Deterministic and Stochastic Bounds in the Phase Retrieval Problem

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Problem Formulation

The phase retrieval problem

- Hilbert space $H = \mathbb{C}^n$, $\mathring{H} = H / T^1$, frame $\mathcal{F} = \{f_1, \ldots, f_m\} \subset \mathbb{C}^n$ and
  \[
  \alpha : \mathring{H} \to \mathbb{R}^m \quad \text{,} \quad \alpha(x) = (|\langle x, f_k \rangle|)_{1 \leq k \leq m}.
  \]
  \[
  \beta : \mathring{H} \to \mathbb{R}^m \quad \text{,} \quad \beta(x) = \left(\langle x, f_k \rangle^2\right)_{1 \leq k \leq m}.
  \]

The frame is said phase retrievable (or that it gives phase retrieval) if $\alpha$ (or $\beta$) is injective.

- The general phase retrieval problem a.k.a. phaseless reconstruction: Decide when a given frame is phase retrievable, and, if so, find an algorithm to recover $x$ from $y = \alpha(x)$ (or from $y = \beta(x)$) up to a global phase factor.
Assume $\mathcal{F}$ is phase retrievable.

Our Problems Today:

1. Are the nonliner maps $\alpha, \beta$ bi-Lipschitz with respect to appropriate metrics?
2. Do they admit left inverses that are globally Lipschitz?
3. What are the Lipschitz constants? What is the structure of local Lipschitz bounds?
4. What is the average performance of any reconstruction scheme (Cramer-Rao Lower Bounds)?

1-3: Worst Case Performance
4: Average Case Performance
Metric Space Structures

Let $H = \mathbb{C}^n$. The quotient space $\hat{H} = \mathbb{C}^n / T^1$, with classes induced by $x \sim y$ if there is real $\varphi$ with $x = e^{i\varphi}y$.

Topologically:

$$\hat{\mathbb{C}}^n = \{0\} \cup \left((0, \infty) \times \mathbb{C}\mathbb{P}^{n-1}\right)$$

with

$$\hat{\mathbb{C}}^n = \hat{\mathbb{C}}^n \setminus \{0\} = (0, \infty) \times \mathbb{C}\mathbb{P}^{n-1}$$

a real analytic manifold of real dimension $2n - 1$.

Another embedding is into the space of symmetric matrices $\text{Sym}(\mathbb{C}^n)$.

Specifically let

$$S^{p,q}(H) = \{ T \in \text{Sym}(H) \mid T \text{ has at most } p \text{ pos.eigs. and } q \text{ neg.eigs.} \}$$

Then:

$$\kappa_\beta : \hat{H} \to S^{1,0}, \quad \hat{x} \mapsto xx^* , \text{ is an embedding.}$$
Fix $1 \leq p \leq \infty$. The matrix-norm induced distance

$$d_p : \hat{H} \times \hat{H} \to \mathbb{R}, \quad d_p(\hat{x}, \hat{y}) = \|xx^* - yy^*\|_p$$

with the $p$-norm of the singular values. In the case $p = 2$ we obtain

$$d_2(x, y) = \sqrt{\|x\|^4 + \|y\|^4 - 2|\langle x, y \rangle|^2}$$

**Lemma (BZ15)**

1. $(d_p)_{1 \leq p \leq \infty}$ are equivalent metrics and the identity map $i : (\hat{H}, d_p) \to (\hat{H}, d_q), \ i(x) = x$ has Lipschitz constant

$$\text{Lip}_{p,q,n}^d = \max(1, 2^{\frac{1}{q} - \frac{1}{p}}).$$

2. The metric space $(\hat{H}, d_p)$ is isometrically isomorphic to $S^{1,0}$ endowed with the $p$-norm via $\kappa_{\beta} : \hat{H} \to S^{1,0}, \ x \mapsto \kappa_{\beta}(x) = xx^*$. 

