Lipschitz analysis of the phase retrieval problem

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Problem Formulation

The phase retrieval problem

- Hilbert space $H = \mathbb{C}^n$, $\hat{H} = H/T^1$, frame $\mathcal{F} = \{f_1, \cdots, f_m\} \subset \mathbb{C}^n$ and

$$\alpha : \hat{H} \to \mathbb{R}^m \quad , \quad \alpha(x) = (|\langle x, f_k \rangle|)_{1 \leq k \leq m} .$$

$$\beta : \hat{H} \to \mathbb{R}^m \quad , \quad \beta(x) = \left(|\langle x, f_k \rangle|^2 \right)_{1 \leq k \leq m} .$$

  The frame is said *phase retrievable* (or that it gives phase retrieval) if α (or β) is injective.
- The general *phase retrieval problem* a.k.a. *phaseless reconstruction*: Decide when a given frame is phase retrievable, and, if so, find an algorithm to recover x from $y = \alpha(x)$ (or from $y = \beta(x)$) up to a global phase factor.

Problem Formulation
Lipschitz Reconstruction

Assume $\mathcal{F}$ is phase retrievable.
Our Problems Today:

1. Are the nonliner maps α, β bi-Lipschitz with respect to appropriate metrics?
2. Do they admit left inverses that are globally Lipschitz?
3. What are the Lipschitz constants? What is the structure of local Lipschitz bounds?
4. What is the average performance of any reconstruction scheme (Cramer-Rao Lower Bounds)?

1-3: Worst Case Performance
4: Average Case Performance

Metric Space Structures

Topological Structures

Let $H = \mathbb{C}^n$. The quotient space $\hat{H} = \mathbb{C}^n / T^1$, with classes induced by $x \sim y$ if there is real φ with $x = e^{i\varphi} y$.
Topologically:

$$\hat{\mathbb{C}}^n = \{0\} \cup \left((0, \infty) \times \mathbb{CP}^{n-1} \right)$$

with

$$\mathring{\hat{\mathbb{C}}}^n = \hat{\mathbb{C}}^n \setminus \{0\} = (0, \infty) \times \mathbb{CP}^{n-1}$$

a real analytic manifold of real dimension $2n - 1$.
Another embedding is into the space of symmetric matrices $Sym(\mathbb{C}^n)$.
Specifically let

$$\mathcal{S}^{1,0}(H) = \{ T = xx^* \in Sym(H) \ , \ x \in \mathbb{C}^n \}$$

Then:

$$\kappa_\beta : \hat{H} \rightarrow \mathcal{S}^{1,0} \quad , \quad \hat{x} \mapsto = xx^* \quad , \quad \text{is an embedding.}$$

Metric Space Structures

The matrix norm-induced metric structure

Fix $1 \leq p \leq \infty$. The *matrix-norm induced distance*

$$d_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ \ d_p(\hat{x}, \hat{y}) = \|xx^* - yy^*\|_p$$

with the p-norm of the singular values. In the case $p = 2$ we obtain

$$d_2(x, y) = \sqrt{\|x\|^4 + \|y\|^4 - 2|\langle x, y\rangle|^2}$$

Lemma (BZ15)

1. *$(d_p)_{1\leq p\leq\infty}$ are equivalent metrics and the identity map $i : (\hat{H}, d_p) \to (\hat{H}, d_q)$, $i(x) = x$ has Lipschitz constant*

$$Lip^d_{p,q,n} = \max(1, 2^{\frac{1}{q} - \frac{1}{p}}).$$

2. *The metric space $(\hat{H}, d_p)$ is isometrically isomorphic to $\mathcal{S}^{1,0}$ endowed with the p-norm via $\kappa_\beta : \hat{H} \to \mathcal{S}^{1,0}$, $x \mapsto \kappa_\beta(x) = xx^*$.*

Metric Space Structures
The natural metric structure

Fix $1 \leq p \leq \infty$. The *natural metric*

$$D_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ \ D_p(\hat{x},\hat{y}) = \min_{\varphi} \left\| x - e^{i\varphi} y \right\|_p$$

with the usual p-norm on $\mathbb{C}^n$. In the case $p=2$ we obtain

$$D_2(\hat{x},\hat{y}) = \sqrt{\|x\|^2 + \|y\|^2 - 2|\langle x, y\rangle|}$$

Lemma (BZ15)

1. $(D_p)_{1\leq p\leq\infty}$ *are equivalent metrics and the identity map* $i : (\hat{H}, D_p) \to (\hat{H}, D_q)$, $i(x) = x$ *has Lipschitz constant*

$$Lip^{D}_{p,q,n} = \max(1, n^{\frac{1}{q}-\frac{1}{p}}).$$

2. *The metric space* $(\hat{H}, D_2)$ *is Lipschitz isomorphic to* $\mathring{S}^{1,0}$ *endowed with the* 2*-norm via* $\kappa_\alpha : \hat{H} \to \mathring{S}^{1,0}$, $x \mapsto \kappa_\alpha(x) = \frac{1}{\|x\|} xx^*$.

Metric Space Structures
Distinct Structures

Two different structures: topologically equivalent, BUT the metrics are NOT equivalent:

Lemma (BZ15)

The identity map $i : (\hat{H}, D_p) \rightarrow (\hat{H}, d_p)$, $i(x) = x$ is continuous but it is not Lipschitz continuous. Likewise, the identity map $i : (\hat{H}, d_p) \rightarrow (\hat{H}, D_p)$, $i(x) = x$ is continuous but it is not Lipschitz continuous. Hence the induced topologies on $(\hat{H}, D_p)$ and $(\hat{H}, d_p)$ are the same, but the corresponding metrics are not Lipschitz equivalent.

