}}), t = t + 1 \). Here \( \gamma \) is the learning rate, and \( \mu_{\text{min}} \) is related to performance.
The IRLS Algorithm

Performance

Let $y_k = |\langle x, f_k \rangle|^2 + \nu_k$. Assume the algorithm is stopped at some $T$ so that

$$J(x^{(T)}, x^{(T-1)}; \lambda, \mu) \leq J(x, x; \lambda, \mu).$$

Denote $\hat{X} = \frac{1}{2}(x^{(T)}x^{(T-1)*} + x^{(T-1)}x^{(T)*})$ and $\hat{x}\hat{x}^* = P_+(\hat{X})$. 
The IRLS Algorithm

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Then the following hold true:

1. **Matrix norm error:**

$$\|\hat{X} - xx^*\|_1 \leq \frac{\lambda}{C_0} + \sqrt{C_0}\|\nu\|$$

2. **Natural distance:**

$$D(\hat{x}, x)^2 = \|\hat{X} - xx^*\|_1 + |\text{eig}_{\text{min}}(\hat{X})| \leq \frac{\lambda}{C_0} + \sqrt{C_0}\|\nu\| + \frac{\|\nu\|^2}{4\mu} + \frac{\lambda\|x\|^2}{2\mu}$$

where $C_0$ is a frame dependent constant (lower Lipschitz constant in $S^{1,1}$).
The algorithm requires $O(m)$ memory. Simulations with $m = Rn$ (complex case) with $n = 1000$ and $R \in \{4, 6, 8, 12\}$. Frame vectors corresponding to masked (windowed) DFT:

$$f_{jn+k} = \frac{1}{\sqrt{8n}} \left(w_j e^{2\pi ik(l-1)/n}\right)_{0 \leq l \leq n-1}, \quad 1 \leq j \leq R, \quad 1 \leq k \leq n$$

$$\begin{bmatrix}
  f_1 \\
  f_2 \\
  \vdots \\
  f_m \\
\end{bmatrix} = \begin{bmatrix}
  \text{Diag}(w^1) & \cdots & \text{Diag}(w^R)
\end{bmatrix} \begin{bmatrix}
  \text{DFT}_n & 0 & 0 \\
  0 & \ddots & 0 \\
  0 & 0 & \text{DFT}_n
\end{bmatrix}$$

Parameters: $\varepsilon = 0.1$, $\gamma = 0.95$, $\mu^{\text{min}} = \frac{\mu^0}{10}$. Power method tolerance: $10^{-8}$
Conjugate gradient tolerance: $10^{-14}$. 
Numerical Simulations

MSE Plots

![Bias/Variance/MSE/CRLB vs SNR](image)
Numerical Simulations

MSE Plots

Bias/Variance/MSE/CRLB vs SNR

MSE vs SNR

Variance vs SNR

Bias vs SNR

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Numerical Simulations

Performance

```
Iterations/Realization vs SNR

SNR (dB)  | Iterations/Realization
---|---
-90      | 50
-80      | 100
-70      | 150
-60      | 200
-50      | 250
-40      | 300
-30      | 350
0        | 400
10       | 450
20       | 500
30       | 550

SNR (dB)  | Iterations/Realization
---|---
-90      | 10
-80      | 50
-70      | 100
-60      | 150
-50      | 200
-40      | 250
-30      | 300
0        | 350
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20       | 450
30       | 500

Gamma = .90
Gamma = .99
Gamma = .95
```
Numerical Simulations

Performance

- Iterations/Realization vs SNR
- Error vs Iteration, Input 1, Weight 1, SNR 0
- Error vs Iteration, Input 1, Weight 1, SNR 30
Numerical Simulations

Performance - 2

<table>
<thead>
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<th>SNR</th>
<th>$10 \cdot \log_{10}(\text{Bias})$</th>
<th>$10 \cdot \log_{10}(\text{Variance})$</th>
<th>$10 \cdot \log_{10}(\text{MSE})$</th>
<th>CRLB</th>
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Numerical Simulations

Performance - 2

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References


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The Phase Retrieval Problem

Existing Algorithms

The IRLS Algorithm

Numerical Results


Radu Balan, Naveed Haghani (UMD)

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