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Bounds in Phase Retrieval
Metric Space Structures

The natural metric structure

Fix $1 \leq p \leq \infty$. The natural metric

$$D_p : \hat{H} \times \hat{H} \to \mathbb{R}, \quad D_p(\hat{x}, \hat{y}) = \min_{\varphi} \|x - e^{i\varphi} y\|_p$$

with the usual $p$-norm on $\mathbb{C}^n$. In the case $p = 2$ we obtain

$$D_2(\hat{x}, \hat{y}) = \sqrt{\|x\|^2 + \|y\|^2 - 2|\langle x, y \rangle|}$$

**Lemma (BZ15)**

1. $(D_p)_{1 \leq p \leq \infty}$ are equivalent metrics and the identity map $i : (\hat{H}, D_p) \to (\hat{H}, D_q)$, $i(x) = x$ has Lipschitz constant

   $$\text{Lip}^D_{p,q,n} = \max(1, n^{\frac{1}{q} - \frac{1}{p}}).$$

2. The metric space $(\hat{H}, D_2)$ is Lipschitz isomorphic to $S^{1,0}$ endowed with the 2-norm via $\kappa_\alpha : \hat{H} \to S^{1,0}$, $x \mapsto \kappa_\alpha(x) = \frac{1}{\|x\|} xx^*$. 
Two different structures: topologically equivalent, BUT the metrics are NOT equivalent:

Lemma (BZ15)

The identity map \( i : (\hat{H}, D_p) \rightarrow (\hat{H}, d_p), \ i(x) = x \) is continuous but it is not Lipschitz continuous. Likewise, the identity map \( i : (\hat{H}, d_p) \rightarrow (\hat{H}, D_p), \ i(x) = x \) is continuous but it is not Lipschitz continuous. Hence the induced topologies on \((\hat{H}, D_p)\) and \((\hat{H}, d_p)\) are the same, but the corresponding metrics are not Lipschitz equivalent.
Main Results

Lipschitz inversion: $\alpha$

Theorem (BZ15)

Assume $\mathcal{F}$ is a phase retrievable frame for $H$. Then:

1. **The map** $\alpha : (\hat{H}, D_2) \rightarrow (\mathbb{R}^m, \| \cdot \|_2)$ **is bi-Lipschitz.** Let $\sqrt{A_0}, \sqrt{B_0}$ denote its Lipschitz constants: for every $x, y \in H$:

   $$A_0 \min_{\varphi} \| x - e^{i\varphi} y \|_2^2 \leq \sum_{k=1}^{m} \| \langle x, f_k \rangle - \langle y, f_k \rangle \|_2^2 \leq B_0 \min_{\varphi} \| x - e^{i\varphi} y \|_2^2.$$

2. **There is a Lipschitz map** $\omega : (\mathbb{R}^m, \| \cdot \|_2) \rightarrow (\hat{H}, D_2)$ **so that:** (i) $\omega(\alpha(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $\text{Lip}(\omega) \leq \frac{4 + 3\sqrt{2}}{\sqrt{A_0}} = \frac{8.24}{\sqrt{A_0}}$. 
Main Results

Lipschitz inversion: $\beta$

**Theorem (BZ15)**

Assume $\mathcal{F}$ is a phase retrievable frame for $H$. Then:

1. The map $\beta : (\hat{H}, d_1) \to (\mathbb{R}^m, \| \cdot \|_2)$ is bi-Lipschitz. Let $\sqrt{a_0}, \sqrt{b_0}$ denote its Lipschitz constants: for every $x, y \in H$:

   $$a_0 \| xx^* - yy^* \|_1^2 \leq \sum_{k=1}^{m} \left| \| \langle x, f_k \rangle \|_2^2 - \| \langle y, f_k \rangle \|_2^2 \right| \leq b_0 \| xx^* - yy^* \|_1^2.$$

2. There is a Lipschitz map $\psi : (\mathbb{R}^m, \| \cdot \|_2) \to (\hat{H}, d_1)$ so that: (i) $\psi(\beta(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $\text{Lip}(\psi) \leq \frac{4 + 3\sqrt{2}}{\sqrt{a_0}} = \frac{8.24}{\sqrt{a_0}}$. 

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Bounds in Phase Retrieval
Main Results
Statistical models

- A general noisy measurement process is given by:

\[ y_k = |\langle x, f_k \rangle + \mu_k|^p + \nu_k, \quad 1 \leq k \leq m, \]

where \((\mu_k)_k, (\nu_k)_k\) are two noise processes.
Main Results
Statistical models

- A general noisy measurement process is given by:
  \[ y_k = |\langle x, f_k \rangle + \mu_k|^p + \nu_k , \quad 1 \leq k \leq m, \]
  where \((\mu_k)_k, (\nu_k)_k\) are two noise processes.