Main Results

Lipschitz inversion: α

Theorem (BZ15)

Assume $\mathcal{F}$ is a phase retrievable frame for H. Then:

1. *The map $\alpha : (\hat{H}, D_2) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz. Let $\sqrt{A_0}, \sqrt{B_0}$ denote its Lipschitz constants: for every $x, y \in H$:*

$$A_0 \min_{\varphi} \|x - e^{i\varphi} y\|_2^2 \leq \sum_{k=1}^{m} \left| |\langle x, f_k\rangle| - |\langle y, f_k\rangle| \right|^2 \leq B_0 \min_{\varphi} \|x - e^{i\varphi} y\|_2^2.$$

2. *There is a Lipschitz map $\omega : (\mathbb{R}^m, \|\cdot\|_2) \to (\hat{H}, D_2)$ so that: (i) $\omega(\alpha(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $Lip(\omega) \leq \frac{4+3\sqrt{2}}{\sqrt{A_0}} = \frac{8.24}{\sqrt{A_0}}$.*

Main Results
Lipschitz inversion: β

Theorem (BZ15)

Assume $\mathcal{F}$ is a phase retrievable frame for H. Then:

1. *The map $\beta : (\hat{H}, d_1) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz. Let $\sqrt{a_0}, \sqrt{b_0}$ denote its Lipschitz constants: for every $x, y \in H$:*

$$a_0 \|xx^* - yy^*\|_1^2 \leq \sum_{k=1}^{m} \left| |\langle x, f_k \rangle|^2 - |\langle y, f_k \rangle|^2 \right|^2 \leq b_0 \|xx^* - yy^*\|_1^2.$$

2. *There is a Lipschitz map $\psi : (\mathbb{R}^m, \|\cdot\|_2) \to (\hat{H}, d_1)$ so that: (i) $\psi(\beta(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $Lip(\psi) \leq \frac{4+3\sqrt{2}}{\sqrt{a_0}} = \frac{8.24}{\sqrt{a_0}}$.*

Main Results
Statistical models

- A general noisy measurement process is given by:

$$y_k = |\langle x, f_k\rangle + \mu_k|^p + \nu_k \quad , \quad 1 \leq k \leq m,$$

where $(\mu_k)_k, (\nu_k)_k$ are two noise processes.

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- AWGN Model: $\mu_k = 0$, $p = 2$ and $\nu_k \sim \mathbb{N}(0, \sigma^2)$ i.i.d.

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- Non-AWGN Model: $\mu_k \sim \mathbb{CN}(0, \rho^2)$, i.i.d. and $\nu_k = 0$,
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Want:

1) Fisher Information Matrix $\mathbb{I} = \mathbb{E}\left[(\nabla_x \, log \, p(y;x))(\nabla_x \, log \, p(y;x))^*\right]$.

2) Cramer-Rao Lower Bounds for unbiased estimators.

Main Results

Fisher Information Matrix

$$\mathbb{I}^{AWGN,real}(x) = \frac{4}{\sigma^2}\sum_{k=1}^{m} |\langle x, f_k\rangle|^2 f_k f_k^T = \frac{4}{\sigma^2}\sum_{k=1}^{m}(f_k f_k^T)xx^T(f_k f_k^T) \quad \text{[Bal12]}.$$

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$$\mathbb{I}^{AWGN,cplx}(x) = \frac{4}{\sigma^2}\sum_{k=1}^{m} \Phi_k \xi\xi^* \Phi_k \quad \text{[Bal13, BCMN13]}.$$

Main Results

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$$\mathbb{I}^{AWGN,real}(x) = \frac{4}{\sigma^2}\sum_{k=1}^{m} |\langle x, f_k\rangle|^2 f_k f_k^T = \frac{4}{\sigma^2}\sum_{k=1}^{m}(f_k f_k^T) x x^T (f_k f_k^T) \quad \text{[Bal12]}.$$

$$\mathbb{I}^{AWGN,cplx}(x) = \frac{4}{\sigma^2}\sum_{k=1}^{m} \Phi_k \xi \xi^* \Phi_k \quad \text{[Bal13, BCMN13]}.$$

$$\begin{aligned}\mathbb{I}^{nonAWGN,cplx}(x) &= \frac{4}{\rho^4}\sum_{k=1}^{m}\left(G_1\left(\frac{\langle \Phi_k\xi,\xi\rangle}{\rho^2}\right)-1\right)\Phi_k\xi\xi^*\Phi_k \\ &= \frac{4}{\rho^2}\sum_{k=1}^{m} G_2\left(\frac{\langle \Phi_k\xi,\xi\rangle}{\rho^2}\right)\frac{1}{\langle \Phi_k\xi,\xi\rangle}\Phi_k\xi\xi^*\Phi_k \quad \text{[Bal15]}.\end{aligned}$$

where

$$G_1(a) = \frac{e^{-a}}{8a^3}\int_0^\infty \frac{I_1^2(t)}{I_0(t)} t^3 e^{-\frac{t^2}{4a}} dt \quad , \quad G_2(a) = a(G_1(a)-1).$$

Main Results

AWGN vs. non-AWGN: Comparisons and Identifiability

Let B be the frame upper bound.

Lemma

$$\frac{\sigma^2}{\rho^4}\left(G_1\left(\frac{B\|x\|^2}{\rho^2}\right)-1\right)\mathbb{I}^{AWGN,cplx}(x) \leq \mathbb{I}^{nonAWGN,cplx}(x) \leq \frac{\sigma^2}{\rho^4}\mathbb{I}^{AWGN,cplx}(x)$$

Main Results

AWGN vs. non-AWGN: Comparisons and Identifiability

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$$\frac{\sigma^2}{\rho^4}\left(G_1\left(\frac{B\|x\|^2}{\rho^2}\right)-1\right)\mathbb{I}^{AWGN,cplx}(x)\leq\mathbb{I}^{nonAWGN,cplx}(x)\leq\frac{\sigma^2}{\rho^4}\mathbb{I}^{AWGN,cplx}(x)$$

Theorem

The following are equivalent:

1. *The frame $\mathcal{F}$ is phase retrievable;*
2. *For every $0\neq x\in\mathbb{C}^n$, $rank(\mathbb{I}^{nonAWGN,cplx}(x))=2n-1$;*
3. *For every $0\neq x\in\mathbb{C}^n$, $rank(\mathbb{I}^{AWGN,cplx}(x))=2n-1$;*

Main Results
The Cramer-Rao Lower Bound

Fix $z_0 \in \mathbb{C}^n$, $\|z_0\| = 1$, let $\zeta_0 = [real(z_0)\ imag(z_0)]^T$ and set

$$\Omega_{z_0} = \{\xi \in \mathbb{R}^{2n}\ ,\ \langle \xi, \zeta_0 \rangle) \geq 0, \langle \xi, J\zeta_0 \rangle) = 0\}.$$

Let $\Pi_{z_0} = 1 - J\zeta_0\zeta_0^* J^*$ with J the symplectic form matrix.

