Lipschitz Extensions in Inverse Problems

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Happy Birthday Akram!
Table of Contents:

1 Framework

2 Metrics on Matrices

3 BiLipschitz Results

4 Proofs
High-Level Problem Formulation

Given: A nonlinear map (analysis) $\alpha : S \rightarrow \mathbb{R}^m$ from a metric space $(S, d)$ to the Euclidean space $(\mathbb{R}^m, \|\cdot\|_2)$.

Wanted: A left inverse $\omega : \mathbb{R}^m \rightarrow S$ that is globally Lipschitz.

Today problems: The case when $S \subset Sym^+(\mathbb{C}^n)$ is a class of psd matrices, or $S \subset \mathbb{R}^n$ is the class of sparse signals.
Quantum Tomography

Setup

A quantum system is characterized by the density matrix $M \in \mathbb{C}^{n \times n}$. Given a set of observables $Y_1, \cdots, Y_m$ that can be measured simultaneously, the problem is to estimate (compute) the density matrix $M = M^* \geq 0$ from noisy measurements:

$$y_k = \text{trace}(MY_k) + \nu_k.$$ 

Constraints: (1) $\text{trace}(M) = 1$ (2) weakly mixed system, i.e. $M$ has low rank, $\text{rank}(M) \leq d$: 

$$S = St^d(\mathbb{C}^n) = \{X = X^* \geq 0 \ , \ \text{trace}(X) = 1 \ , \ \text{rank}(X) \leq d\}.$$
Scene Understanding from Power Measurements

Setup

Mixing model: $d$ decorrelated sources (acoustic, RF, etc) monitored by $n$ sensors. A subset $S$ of all possible ordered pairs $\{(i,j) ; 1 \leq i \leq j \leq n\}$ of sensors determines signal covariance, i.e. the measurements are:

$$y_{\alpha} = \mathbb{E}[x_i x_j] + \nu_{\alpha} = R_{i,j} + \nu_{\alpha}.$$  

for $\alpha = (i,j) \in S$ and $R = \mathbb{E}[xx^*]$ is the $n \times n$ cov. matrix of rank $d$.

The problem is to estimate $R$ from $\{y_{\alpha} , \alpha \in S\}$ ($|S| = m$).

Here: $S = S_{d,0} = \{X = X^* \succeq 0 \ , \ \text{rank}(X) \leq d\}$. 

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Lipschitz
Compressive Sampling Scenario

Setup

Signal Model: \( x: \) \( d \)-sparse \( \mathbb{R}^n \)-vector.

Measurement Model:

\[
y = Ax + \nu \in \mathbb{R}^m.
\]

Here:

\[
S = \mathbb{R}^n_d = \{ x \in \mathbb{R}^n , \| x \|_0 \leq d \}.
\]
Notations

\( H = \mathbb{F}^n \) a finite dimensional Euclidean space, with \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{F} = \mathbb{C} \).

- \( \text{Sym}(H) = \{ T \in H^{n \times n} , \ T = T^* \} \)
- Convex cone of PSD: \( \text{Sym}^+(H) = \{ T \in \text{Sym}(H) , \ T = T^* \geq 0 \} \)
- Quantum states: \( \text{St}(H) = \{ T \in \text{Sym}^+(H) , \ \text{trace}(T) = 1 \} \)
- Low-rank quantum states
  \( \text{St}^r(H) = \{ T \in \text{Sym}^+(H) , \ \text{trace}(T) = 1 , \ \text{rank}(T) \leq r \} \)
- Cone of low-rank mixed signature matrices:
  \( \mathbb{S}^{p,q} = \{ T \in \text{Sym}(H) , \ T \ \text{has at most} \ p \ \text{positive and} \ q \ \text{negative eigenvalues} \} \)
  In particular \( \mathbb{S}^{1,0} = \{ xx^* , \ x \in H \} \), set of rank (at most) one PSDs.
- Cone of sparse signals:
  \( H_d = \mathbb{R}^n_d = \{ x \in H = \mathbb{R}^n , \ \| x \|_0 \leq d \} \).
Problem Formulation
Models