- AWGN Model: \(\mu_k = 0, p = 2\) and \(\nu_k \sim \mathcal{N}(0, \sigma^2)\) i.i.d.
  \[ y_k = |\langle x, f_k \rangle|^2 + \nu_k , \quad 1 \leq k \leq m. \]
Main Results

Statistical models

- A general noisy measurement process is given by:
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- AWGN Model: \(\mu_k = 0, \ p = 2\) and \(\nu_k \sim \mathcal{N}(0, \sigma^2)\) i.i.d.
  \[ y_k = |\langle x, f_k \rangle|^2 + \nu_k, \quad 1 \leq k \leq m. \]
- Non-AWGN Model: \(\mu_k \sim \mathbb{C}\mathcal{N}(0, \rho^2)\), i.i.d. and \(\nu_k = 0,\)
  \[ y_k = |\langle x, f_k \rangle + \mu_k|^p, \quad 1 \leq k \leq m. \]
Main Results

Statistical models

- A general noisy measurement process is given by:
  \[ y_k = |\langle x, f_k \rangle + \mu_k|^p + \nu_k , \quad 1 \leq k \leq m, \]
  where \((\mu_k)_k, (\nu_k)_k\) are two noise processes.

- AWGN Model: \(\mu_k = 0, \ p = 2\) and \(\nu_k \sim \mathcal{N}(0, \sigma^2)\) i.i.d.
  \[ y_k = |\langle x, f_k \rangle|^2 + \nu_k , \quad 1 \leq k \leq m. \]

- Non-AWGN Model: \(\mu_k \sim \mathcal{CN}(0, \rho^2)\), i.i.d. and \(\nu_k = 0, \)
  \[ y_k = |\langle x, f_k \rangle + \mu_k|^p , \quad 1 \leq k \leq m. \]

Want:

1) Fisher Information Matrix \( \mathbb{I} = \mathbb{E} \left[ (\nabla_x \log p(y; x)) (\nabla_x \log p(y; x))^* \right]. \)

2) Cramer-Rao Lower Bounds for unbiased estimators.
Main Results
Fisher Information Matrix

\[ I_{\text{AWGN, real}}^{\text{AWGN, real}}(x) = \frac{4}{\sigma^2} \sum_{k=1}^{m} |\langle x, f_k \rangle|^2 f_k f_k^T = \frac{4}{\sigma^2} \sum_{k=1}^{m} (f_k f_k^T) x x^T (f_k f_k^T) \] [Bal12].
Main Results
Fisher Information Matrix

\[ I^{AWGN, real}(x) = \frac{4}{\sigma^2} \sum_{k=1}^{m} |\langle x, f_k \rangle|^2 f_k f_k^T = \frac{4}{\sigma^2} \sum_{k=1}^{m} (f_k f_k^T) x x^T (f_k f_k^T) \quad [Bal12]. \]

\[ I^{AWGN, cplx}(x) = \frac{4}{\sigma^2} \sum_{k=1}^{m} \Phi_k \xi \xi^* \Phi_k \quad [Bal13, BCMN13]. \]
Main Results

Fisher Information Matrix

\[ \mathbb{I}_{AWGN, real}(x) = \frac{4}{\sigma^2} \sum_{k=1}^{m} |\langle x, f_k \rangle|^2 f_k f_k^T = \frac{4}{\sigma^2} \sum_{k=1}^{m} (f_k f_k^T)x x^T (f_k f_k^T) \]  [Bal12].

\[ \mathbb{I}_{AWGN, cplx}(x) = \frac{4}{\sigma^2} \sum_{k=1}^{m} \Phi_k \xi \xi^* \Phi_k \]  [Bal13, BCMN13].

\[ \mathbb{I}_{nonAWGN, cplx}(x) = \frac{4}{\rho^4} \sum_{k=1}^{m} \left( G_1 \left( \frac{\langle \Phi_k \xi, \xi \rangle}{\rho^2} \right) - 1 \right) \Phi_k \xi \xi^* \Phi_k \]

\[ = \frac{4}{\rho^2} \sum_{k=1}^{m} G_2 \left( \frac{\langle \Phi_k \xi, \xi \rangle}{\rho^2} \right) \frac{1}{\langle \Phi_k \xi, \xi \rangle} \Phi_k \xi \xi^* \Phi_k \]  [Bal15].

where

\[ G_1(a) = \frac{e^{-a}}{8a^3} \int_0^\infty \frac{l_1^2(t)}{l_0(t)} t^3 e^{-\frac{t^2}{4a}} dt , \quad G_2(a) = a(G_1(a) - 1). \]
Main Results

AWGN vs. non-AWGN: Comparisons and Identifiability

Let $B$ be the frame upper bound.