Theorem (Bal13,Bal16)

Assume a measurement model $y_k = |\langle x, f_k \rangle + \mu_k|^p + \nu_k$ *with* $\xi = [real(x)\ imag(x)]^T \in \mathring{\Omega}_{z_0}$. *Then the covariance of any unbiased estimator* $\omega : \mathbb{R}^m \to \mathbb{R}^{2n}$ *is bounded below by*

$$Cov[\omega(y); \xi] \geq (\Pi_{z_0} \mathbb{I}(\xi) \Pi_{z_0})^\dagger = L^T (\mathbb{I}(xi))^\dagger L$$

where $L = I - \frac{1}{\langle \xi, \zeta_0 \rangle} J\zeta_0 \xi^T J^T$. *If the reference vector happens to satisfy* $\zeta_0 = \frac{\xi}{\|\xi\|}$ *then*

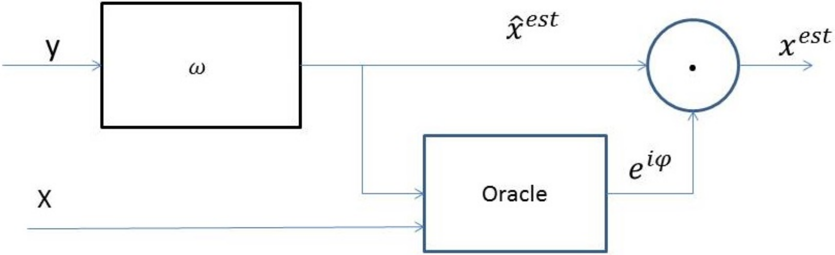
$$Cov[\omega(y); \xi] \geq (\mathbb{I}(\xi))^\dagger .$$

Main Results

The Cramer-Rao Lower Bound - The Oracle-Based Estimator

A different case is the oracle-based estimator:

$$x^{est} : \mathbb{R}^m \to \widehat{\mathbb{C}^n} \ , \ x^{est}(y) = e^{i\varphi(x,\hat{x}^{est})}\hat{x}^{est}$$



Main Results

The Cramer-Rao Lower Bound - The Oracle-Based Estimator (2)

Theorem

Consider the oracle based estimator. ω and assume it is unbiased. Then for any $\xi \neq 0$ the covariance matrix is bounded below as follows

$$Cov[\tilde{\omega}(y);\xi] \geq (I-\Delta)(\mathbb{I}(\xi))^{\dagger}(I-\Delta) \tag{3.1}$$

where

$$\begin{aligned}\Delta &= \mathbb{E}\Bigg[\frac{(\langle\omega,J\xi\rangle)^2}{((\langle\omega,\xi\rangle)^2+(\langle\omega,J\xi\rangle)^2)^{3/2}}\omega\omega^T \\ &+\frac{\langle\omega,\xi\rangle\langle\omega,J\xi\rangle}{((\langle\omega,\xi\rangle)^2+(\langle\omega,J\xi\rangle)^2)^{3/2}}(J\omega\omega^T+\omega\omega^T J^T) \\ &+\frac{(\langle\omega,\xi\rangle)^2}{((\langle\omega,\xi\rangle)^2+(\langle\omega,J\xi\rangle)^2)^{3/2}}J\omega\omega^T J^T\Bigg].\end{aligned} \tag{3.2}$$

The matrix Δ satisfies:

$$\Delta=\Delta^T \geq \frac{1}{\|\xi\|^2}J\xi\xi^T J^T \geq 0 \quad , \quad \Delta J\xi = J\xi \quad , \quad \Delta\xi = 0.$$

Main Results
Prior Works

Prior literature:

- 2012: **B.**: Cramer-Rao lower bound in the real case; **Eldar&Mendelson** : map α in the real case

$$\|\alpha(x) - \alpha(y)\| \geq C\|x - y\|\|x + y\|.$$

- 2013: **Bandeira,Cahill,Mixon,Nelson**: improved the estimate of C. **B.**: β bi-Lipschitz in real and complex case.
- 2014: **B.&Yang**: Find the exact Lipschitz constant for α in the real case - the constants A_0, B_0; **B.&Z.**:constructed a Lipschitz left inverse for β.
- 2015: **B.&Z.**: Proved α is bi-Lipschitz in the complex case; constructed a Lipschitz left inverse. **B.**: lower Lipschitz constant A_0 connected to CRLB of a non-AWGN model.

Main Results

Key relationship between deterministic and stochastic bounds

The central object: $\mathcal{R}(\xi) = \sum_{k=1}^{m} \Phi_k \xi \xi^T \Phi_k$.

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Key relationship between deterministic and stochastic bounds

The central object: $\mathcal{R}(\xi) = \sum_{k=1}^{m} \Phi_k \xi \xi^T \Phi_k$.
The lower Lipschitz bound for β map is:

$$a_0 = \min_{\|\xi\|=1} \lambda_{2n-1}(\mathcal{R}(\xi)).$$

The Fisher information matrix for the AWGN model:

$$\mathbb{I}^{AWGN,cplx}(x) = \frac{4}{\sigma^2}\mathcal{R}(\xi).$$

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The Fisher information matrix for the AWGN model:

$$\mathbb{I}^{AWGN,cplx}(x) = \frac{4}{\sigma^2}\mathcal{R}(\xi).$$

Best inversion scheme ψ that is lossless in the absence of noise achieves:

$$d_1(\psi(c), \psi(d))^2 \leq \frac{68}{a_0}\|c-d\|_2^2.$$

An efficient estimator (i.e. unbiased that achieves CRLB) ω^0 achieves:

$$\mathbb{E}\left[\|\omega^0(y) - x\|_2^2; x\right] \leq \frac{(2n-1)\sigma^2}{4a_0\|x\|^2} = \frac{2n-1}{4a_0\, SNR}.$$

Proofs
Overview

Deterministic bounds: The proofs involve several steps (details in [BZ15]).