Forward maps:

\[ \alpha : \text{Sym}^{+}(H) \rightarrow \mathbb{R}^{m} \, , \, (\alpha(X))_{k} = \sqrt{\text{trace}(XF_{k})} = \sqrt{\langle X, F_{k} \rangle} \]

\[ \beta : \text{Sym}^{+}(H) \rightarrow \mathbb{R}^{m} \, , \, (\beta(X))_{k} = \text{trace}(XF_{k}) =: \langle X, F_{k} \rangle \]

where \( F_{1}, \cdots, F_{m} \in \text{Sym}^{+}(H) \) are fixed PSD matrices.

\[ \gamma : H_{d} \rightarrow \mathbb{R}^{m} \, , \, \gamma(x) = Ax \]

where \( A \in \mathbb{R}^{m \times n} \) is a "fat" measurement matrix (\( n > m \geq 2d \)).
Problem Formulation
Models

Forward maps:

\[ \alpha : Sym^+(H) \to \mathbb{R}^m, \quad (\alpha(X))_k = \sqrt{\text{trace}(X F_k)} = \sqrt{\langle X, F_k \rangle} \]
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where \( F_1, \cdots, F_m \in Sym^+(H) \) are fixed PSD matrices.

\[ \gamma : H_d \to \mathbb{R}^m, \quad \gamma(x) = Ax \]

where \( A \in \mathbb{R}^{m \times n} \) is a "fat" measurement matrix \( (n > m \geq 2d) \).

Spaces:

- Phase Retrieval: \( S = \mathbb{S}^{1,0} = \{xx^* \mid x \in H\} \) or \( S = \hat{H} = H/T \).
- Quantum Tomography:
  \[ S = St^r(H) = \{X = X^* \geq 0, \text{trace}(X) = 1, \text{rank}(X) \leq r\} \].
- Covariance Matrix Estimation: \( S = \mathbb{S}^{d,0} \).
- Sparse Signal Estimation: \( S = \mathbb{R}^n_d \).
Problem Formulation

The phase retrieval problem

Hilbert space $H = \mathbb{C}^n$, $\hat{H} = H/T^1$, frame $\mathcal{F} = \{f_1, \cdots, f_m\} \subset \mathbb{C}^n$ and

$$\alpha : \hat{H} \to \mathbb{R}^m \ , \ (\alpha(x))_k = |\langle x, f_k \rangle| = \sqrt{\langle xx^*, f_k f_k^* \rangle}.$$ 

$$\beta : \hat{H} \to \mathbb{R}^m \ , \ (\beta(x))_k = |\langle x, f_k \rangle|^2 = \langle xx^*, f_k f_k^* \rangle.$$ 

Assume $\alpha, \beta$ are injective, the problem is to construct global Lipschitz inverses and to study their Lipschitz constants.
Problem Formulation
Lipschitz reconstruction: the general case

Assume the maps $\alpha, \beta, \gamma : S \to \mathbb{R}^m$ are injective, where

$$(\alpha(X))_k = \sqrt{\text{trace}(XF_k)} , \quad (\beta(X))_k = \text{trace}(XF_k) , \quad \gamma(x) = Ax.$$ 

Our Problem Today:

Construct Lipschitz maps $\omega, \psi, \theta : \mathbb{R}^m \to S$ so that $\omega \circ \alpha = 1_X$, $\psi \circ \beta = 1_X$, $\theta \circ \gamma = 1_S$. Determine $\text{Lip}(\omega)$, $\text{Lip}(\psi)$ and $\text{Lip}(\theta)$. 

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Metric Structures on \( \hat{H} \) and \( \text{Sym}(H) \)

Norm Induced Metric

Fix \( 1 \leq p \leq \infty \). The *matrix-norm induced distance* on \( \text{Sym}(H) \):

\[
d_p : \text{Sym}(H) \times \text{Sym}(H) \rightarrow \mathbb{R}, \quad d_p(X, Y) = \|X - Y\|_p,
\]

the \( p \)-norm of singular values (nuclear \( p = 1 \), Frobenius \( p = 2 \), operator \( p = \infty \)).