Lemma

$$
\frac{\sigma^2}{\rho^4} \left( G_1\left( \frac{B\|x\|^2}{\rho^2} \right) - 1 \right) \mathbb{I}_{AWGN,cplx}(x) \leq \mathbb{I}_{nonAWGN,cplx}(x) \leq \frac{\sigma^2}{\rho^4} \mathbb{I}_{AWGN,cplx}(x)
$$
Main Results
AWGN vs. non-AWGN: Comparisons and Identifiability

Let $B$ be the frame upper bound.

Lemma

\[
\frac{\sigma^2}{\rho^4} \left( G_1 \left( \frac{B \|x\|^2}{\rho^2} \right) - 1 \right) \leq I_{AWGN, cplx}^\sigma(x) \leq I_{nonAWGN, cplx}^\sigma(x) \leq \frac{\sigma^2}{\rho^4} I_{AWGN, cplx}^\sigma(x)
\]

Theorem

The following are equivalent:

1. The frame $\mathcal{F}$ is phase retrievable;
2. For every $0 \neq x \in \mathbb{C}^n$, $\text{rank}(I_{nonAWGN, cplx}^\sigma(x)) = 2n - 1$;
3. For every $0 \neq x \in \mathbb{C}^n$, $\text{rank}(I_{AWGN, cplx}^\sigma(x)) = 2n - 1$;

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Main Results

The Cramer-Rao Lower Bound

Fix $z_0 \in \mathbb{C}^n$, $\|z_0\| = 1$, let $\zeta_0 = [\text{real}(z_0) \; \text{imag}(z_0)]^T$ and set

$$\Omega_{z_0} = \{\xi \in \mathbb{R}^{2n}, \langle \xi, \zeta_0 \rangle \geq 0, \langle \xi, J\zeta_0 \rangle = 0\}.$$

Let $\Pi_{z_0} = 1 - J\zeta_0\zeta_0^*J^*$ with $J$ the symplectic form matrix.

Theorem

Assume a measurement model $y_k = |\langle x, f_k \rangle + \mu_k|^p + \nu_k$ with $\xi = [\text{real}(x) \; \text{imag}(x)]^T \in \Omega_{z_0}$. Then the covariance of any unbiased estimator $\omega : \mathbb{R}^m \rightarrow \mathbb{C}^n$ is bounded below by

$$\text{Cov}[\omega(y); \xi] \geq (\Pi_{z_0}\Pi(\xi)\Pi_{z_0})^\dagger.$$

If one chooses the global phase so that $\langle \omega(y), x \rangle \geq 0$ then

$$\text{Cov}[\omega(y); \xi] \geq (\Pi(\xi))^\dagger.$$
Prior literature:

- **2012**: B.: Cramer-Rao lower bound in the real case; Eldar&Mendelson: map $\alpha$ in the real case
  \[ \|\alpha(x) - \alpha(y)\| \geq C\|x - y\|\|x + y\|. \]

- **2013**: Bandeira,Cahill,Mixon,Nelson: improved the estimate of $C$. B.: $\beta$ bi-Lipschitz in real and complex case.

- **2014**: B.&Yang: Find the exact Lipschitz constant for $\alpha$ in the real case - the constants $A_0, B_0$; B.&Z.: constructed a Lipschitz left inverse for $\beta$.

- **2015**: B.&Z.: Proved $\alpha$ is bi-Lipschitz in the complex case; constructed a Lipschitz left inverse. B.: lower Lipschitz constant $A_0$ connected to CRLB of a non-AWGN model.
Main Results
Key relationship between deterministic and stochastic bounds

The central object: \( R(\xi) = \sum_{k=1}^{m} \Phi_k \xi \xi^T \Phi_k. \)
Main Results

Key relationship between deterministic and stochastic bounds

The central object: \( R(\xi) = \sum_{k=1}^{m} \Phi_k \xi \xi^T \Phi_k \).