1. Part 1: Injectivity $\longrightarrow$ bi-Lipschitz: Upper bounds are not too hard; lower bounds: relatively easy for β (the "square" map), but relatively hard for α.
2. Part 2: Left inverse construction is done in three steps:
   1. The left inverse is first extended to $\mathbb{R}^m$ into $Sym(H)$ using Kirszbraun's theorem;
   2. Then we show that $\mathcal{S}^{1,0}(H)$ is a Lipschitz retract in $Sym(H)$;
   3. The proof is concluded by composing the two maps.

The stochastic bounds: Direct computations and a bit of luck! [Bal15]

Proofs

Part 1a: Bi-Lipschitzianity of α

$$\alpha : \hat{H} \to \mathbb{R}^m \quad , \quad \alpha(x) = (|\langle x, f_k\rangle|)_{1\leq k\leq m}$$

The homogeneity of α shows that

$$L(x,y) = \frac{\|\alpha(x)-\alpha(y)\|}{D_p(x,y)}$$

is homogeneous of degree 0: $L(tx,ty) = L(x,y)$, for every $t > 0$.
This reduces the problem to the unit ball: $1 = \|x\| \geq \|y\|$.
The upper bound was computed in [BCMN13]:

$$\sup_{x\neq y} \frac{\|\alpha(x)-\alpha(y)\|^2}{D_2(x,y)^2} = B \ \ \text{(upper frame bound)}.$$

Proofs

Part 1a: Bi-Lipschitzianity of α - cont'd

A compactness argument shows the lower bound is positive if and only if the local lower bound is positive:

$$\inf_{\|z\|=1} \lim_{r\to 0} \inf_{\substack{x,y\in\hat{H} \\ D_2(x,z)<r \\ D_2(y,z)<r}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2} > 0.$$

This bound is computed explicitely and shown positive: Computations involve the realification framework and other delicate nonlinear expansions.

Proofs

Part 1b: Bi-Lipschitzianity of β

Key Remark (B.Bodmann,Casazza,Edidin - 2007): The nonlinear map β is the restriction of the linear map

$$\mathbb{A} : Sym(H) \to \mathbb{R}^m \quad , \quad \mathbb{A}(T) = (\langle Tf_k, f_k \rangle)_{1 \leq k \leq m}$$

Specifically: $\beta(x) = \mathbb{A}(xx^*) = (|\langle x, f_k \rangle|^2)_{1 \leq k \leq m}$.

$$\begin{aligned} \|\beta(x) - \beta(y)\| = \|\mathbb{A}(xx^*) - \mathbb{A}(yy^*)\| &= \|\mathbb{A}(xx^* - yy^*)\| \\ &= \|xx^* - yy^*\| \|\mathbb{A}\left(\frac{xx^* - yy^*}{\|xx^* - yy^*\|}\right)\| \end{aligned}$$

$$a_0 = \min_{T \in \mathcal{S}^{1,1}, \|T\|_1 = 1} \|\mathbb{A}(T)\| > 0 \quad , \quad b_0 = \max_{T \in \mathcal{S}^{1,1}, \|T\|_1 = 1} \|\mathbb{A}(T)\|$$

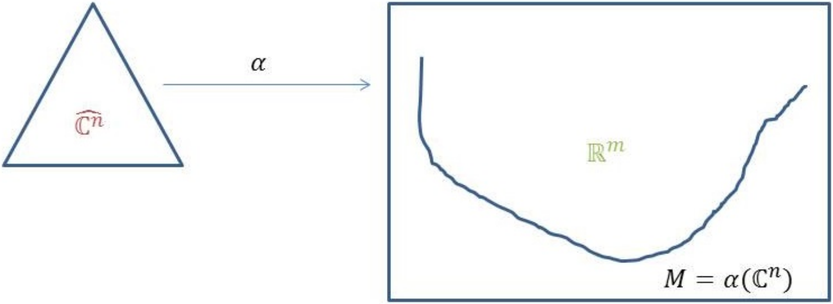
Proofs

Part 2a: Extension of the inverse for α

We know $\alpha : (\hat{H}, D_2) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz:

$$A_0 D_1(x,y)^2 \leq \|\alpha(x) - \alpha(y)\|^2 \leq B_0 D_2(x,y)^2$$

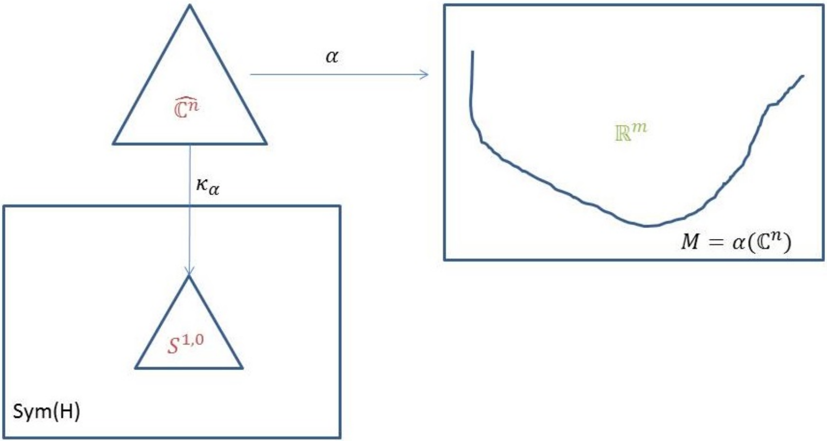
Let $M = \alpha(\hat{H}) \subset \mathbb{R}^m$.



Proofs

Part 2a: Extension of the inverse for α

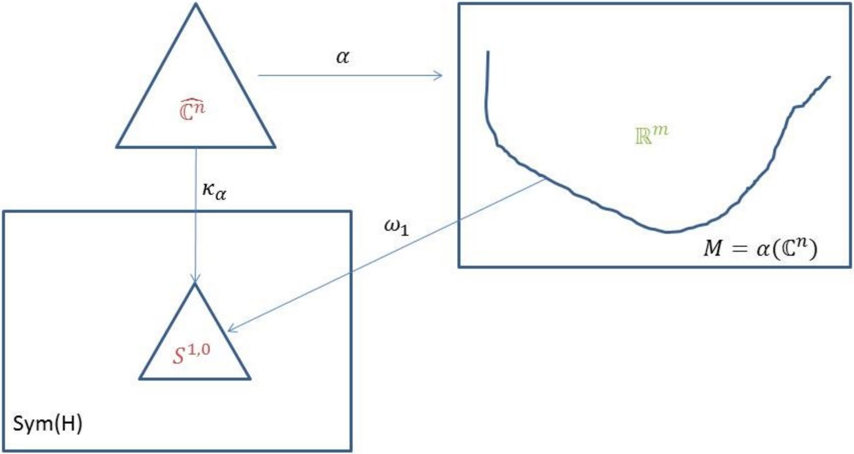
First identify $\hat{H}$ with $\mathcal{S}^{1,0}(H)$.