On \( \hat{H} = H/ T^1 \) it induces the metric

\[
d_p : \hat{H} \times \hat{H} \rightarrow \mathbb{R}, \quad d_p(\hat{x}, \hat{y}) = \|xx^* - yy^*\|_p
\]

so that \( d_p(\hat{x}, \hat{y}) = d_p(xx^*, yy^*) \). In the case \( p = 2 \) we obtain

\[
d_2(X, Y) = \|X - Y\|_F, \quad d_2(x, y) = \sqrt{\|x\|^4 + \|y\|^4 - 2|\langle x, y \rangle|^2}
\]
Metric Structures on $\hat{H}$ and $\text{Sym}(H)$

The Natural Metric

The *natural metric*

$$D_p : \hat{H} \times \hat{H} \to \mathbb{R}, \quad D_p(\hat{x}, \hat{y}) = \min_\varphi \| x - e^{i\varphi} y \|_p$$

with the usual $p$-norm on $\mathbb{C}^n$. In the case $p = 2$ we obtain

$$D_2(\hat{x}, \hat{y}) = \sqrt{\|x\|^2 + \|y\|^2 - 2|\langle x, y \rangle|}$$

On $\text{Sym}^+(H)$, the ”natural” metric lifts to

$$D_p : \text{Sym}^+(H) \times \text{Sym}^+(H) \to \mathbb{R}, \quad D_p(X, Y) = \min_{V, W \text{ s.t. } VV^* = X, WW^* = Y} \| V - W \|_p.$$
Metric Structures on $\text{Sym}(H)$

Natural metric vs. Bures/Helinger

Let $X, Y \in \text{Sym}^+(H)$. For the natural distance we choose $p = 2$:

$$D_{\text{natural}}(X, Y) = \min_{VV^* = X, WW^* = Y} \|V - W\|_F$$

Fact:

$$D_{\text{natural}}(X, Y) = \min_{U \in U(n)} \|X^{1/2} - Y^{1/2}U\|_F = \sqrt{\text{tr}(X) + \text{tr}(Y) - 2\|X^{1/2}Y^{1/2}\|_1}$$
Metric Structures on $\text{Sym}(H)$

Natural metric vs. Bures/Helinger

Let $X, Y \in \text{Sym}^+(H)$. For the natural distance we choose $p = 2$:

$$D_{\text{natural}}(X, Y) = \min_{VV^* = X, WW^* = Y} \| V - W \|_F$$

Fact:

$$D_{\text{natural}}(X, Y) = \min_{U \in U(n)} \left\| X^{1/2} - Y^{1/2} U \right\|_F = \sqrt{\text{tr}(X) + \text{tr}(Y) - 2 \| X^{1/2} Y^{1/2} \|_1}$$

Another distance: Bures/Helinger distance:

$$D_{\text{Bures}}(X, Y) = \| X^{1/2} - Y^{1/2} \|_F = d_2(X^{1/2}, Y^{1/2})$$

A consequence of the Arithmetic-Geometric Mean Inequality [BhatiaKittaneh00]:

$$\frac{1}{2} \| X^{1/2} - Y^{1/2} \|_F^2 \leq \min_{U \in U(n)} \| X^{1/2} - Y^{1/2} U \|_F^2 \leq \| X^{1/2} - Y^{1/2} \|_F^2$$
Stability Results for the forward maps
Bi-Lipschitz properties of $\alpha$ and $\beta$

Fix a closed subset $S \subset Sym^+(H)$. For instance $S = St(H)$, or $S = S^{r,0}$, or $S = St^r(H) = St(H) \cap S^{r,0}$.

**Theorem**

Assume $F = \{F_1, \cdots, F_m\} \subset Sym^+(H)$ so that $\alpha|_S$ and $\beta|_S$ are injective. Then there are constants $a_0, A_0, b_0, B_0 > 0$ so that for every $X, Y \in S$,

$$A_0 \| X^{1/2} - Y^{1/2} \|_F^2 \leq \sum_{k=1}^m \left| \sqrt{\langle X, F_k \rangle} - \sqrt{\langle Y, F_k \rangle} \right|^2 \leq B_0 \| X^{1/2} - Y^{1/2} \|_F^2$$

$$a_0 \| X - Y \|_F^2 \leq \sum_{k=1}^m |\langle X, F_k \rangle - \langle Y, F_k \rangle|^2 \leq b_0 \| X - Y \|_F^2.$$
Stability Results for the inverse map
Lipschitz inversion of $\alpha$ and $\beta$ on Quantum States

Consider the measurement maps

$$\alpha, \beta : (St^r(H), d_1) \rightarrow (\mathbb{R}^m, \| \cdot \|_2) , \ (\alpha(T))_k = \sqrt{tr(TF_k)} , \ (\beta(T))_k = tr(TF_k)$$

where $St^r(H) = \{ T = T^* \geq 0 , \ tr(T) = 1 , \ rank(T) \leq r \}$. If $r = n := dim(H)$ then $St^n(H) = St(H)$ is a compact convex set, hence a Lipschitz retract.