The lower Lipschitz bound for \( \beta \) map is:

\[
a_0 = \min_{\|\xi\|=1} \lambda_{2n-1}(R(\xi)).
\]

The Fisher information matrix for the AWGN model:

\[
I^{AWGN,\text{cplx}}(x) = \frac{4}{\sigma^2} R(\xi).
\]
Main Results

Key relationship between deterministic and stochastic bounds

The central object: \( R(\xi) = \sum_{k=1}^{m} \Phi_k \xi \xi^T \Phi_k \).

The lower Lipschitz bound for \( \beta \) map is:

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a_0 = \min_{\|\xi\|=1} \lambda_{2n-1}(R(\xi)).
\]

The Fisher information matrix for the AWGN model:

\[
I_{AWGN,\text{cplx}}(x) = \frac{4}{\sigma^2} R(\xi).
\]

Best inversion scheme \( \psi \) that is lossless in the absence of noise achieves:

\[
d_1(\psi(c), \psi(d))^2 \leq \frac{68}{a_0} \|c - d\|_2^2.
\]

An efficient estimator (i.e. unbiased that achieves CRLB) \( \omega^0 \) achieves:

\[
\mathbb{E} \left[ \|\omega^0(y) - x\|_2^2; x \right] \leq \frac{(2n - 1) \sigma^2}{4a_0 \|x\|^2} = \frac{2n - 1}{4a_0 \text{SNR}}.
\]
Deterministic bounds: The proofs involve several steps (details in [BZ15]).

1. **Part 1: Injectivity → bi-Lipschitz:** Upper bounds are not too hard; lower bounds: relatively easy for $\beta$ (the "square" map), but relatively hard for $\alpha$.

2. **Part 2: Left inverse construction is done in three steps:**
   1. The left inverse is first extended to $\mathbb{R}^m$ into $\text{Sym}(H)$ using Kirszbraun’s theorem;
   2. Then we show that $S^{1,0}(H)$ is a Lipschitz retract in $\text{Sym}(H)$;
   3. The proof is concluded by composing the two maps.

The stochastic bounds: Direct computations and a bit of luck! [Bal15]
Part 1a: Bi-Lipschitzianity of $\alpha$

\[ \alpha : \hat{H} \to \mathbb{R}^m , \quad \alpha(x) = (|\langle x, f_k \rangle|)_{1 \leq k \leq m} \]

The homogeneity of $\alpha$ shows that

\[ L(x, y) = \frac{\|\alpha(x) - \alpha(y)\|}{D_p(x, y)} \]

is homogeneous of degree 0: $L(tx, ty) = L(x, y)$, for every $t > 0$. This reduces the problem to the unit ball: $1 = \|x\| \geq \|y\|$.

The upper bound was computed in [BCMN13]:

\[ \sup_{x \neq y} \frac{\|\alpha(x) - \alpha(y)\|^2}{D_2(x, y)^2} = B \quad \text{(upper frame bound)}. \]
A compactness argument shows the lower bound is positive if and only if the local lower bound is positive:

\[ \inf_{\|z\|=1} \lim_{r \to 0} \inf_{x,y \in \hat{H}, \quad D_2(x,z) < r, \quad D_2(y,z) < r} \frac{\|\alpha(x) - \alpha(y)\|_2^2}{D_2(x,y)^2} > 0. \]

This bound is computed explicitly and shown positive: Computations involve the realification framework and other delicate nonlinear expansions.
Proofs

Part 1b: Bi-Lipschitzianity of \( \beta \)

Key Remark (B. Bodmann, Casazza, Edidin - 2007): The nonlinear map \( \beta \) is the restriction of the linear map

\[
A : \text{Sym}(H) \to \mathbb{R}^m, \quad A(T) = (\langle Tf_k, f_k \rangle)_{1 \leq k \leq m}
\]

Specifically:

\[
\beta(x) = A(xx^*) = (|\langle x, f_k \rangle|^2)_{1 \leq k \leq m}.
\]

\[
\|\beta(x) - \beta(y)\| = \|A(xx^*) - A(yy^*)\| = \|A(xx^* - yy^*)\|
\]

\[
= \|xx^* - yy^*\| A\left(\frac{xx^* - yy^*}{\|xx^* - yy^*\|}\right)
\]

\[
a_0 = \min_{T \in S^{1,1}, \|T\|_1 = 1} \|A(T)\| > 0, \quad b_0 = \max_{T \in S^{1,1}, \|T\|_1 = 1} \|A(T)\|
\]
Proofs