Proofs

Part 2a: Extension of the inverse for α

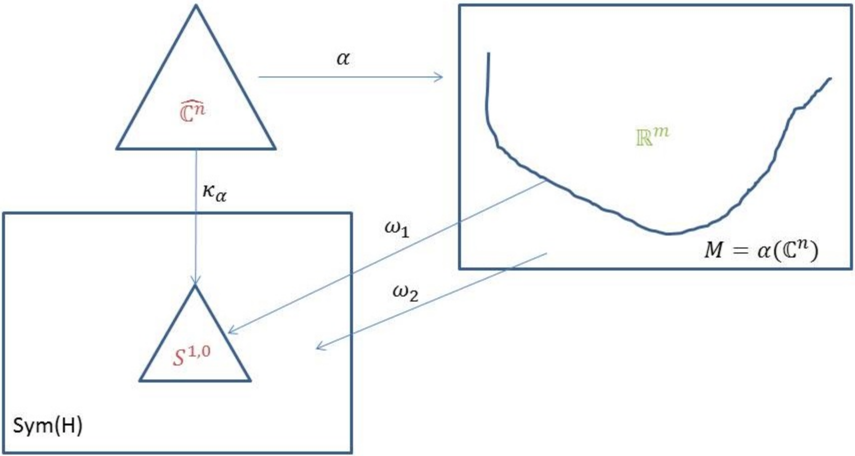
Then construct the local left inverse $\omega_1 : M \to \hat{H}$ with $Lip(\omega_1) = \frac{1}{\sqrt{A_0}}$.



Proofs

Part 2a: Extension of the inverse for α

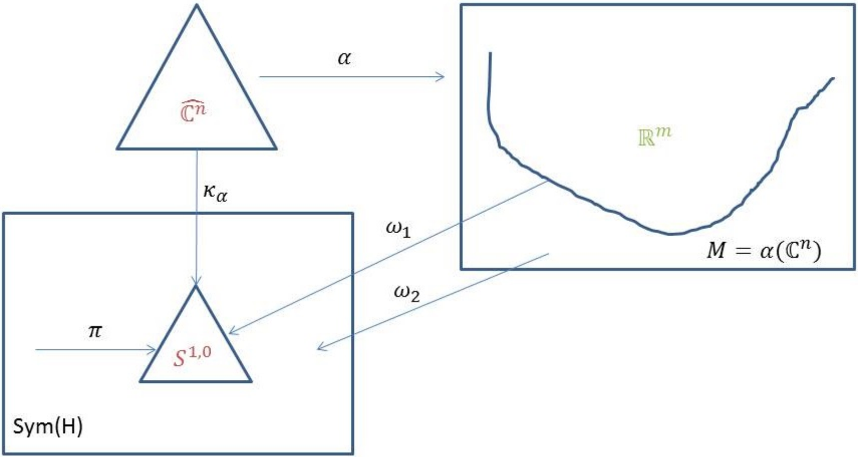
Use Kirszbraun's theorem to extend isometrically $\omega_2 : \mathbb{R}^m \to Sym(H)$.



Proofs

Part 2a: Extension of the inverse for α

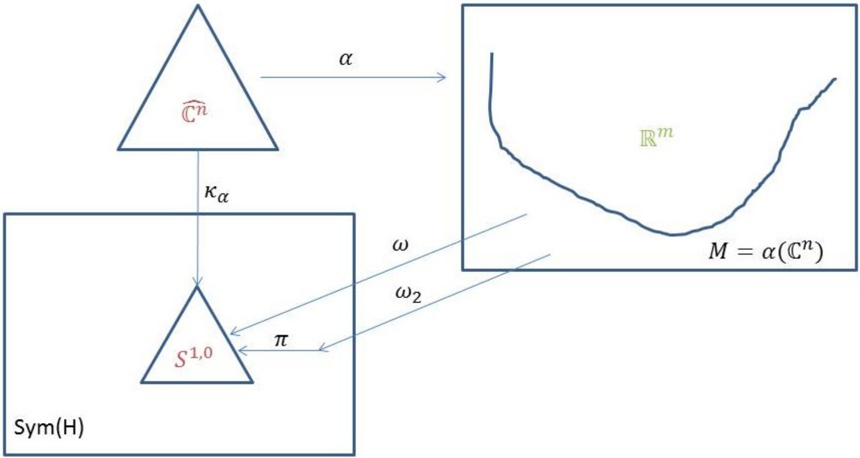
Construct a Lipschitz "projection" $\pi : Sym(H) \to \mathcal{S}^{1,0}(H)$.



Proofs

Part 2a: Extension of the inverse for α

Compose the two maps to get $\omega : \mathbb{R}^m \to \mathcal{S}^{1,0}$, $\omega = \pi \circ \omega_2$.