If $r < n$ then $St^r(H)$ is not contractible hence not a Lipschitz retract ($St^1(H) = P(H)$).
Stability Results for the inverse map
Lipschitz inversion of $\alpha$ and $\beta$ on Quantum States

Consider the measurement maps

$$\alpha, \beta : (\text{St}^r(H), d_1) \to (\mathbb{R}^m, \| \cdot \|_2) \text{, } (\alpha(T))_k = \sqrt{\text{tr}(TF_k)}, (\beta(T))_k = \text{tr}(TF_k)$$

where $\text{St}^r(H) = \{ T = T^* \geq 0 \text{, } \text{tr}(T) = 1 \text{, } \text{rank}(T) \leq r \}$. 

If $r = n := \text{dim}(H)$ then $\text{St}^n(H) = \text{St}(H)$ is a compact convex set, hence a Lipschitz retract. 

If $r < n$ then $\text{St}^r(H)$ is not contractible hence not a Lipschitz retract ($\text{St}^1(H) = P(H)$). Consequence:

**Theorem**

*Fix $1 \leq r < n$. For any set of matrices $F_1, \cdots, F_m \in \text{Sym}^+(H)$ there are no continuous maps $\omega : \mathbb{R}^m \to \text{St}^r(H)$ or $\psi : \mathbb{R}^m \to \text{St}^r(H)$ so that $\omega(\alpha(T)) = T$ for every $T \in \text{Sym}^+(H)$, or $\psi(\beta(T)) = T$ for every $T \in \text{Sym}^+(H)$.***
Lipschitz inversion of $\alpha$ on $\mathbb{S}^{r,0}$

**Theorem**

Assume the map

$$\alpha : (\mathbb{S}^{r,0}(H), D_{Bures}) \rightarrow (\mathbb{R}^m, \| \cdot \|_2), \quad (\alpha(T))_k = \sqrt{\text{trace}(TF_k)}$$

is injective, where $\mathbb{S}^{r,0}(H) = \{ T = T^* \geq 0, \text{rank}(T) \leq r \}$. Then there exists a Lipschitz map $\omega : \mathbb{R}^m \rightarrow \mathbb{S}$ so that $\omega(\alpha(T)) = T$ for every $T \in \mathbb{S}^{r,0}$, and

$$\text{Lip}(\omega) = \sup_{c \neq d \in \mathbb{R}^m} \frac{\| (\omega(c))^{1/2} - (\omega(d))^{1/2} \|_F}{\| c - d \|_2} \leq \frac{\sqrt{r + 1}}{\sqrt{A_0}}.$$
Lipschitz inversion of $\beta$ on $S^{r,0}$

**Theorem**

Assume the map

$$\beta : (S^{r,0}(H), \| \cdot \|_F) \to (\mathbb{R}^m, \| \cdot \|_2), \quad (\beta(T))_k = \text{trace}(TF_k)$$

is injective, where $S^{r,0}(H) = \{ T = T^* \geq 0, \text{ rank}(T) \leq r \}$. Then there exists a Lipschitz map $\psi : \mathbb{R}^m \to S$ so that $\psi(\beta(T)) = T$ for every $T \in S^{r,0}$, and

$$\text{Lip}(\psi) = \sup_{c \neq d \in \mathbb{R}^m} \frac{\| \psi(c) - \psi(d) \|_F}{\| c - d \|_2} \leq \frac{\sqrt{r + 1}}{\sqrt{a_0}}.$$
Phase Retrieval: Lipschitz inversion of $\alpha$

Theorem (B.Li18, B.Zou15, BWang15, BCMN14)

Assume $\mathcal{F}$ is a phase retrievable frame for $H$. Then:

1. The map $\alpha : (\hat{\mathbb{C}}^n, D_2) \to (\mathbb{R}^m, \| \cdot \|_2)$ is bi-Lipschitz. Let $\sqrt{A_0}$, $\sqrt{B_0}$ denote its Lipschitz constants: for every $x, y \in \mathbb{C}^n$:

\[
A_0 \min_{\varphi} \| x - e^{i\varphi} y \|_2^2 \leq \sum_{k=1}^{m} \| \langle x, f_k \rangle - \langle y, f_k \rangle \|_2^2 \leq B_0 \min_{\varphi} \| x - e^{i\varphi} y \|_2^2.
\]

2. $B_0 = B$, the frame upper bound.