Part 2a: Extension of the inverse for $\alpha$

We know $\alpha : (\hat{H}, D_2) \to (\mathbb{R}^m, \| \cdot \|_2)$ is bi-Lipschitz:

$$A_0 D_1(x, y)^2 \leq \|\alpha(x) - \alpha(y)\|^2 \leq b_0 D_2(x, y)^2$$

Let $M = \alpha(\hat{H}) \subset \mathbb{R}^m$. 

\[ M = \beta(\mathbb{C}^n) \]
Proofs
Part 2a: Extension of the inverse for $\alpha$

First identify $\hat{H}$ with $S^{1,0}(H)$.
Proofs

Part 2a: Extension of the inverse for $\alpha$

Then construct the local left inverse $\omega_1 : M \to \hat{H}$ with $\text{Lip}(\omega_1) = \frac{1}{\sqrt{A_0}}$. 

![Diagram of mathematical structures and mappings]
Proofs

Part 2a: Extension of the inverse for $\alpha$

Use Kirszbraun’s theorem to extend isometrically $\omega_2 : \mathbb{R}^m \to \text{Sym}(H)$.
Proofs

Part 2a: Extension of the inverse for $\alpha$

Construct a Lipschitz ”projection” $\pi : \text{Sym}(H) \to S^{1,0}(H)$.
Proofs

Part 2a: Extension of the inverse for $\alpha$

Compose the two maps to get $\omega : \mathbb{R}^m \rightarrow S^{1,0}$, $\omega = \pi \circ \omega_2$. 

\[
\begin{array}{c}
\text{Sym}(H) \\
\downarrow \psi \\
\downarrow \psi_2 \\
S^{1,0} \\
\downarrow \pi \\
\downarrow \kappa_\beta \\
\hat{\mathbb{C}}^n \\
\downarrow \beta \\
\mathbb{R}^m \\
\end{array}
\] 

\[
M = \beta(\mathbb{C}^n)
\]
Proofs

Part 2b: Extension of the inverse for $\beta$

We know $\beta : (\hat{H}, d_1) \rightarrow (\mathbb{R}^m, \| \cdot \|_2)$ is bi-Lipschitz:

$$a_0 d_1(x, y)^2 \leq \| \beta(x) - \beta(y) \|^2 \leq b_0 d_1(x, y)^2.$$  

Let $M = \beta(\hat{H}) \subset \mathbb{R}^m.$
Proofs

Part 2b: Extension of the inverse for $\beta$

First identify $\hat{H}$ with $S^{1,0}(H)$.
Proofs
Part 2b: Extension of the inverse for $\beta$

Then construct the local left inverse $\psi_1 : M \to \hat{H}$ with $Lip(\psi_1) = \frac{1}{\sqrt{a_0}}$. 

![Diagram showing the relationship between $\mathbb{C}^n$, $\kappa_\beta$, $S^{1,0}$, $\psi_1$, $M = \beta(\mathbb{C}^n)$, and $\hat{H}$ with an arrow labeled $\beta$.](image)
Proofs

Part 2b: Extension of the inverse for $\beta$

Use Kirszbraun’s theorem to extend isometrically $\psi_2 : \mathbb{R}^m \rightarrow \text{Sym}(H)$. 

![Diagram showing the extension of $\psi_2$ through Kirszbraun's theorem]
Proofs

Part 2b: Extension of the inverse for $\beta$

Construct a Lipschitz "projection" $\pi : \text{Sym}(H) \to S^{1,0}(H)$. 
Proofs

Part 2b: Extension of the inverse for $\beta$

Compose the two maps to get $\psi : \mathbb{R}^m \rightarrow S^{1,0}$, $\psi = \pi \circ \psi_2$. 