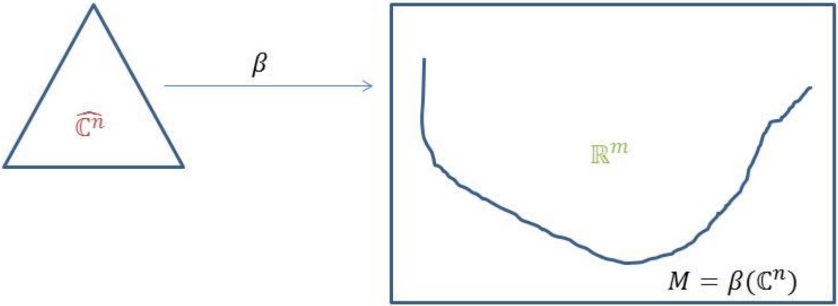
Proofs

Part 2b: Extension of the inverse for β

We know $\beta : (\hat{H}, d_1) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz:

$$a_0 d_1(x,y)^2 \le \|\beta(x) - \beta(y)\|^2 \le b_0 d_1(x,y)^2.$$

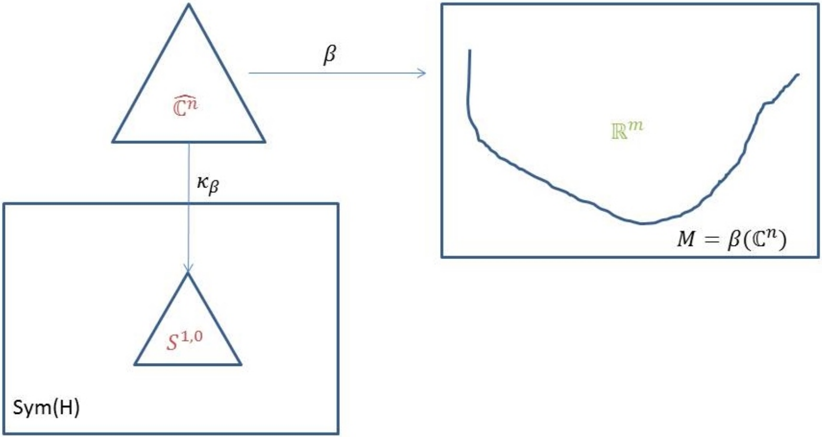
Let $M = \beta(\hat{H}) \subset \mathbb{R}^m$.



Proofs

Part 2b: Extension of the inverse for β

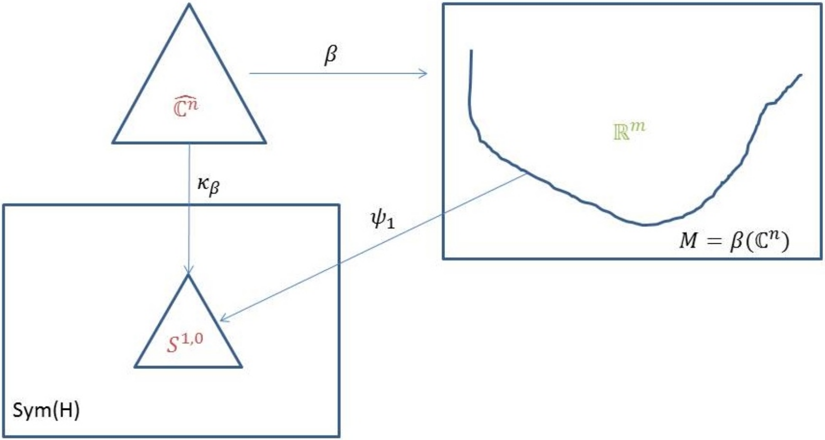
First identify $\hat{H}$ with $\mathcal{S}^{1,0}(H)$.



Proofs

Part 2b: Extension of the inverse for β

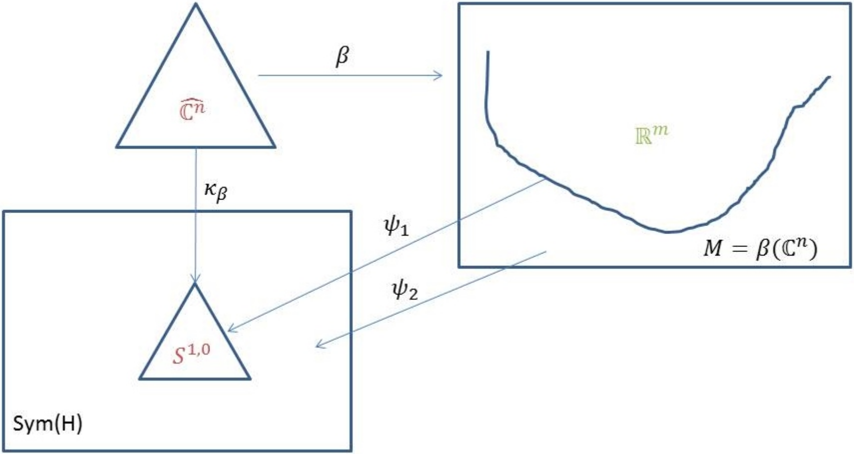
Then construct the local left inverse $\psi_1 : M \to \hat{H}$ with $Lip(\psi_1) = \frac{1}{\sqrt{a_0}}$.



Proofs

Part 2b: Extension of the inverse for β

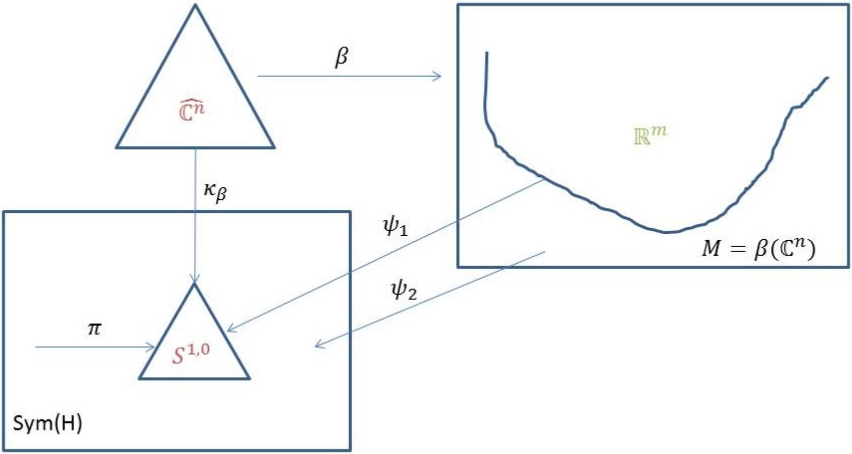
Use Kirszbraun's theorem to extend isometrically $\psi_2 : \mathbb{R}^m \to Sym(H)$.