3. In the real case: $A_0 = \min_{I \subset [m]} A[I] + A[I^c]$.

4. There is a Lipschitz map $\omega : (\mathbb{R}^m, \| \cdot \|_2) \to (\hat{H}, D_2)$ so that: (i) $\omega(\alpha(x)) = x$ for every $x \in \hat{\mathbb{C}}^n$, and (ii) its Lipschitz constant is $\text{Lip}(\omega) \leq \frac{2}{\sqrt{A_0}}$. 

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Phase Retrieval: Lipschitz inversion of $\beta$

Theorem (B.Li18, B.Zou15, BWang15, BCMN14)

Assume $F$ is a phase retrievable frame for $H$. Then:

1. The map $\beta : (\hat{\mathbb{C}}^n, d_1) \rightarrow (\mathbb{R}^m, \| \cdot \|_2)$ is bi-Lipschitz. Let $\sqrt{a_0}, \sqrt{b_0}$ denote its Lipschitz constants: for every $x, y \in \mathbb{C}^n$:

$$a_0 \| xx^* - yy^* \|_1^2 \leq \sum_{k=1}^{m} \left| \langle x, f_k \rangle^2 - \langle y, f_k \rangle^2 \right|^2 \leq b_0 \| xx^* - yy^* \|_1^2.$$

2. $b_0 = \max_{\|x\|=1} \|Fx\|_4^4$.

3. There is a Lipschitz map $\psi : (\mathbb{R}^m, \| \cdot \|_2) \rightarrow (\hat{H}, d_1)$ so that: (i) $\psi(\beta(x)) = x$ for every $x \in \hat{\mathbb{C}}^n$, and (ii) its Lipschitz constant is $\text{Lip}(\psi) \leq \frac{2}{\sqrt{a_0}}$. 
Global Lipschitz inversion in Compressive Sampling

**Theorem**

Assume that every $2d$ columns of the $m \times n$ matrix $A$ are linearly independent. Let $c_0 = \min_{|I|=2d} \sigma_{2d}(A[I])$ (square root of the smallest lower Riesz bound among all possible combinations of $2d$ columns). Let $\gamma : \mathbb{R}_d^n \to \mathbb{R}^m$, $\gamma(x) = Ax$, where $\mathbb{R}_d^n$ denotes the space of $d$-sparse signals in $\mathbb{R}^n$. Then

1. For every $x, y \in \mathbb{R}_d^n$, $\|\gamma(x) - \gamma(y)\|_0 \geq c_0 \|x - y\|_2$.
2. There is a Lipschitz maps $\theta : \mathbb{R}^m \to \mathbb{R}_d^n$ so that: (i) $\theta(\gamma(x)) = x$ for all $x \in \mathbb{R}_d^n$; (ii) $\text{Lip}(\theta) \leq \frac{\sqrt{d+1}}{c_0}$.

Note: Same bounds for $\mathbb{C}_d^n$. 
The extension mechanism involves three steps:

1. Embed the metric space \((S, d)\) into a Hilbert space \(K\) \((\text{Sym}(H)\) or \(H)\);
2. Use Kirszbraun’s theorem to obtain an isometric extension;
3. Construct and apply a Lipschitz projection in \(K\) onto the image of \((S, d)\).

We exemplify this mechanism on the phase retrieval (PR) problem. The Low-Rank PSD Case: Similar to the PR case; different Lipschitz retraction for \(\mathbb{S}^{r,0}(H)\). Same for the compressive sampling problem. Note: The same mechanism works in the Johnson-Lindenstrauss theorem.
PR Inversion
Extension of the inverse for $\alpha$

We know $\alpha : (\hat{H}, D_2) \to (\mathbb{R}^m, \| \cdot \|_2)$ is bi-Lipschitz:

$$A_0D_2(x, y)^2 \leq \|\alpha(x) - \alpha(y)\|^2 \leq b_0D_2(x, y)^2$$

Let $M = \alpha(\hat{H}) \subset \mathbb{R}^m$. 
PR Inversion
Extension of the inverse for $\alpha$: Step 1

First identify (=embed) $\hat{H}$ with $S^{1,0}(H)$. 