$\beta$ 

$M = \beta(\mathbb{C}^n)$ 

$\kappa_\beta$ 

$\pi$ 

$\psi$ 

$\psi_2$ 

$\mathbb{C}^n$ 

$\text{Sym}(H)$ 

$S^{1,0}$ 

$\mathbb{R}^m$
Proofs

Part 2: $S^{1,0}(H)$ as Lipschitz retract in $\text{Sym}(H)$

Lemma

Consider the spectral decomposition of the self-adjoint operator $A$ in $\text{Sym}(H)$, $A = \sum_{k=1}^{d} \lambda_{m(k)} P_k$. Then the map

$$\pi : \text{Sym}(H) \to S^{1,0}(H), \quad \pi(A) = (\lambda_1 - \lambda_2)P_1$$

satisfies the following two properties:

1. for $1 \leq p \leq \infty$, it is Lipschitz continuous from $(\text{Sym}(H), \| \cdot \|_p)$ to $(S^{1,0}(H), \| \cdot \|_p)$ with Lipschitz constant less than or equal to $3 + 2^{1 + \frac{1}{p}}$;

2. $\pi(A) = A$ for all $A \in S^{1,0}(H)$.

Assume simple top eigenvalues (otherwise the bound is immediate):
\[ \pi(A) = (\lambda_1 - \lambda_2)P_1, \quad \pi(B) = (\mu_1 - \mu_2)Q_1. \]
Then:
\[
\| \pi(A) - \pi(B) \|_p \leq (\lambda_1 - \lambda_2) \| P_1 - Q_1 \|_p + |\lambda_1 - \mu_1| + |\lambda_2 - \mu_2| \\
\leq (\lambda_1 - \lambda_2) \| P_1 - Q_1 \|_p + 2 \| A - B \|_p.
\]

\[
\| P_1 - Q_1 \|_p \leq \frac{1}{2\pi} \int_I \| (R_A - R_B)(\gamma(t)) \|_p |\gamma'(t)| dt
\]

\[
R_A(z) = (A - zI)^{-1}, \quad R_B(z) = (B - zI)^{-1}.
\]

\[
(R_A - R_B)(z) = \sum_{n \geq 1} (-1)^n (R_A(z)(B - A))^n R_A(z).
\]

\[
\| (R_A - R_B)(\gamma(t)) \|_p \leq \sum_{n \geq 1} \| R_A(\gamma(t)) \|_\infty^{n+1} \| A - B \|_p^n
\]

\[
= \frac{\| R_A(\gamma(t)) \|_\infty^2 \| A - B \|_p}{1 - \| R_A(\gamma(t)) \|_\infty \| A - B \|_p} < \frac{\| A - B \|_p}{\text{dist}^2(\gamma(t), \text{Spec}(A))}.
\]
The analysis requires a deeper understanding of local behavior.

1. The *global lower and upper Lipschitz bounds*:

\[
A_0 = \inf_{x, y \in \hat{H}} \frac{\|\alpha(x) - \alpha(y)\|^2}{D_2(x, y)^2}, \quad B_0 = \sup_{x, y \in \hat{H}} \frac{\|\alpha(x) - \alpha(y)\|^2}{D_2(x, y)^2}
\]

2. The *type I local lower and upper Lipschitz bounds* at \( z \in \hat{H} \):

\[
A(z) = \lim_{r \to 0} \inf_{\substack{x, y \in \hat{H} \\:\text{ such that } \quad D_2(x, y) < r \\text{ and } \quad D_2(x, z) < r}} \frac{\|\alpha(x) - \alpha(y)\|^2}{D_2(x, y)^2}, \quad B(z) = \lim_{r \to 0} \sup_{\substack{x, y \in \hat{H} \\:\text{ such that } \quad D_2(x, y) < r \\text{ and } \quad D_2(x, z) < r}} \frac{\|\alpha(x) - \alpha(y)\|^2}{D_2(x, y)^2}
\]

3. The *type II local lower and upper Lipschitz bounds* at \( z \in \hat{H} \):

\[
\tilde{A}(z) = \lim_{r \to 0} \inf_{x \in \hat{H} \\:\text{ such that } \quad D_2(x, z) < r} \frac{\|\alpha(x) - \alpha(z)\|^2}{D_2(x, z)^2}, \quad \tilde{B}(z) = \lim_{r \to 0} \sup_{x \in \hat{H} \\:\text{ such that } \quad D_2(x, z) < r} \frac{\|\alpha(x) - \alpha(z)\|^2}{D_2(x, y)^2}
\]
Part 1: Bi-Lipschitzianity of $\alpha$ -cont’d