Proofs

Part 2b: Extension of the inverse for β

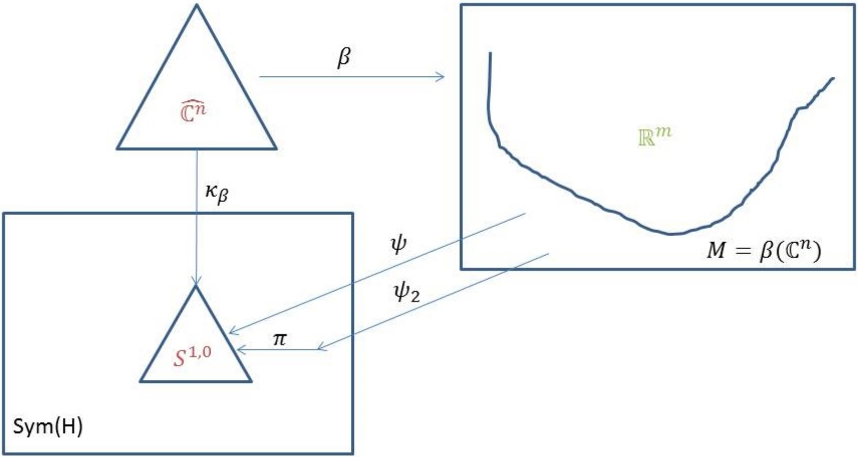
Construct a Lipschitz "projection" $\pi : Sym(H) \rightarrow \mathcal{S}^{1,0}(H)$.



Proofs

Part 2b: Extension of the inverse for β

Compose the two maps to get $\psi : \mathbb{R}^m \to \mathcal{S}^{1,0}$, $\psi = \pi \circ \psi_2$.



Proofs

Part 2: $\mathcal{S}^{1,0}(H)$ as Lipschitz retract in $Sym(H)$

Lemma

Consider the spectral decomposition of the self-adjoint operator A in $Sym(H)$, $A = \sum_{k=1}^{d} \lambda_{m(k)} P_k$. Then the map

$$\pi : Sym(H) \to \mathcal{S}^{1,0}(H) \quad , \quad \pi(A) = (\lambda_1 - \lambda_2) P_1$$

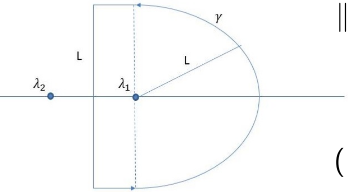
satisfies the following two properties:

1. *for $1 \leq p \leq \infty$, it is Lipschitz continuous from $(Sym(H), \|\cdot\|_p)$ to $(\mathcal{S}^{1,0}(H), \|\cdot\|_p)$ with Lipschitz constant less than or equal to $3 + 2^{1+\frac{1}{p}}$;*
2. *$\pi(A) = A$ for all $A \in \mathcal{S}^{1,0}(H)$.*

Proof uses Weyl's inequality and spectral formula on a complex integration contour by Zwald & Blanchard (2006).

Assume simple top eigenvalues (otherwise the bound is immediate):
$\pi(A) = (\lambda_1 - \lambda_2)P_1$, $\pi(B) = (\mu_1 - \mu_2)Q_1$. Then:

$$\begin{aligned}\|\pi(A) - \pi(B)\|_p &\leq (\lambda_1 - \lambda_2)\|P_1 - Q_1\|_p + |\lambda_1 - \mu_1| + |\lambda_2 - \mu_2| \\ &\leq (\lambda_1 - \lambda_2)\|P_1 - Q_1\|_p + 2\|A - B\|_p.\end{aligned}$$



$$\|P_1 - Q_1\|_p \leq \frac{1}{2\pi}\int_I \|(R_A - R_B)(\gamma(t))\|_p |\gamma'(t)| dt$$

$$R_A(z) = (A - zI)^{-1},\ R_B(z) = (B - zI)^{-1}.$$

$$(R_A - R_B)(z) = \sum_{n\geq 1}(-1)^n (R_A(z)(B - A))^n R_A(z).$$

$$\begin{aligned}\|(R_A - R_B)(\gamma(t))\|_p &\leq \sum_{n\geq 1}\|R_A(\gamma(t))\|_\infty^{n+1}\|A - B\|_p^n \\ &= \frac{\|R_A(\gamma(t))\|_\infty^2 \|A - B\|_p}{1 - \|R_A(\gamma(t))\|_\infty \|A - B\|_p} < \frac{\|A - B\|_p}{dist^2(\gamma(t), Spec(A))}.\end{aligned}$$

Proofs

Part 1: Bi-Lipschitzianity of α-cont'd

The analysis requires a deeper understanding of local behavior.

1. The *global lower* and *upper Lipschitz bounds*:

$$A_0 = \inf_{x,y\in\hat{H}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2} \ , \ B_0 = \sup_{x,y\in\hat{H}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2}$$

2. The *type I local lower* and *upper Lipschitz bounds* at $z\in\hat{H}$:

$$A(z) = \lim_{r\to 0} \inf_{\substack{x,y\in\hat{H} \\ D_2(x,z)<r \\ D_2(y,z)<r}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2} \ , \ B(z) = \lim_{r\to 0} \sup_{\substack{x,y\in\hat{H} \\ D_2(x,z)<r \\ D_2(y,z)<r}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2}$$

3. The *type II local lower* and *upper Lipschitz bounds* at $z\in\hat{H}$:

$$\tilde{A}(z) = \lim_{r\to 0} \inf_{\substack{x\in\hat{H} \\ D_2(x,z)<r}} \frac{\|\alpha(x)-\alpha(z)\|_2^2}{D_2(x,z)^2} \ , \ \tilde{B}(z) = \lim_{r\to 0} \sup_{\substack{x\in\hat{H} \\ D_2(x,z)<r}} \frac{\|\alpha(x)-\alpha(z)\|_2^2}{D_2(x,y)^2}$$

Proofs

Part 1: Bi-Lipschitzianity of α -cont'd

We need to analyze the real structure of $\hat{H}$.
Let $\varphi_1, \cdots, \varphi_m, \zeta \in \mathbb{R}^{2n}$, $\Phi_1, \cdots, \Phi_m \in Sym(\mathbb{R}^{2n})$, $J \in \mathbb{R}^{2n \times 2n}$ defined by:

$$\Phi_k = \varphi_k \varphi_k^T + J \varphi_k \varphi_k^T J^T, \varphi_k = \begin{bmatrix} real(f_k) \\ imag(f_k) \end{bmatrix}, J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \zeta = \begin{bmatrix} real(z) \\ imag(z) \end{bmatrix}$$