$\hat{C}^n \xrightarrow{\alpha} \hat{H} \xrightarrow{\kappa_\alpha} S^{1,0} \xrightarrow{M = \alpha(C^n)} \mathbb{R}^m$
PR Inversion
Extension of the inverse for $\alpha$: Step 1

Then construct the local left inverse $\omega_1 : M \to \hat{H}$ with $\text{Lip}(\omega_1) = \frac{1}{\sqrt{A_0}}$. 
PR Inversion
Extension of the inverse for $\alpha$: Step 2

Use Kirszbraun’s theorem to extend isometrically $\omega_2 : \mathbb{R}^m \to \text{Sym}(H)$. 
PR Inversion
Extension of the inverse for $\alpha$: Step 3

Construct a Lipschitz "projection" $\pi : \text{Sym}(H) \rightarrow S^{1,0}(H)$.
PR Inversion
Extension of the inverse for $\alpha$: Final process

Compose the two maps to get $\omega: \mathbb{R}^m \to S^{1,0}$, $\omega = \pi \circ \omega_2$. 

$M = \alpha(\mathbb{C}^n)$
Part 2: $S^{1,0}(H)$ as Lipschitz retract in $Sym(H)$

Lemma

Consider the spectral decomposition of the self-adjoint operator $A$ in $Sym(H)$, $A = \sum_{k=1}^{d} \lambda_{m(k)} P_k$. Then the map

$$\pi : Sym(H) \to S^{1,0}(H) \quad , \quad \pi(A) = (\lambda_1 - \lambda_2)P_1$$

satisfies the following two properties:

1. $\pi : (Sym(H), \| \cdot \|_F) \to (S^{1,0}(H), \| \cdot \|_F)$ is Lipschitz with $Lip(\pi) = \sqrt{2}$. \\
2. $\pi(A) = A$ for all $A \in S^{1,0}(H)$.

In [B.Zou'15] paper we proved, for $\pi : (Sym(H), d_{p}) \to (S^{1,0}(H), d_{p})$, $Lip(\pi) \leq 3 + 2^{1+\frac{1}{p}}$.

Recently [March 2018], Wenbo Li [AMSC/UMD] proved $Lip(\pi) = 2$ for $p = \infty$. 

Radu Balan (UMD) Lipschitz
$\mathbb{S}^{r,0}(H)$ as Lipschitz retract in $\text{Sym}(H)$

**Lemma**

Consider the nonlinear projector $P_+$ onto the cone of PSD matrices $\text{Sym}^+(H)$. Then the map

$$
\pi_r : \text{Sym}(H) \rightarrow \mathbb{S}^{1,0}(H) \quad , \quad \pi(A) = P_+(A - \lambda_{r+1}(A)I)
$$

satisfies the following two properties:

1. $\pi_r : (\text{Sym}(H), \| \cdot \|_F) \rightarrow (\mathbb{S}^{r,0}(H), \| \cdot \|_F)$ is Lipschitz with $\text{Lip}(\pi_r) = \sqrt{r + 1}$.

2. $\pi_r(A) = A$ for all $A \in \mathbb{S}^{r,0}(H)$. 
Consider the nonlinear soft thresholding operator $\tau_\theta(t) = \text{sign}(t)[|t| - \theta]_+$. Consider the map

$$P_d : \mathbb{R}^n \rightarrow \mathbb{R}^n_d, \quad (P_d(x))_k = \tau_\theta(x_k), \quad \theta = |\tilde{x}_{d+1}|$$

where $\tilde{x}_{d+1}$ is the $d + 1^{st}$ largest entry in magnitude. Then $P_d$ satisfies the following two properties:

1. $P_d : (H, \| \cdot \|_2) \rightarrow (H_d, \| \cdot \|_2)$ is Lipschitz with $\text{Lip}(P_d) = \sqrt{d + 1}$.
2. $P_d(x) = x$ for all $x \in H_d$. 

$H_d = \mathbb{R}^n_d$ as Lipschitz retract in $H = \mathbb{R}^n$
THANK YOU!!

Questions?
References