We need to analyze the real structure of $\hat{H}$. Let $\varphi_1, \ldots, \varphi_m, \zeta \in \mathbb{R}^{2n}$, $\Phi_1, \ldots, \Phi_m \in \text{Sym}(\mathbb{R}^{2n})$, $J \in \mathbb{R}^{2n \times 2n}$ defined by:

$$\Phi_k = \varphi_k \varphi_k^T + J \varphi_k \varphi_k^T J^T, \varphi_k = \begin{bmatrix} \text{real}(f_k) \\ \text{imag}(f_k) \end{bmatrix}, J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \zeta = \begin{bmatrix} \text{real}(z) \\ \text{imag}(z) \end{bmatrix}.$$

Key relations: $\langle z, f_k \rangle = \langle \zeta, \varphi_k \rangle + i \langle \zeta, J \varphi_k \rangle$, $|\langle z, f_k \rangle| = \sqrt{\langle \Phi_k \zeta, \zeta \rangle}$. Consider the following objects:

$$\mathcal{R} : \mathbb{R}^{2n} \to \text{Sym}(\mathbb{R}^{2n}) \ , \ \mathcal{R}(\xi) = \sum_{k=1}^{m} \Phi_k \xi \xi^T \Phi_k, \xi \in \mathbb{R}^{2n}$$

$$\mathcal{S} : \mathbb{R}^{2n} \to \text{Sym}(\mathbb{R}^{2n}) \ , \ \mathcal{S}(\xi) = \sum_{k: \Phi_k \xi \neq 0} \frac{1}{\langle \Phi_k \xi, \xi \rangle} \Phi_k \xi \xi^T \Phi_k, \xi \in \mathbb{R}^{2n}$$
Problems
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Proofs
Lipschitz bounds for $\alpha$

**Theorem (BZ15)**

Assume $\mathcal{F}$ is phase retrievable for $H = \mathbb{C}^n$ and $A, B$ are its optimal frame bounds. Then:

1. For every $0 \neq z \in \mathbb{C}^n$, $A(z) = \lambda_{2n-1} \left( S(\zeta) \right)$ (the next to the smallest eigenvalue);
2. $A_0 = A(0) > 0$;
3. For every $z \in \mathbb{C}^n$, $\tilde{A}(z) = \lambda_{2n-1} \left( S(\zeta) + \sum_{k: \langle z, f_k \rangle = 0} \Phi_k \right)$ (the next to the smallest eigenvalue);
4. $\tilde{A}(0) = A$, the optimal lower frame bound;
5. For every $z \in \mathbb{C}^n$, $B(z) = \tilde{B}(z) = \lambda_1 \left( S(\zeta) + \sum_{k: \langle z, f_k \rangle = 0} \Phi_k \right)$ (the largest eigenvalue);
6. $B_0 = B(0) = \tilde{B}(0) = B$, the optimal upper frame bound;
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**Theorem (cont’d)**

7. For every $0 \neq z \in \mathbb{C}^n$, $a(z) = \tilde{a}(z) = \lambda_{2n-1}(\mathcal{R}(\zeta))/\|z\|^2$ (the next to the smallest eigenvalue);

8. For every $0 \neq z \in \mathbb{C}^n$, $b(z) = \tilde{b}(z) = \lambda_1(\mathcal{R}(\zeta))/\|z\|^2$ (the largest eigenvalue);

9. $a_0 = \min_{\|\xi\|=1} \lambda_{2n-1}(\mathcal{R}(\xi))$ is also the largest constant to that $\mathcal{R}(\xi) \geq a_0(\|\xi\|^2 I - J\xi\xi^T J^T)$;

10. $b(0) = \tilde{b}(0) = b_0 = \max_{\|\xi\|=1} \lambda_1(\mathcal{R}(\xi))$ is also the 4th power of the frame analysis operator norm $T : (\mathbb{C}^n, \|\cdot\|_2) \rightarrow (\mathbb{R}^m, \|\cdot\|_4)$: $b_0 = \|T\|_{B(l^2,l^4)}^4 = \max_{\|x\|_2=1} \sum_{k=1}^{m} |\langle x, f_k \rangle|^4$;

11. $\tilde{a}(0)$ is given by $\tilde{a}(0) = \min_{\|z\|=1} \sum_{k=1}^{m} |\langle z, f_k \rangle|^4$. 

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References