Key relations: $\langle z, f_k \rangle = \langle \zeta, \varphi_k \rangle + i \langle \zeta, J\varphi_k \rangle$, $|\langle z, f_k \rangle| = \sqrt{\langle \Phi_k \zeta, \zeta \rangle}$.
Consider the following objects:

$$\mathcal{R} : \mathbb{R}^{2n} \to Sym(\mathbb{R}^{2n}) \quad , \quad \mathcal{R}(\xi) = \sum_{k=1}^{m} \Phi_k \xi \xi^T \Phi_k \ , \ \xi \in \mathbb{R}^{2n}$$

$$\mathcal{S} : \mathbb{R}^{2n} \to Sym(\mathbb{R}^{2n}) \quad , \quad \mathcal{S}(\xi) = \sum_{k : \Phi_k \xi \neq 0} \frac{1}{\langle \Phi_k \xi, \xi \rangle} \Phi_k \xi \xi^T \Phi_k \ , \ \xi \in \mathbb{R}^{2n}$$

Proofs

Lipschitz bounds for α

Theorem (BZ15)

Assume $\mathcal{F}$ is phase retrievable for $H=\mathbb{C}^n$ and A, B are its optimal frame bounds. Then:

1. *For every $0 \neq z \in \mathbb{C}^n$, $A(z)=\lambda_{2n-1}\left(\mathcal{S}(\zeta)\right)$ (the next to the smallest eigenvalue);*
2. $A_0 = A(0) > 0$;
3. *For every $z \in \mathbb{C}^n$, $\tilde{A}(z)=\lambda_{2n-1}\left(\mathcal{S}(\zeta)+\sum_{k:\langle z,f_k\rangle=0}\Phi_k\right)$ (the next to the smallest eigenvalue);*
4. *$\tilde{A}(0)=A$, the optimal lower frame bound;*
5. *For every $z \in \mathbb{C}^n$, $B(z)=\tilde{B}(z)=\lambda_1\left(\mathcal{S}(\zeta)+\sum_{k:\langle z,f_k\rangle=0}\Phi_k\right)$ (the largest eigenvalue);*
6. *$B_0=B(0)=\tilde{B}(0)=B$, the optimal upper frame bound;*

Proofs

Lipschitz bounds for β

Theorem (cont'd)

7. *For every* $0 \neq z \in \mathbb{C}^n$, $a(z) = \tilde{a}(z) = \lambda_{2n-1}(\mathcal{R}(\zeta))/\|z\|^2$ *(the next to the smallest eigenvalue);*
8. *For every* $0 \neq z \in \mathbb{C}^n$, $b(z) = \tilde{b}(z) = \lambda_1(\mathcal{R}(\zeta))/\|z\|^2$ *(the largest eigenvalue);*
9. $a_0 = \min_{\|\xi\|=1} \lambda_{2n-1}(\mathcal{R}(\xi))$ *is also the largest constant to that* $\mathcal{R}(\xi) \geq a_0(\|\xi\|^2 I - J\xi\xi^T J^T)$*;*
10. $b(0) = \tilde{b}(0) = b_0 = \max_{\|\xi\|=1} \lambda_1(\mathcal{R}(\xi))$ *is also the* 4^{th} *power of the frame analysis operator norm* $T : (\mathbb{C}^n, \|\cdot\|_2) \rightarrow (\mathbb{R}^m, \|\cdot\|_4)$*:* $b_0 = \|T\|^4_{B(l^2,l^4)} = \max_{\|x\|_2=1} \sum_{k=1}^m |\langle x, f_k\rangle|^4$*;*
11. $\tilde{a}(0)$ *is given by* $\tilde{a}(0) = \min_{\|z\|=1} \sum_{k=1}^m |\langle z, f_k\rangle|^4$.

References

[BCE06] R. Balan, P. Casazza, D. Edidin, On signal reconstruction without phase, Appl.Comput.Harmon.Anal. **20** (2006), 345–356.

[BBCE07] R. Balan, B. Bodmann, P. Casazza, D. Edidin, Painless reconstruction from Magnitudes of Frame Coefficients, J.Fourier Anal.Applic., **15** (4) (2009), 488–501.

[B12] R. Balan, Reconstruction of Signals from Magnitudes of Frame Representations, arXiv submission arXiv:1207.1134

[B13] R. Balan, Reconstruction of Signals from Magnitudes of Redundant Representations: The Complex Case, available online arXiv:1304.1839v1, Found.Comput.Math. 2015, http://dx.doi.org/10.1007/s10208-015-9261-0

[BW14] R. Balan and Y. Wang, Invertibility and Robustness of Phaseless Reconstruction, available online arXiv:1308.4718v1, Appl. Comp. Harm. Anal., 38 (2015), 469–488.

[BZ14] R. Balan, D. Zou, On Lipschitz inversion of nonlinear redundant representations, to appear in Contemporary Mathematics 2015.

[BZ15] R. Balan, D. Zou, On Lipschitz Analysis and Lipschitz Synthesis for the Phase Retrieval Problem, available online arXiv 1506.02092v1 [mathFA], 6 June 2015; to appear in Lin. Alg. and Appl. 2016.

[Bal15] R. Balan, The Fisher Information Matrix and the Cramer-Rao Lower Bound in a Non-Additive White Gaussian Noise Model for the Phase Retrieval Problem, proceedings of SampTA 2015.

[BCMN13] A. S. Bandeira, J. Cahill, D. Mixon, A. A. Nelson, Saving phase: Injectivity and Stability for phase retrieval, arXiv submission , arXiv: 1302.4618, Appl. Comp. Harm. Anal. 37 (1) (2014), 106–125.

[EM12] Y. C. Eldar, S. Mendelson, *Phase retrieval: Stability and recovery guarantees*, available online: arXiv:1211.0872, Appl. Comp. Harm. Anal. (2014), .

[HG13] M.J. Hirn, E. Le Gruyer, *A general theorem of existence of quasi absolutely minimal Lipschitz extensions*, arXiv:1211.5700v2 [math.FA], 8 Aug 2013.

[ZB06] L. Zwald, G. Blanchard, *On the convergence of eigenspaces in kernel Principal Component Analysis*, Proc. NIPS 05, vol. 18, 1649-1656, MIT Press, 2006.