Lipschitz Properties of General Convolutional Neural Networks

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Consider a nonlinear function between two metric spaces,

\[ \mathcal{F} : (X, d_X) \rightarrow (Y, d_Y). \]
Problem Formulation

Lipschitz analysis of nonlinear systems

\[ \mathcal{F}: (X, d_X) \rightarrow (Y, d_Y) \]

\( \mathcal{F} \) is called *Lipschitz* with constant \( C \) if for any \( f, \tilde{f} \in X \),

\[ d_Y(\mathcal{F}(f), \mathcal{F}(\tilde{f})) \leq C \, d_X(f, \tilde{f}) \]

The optimal (i.e. smallest) Lipschitz constant is denoted \( \text{Lip}(\mathcal{F}) \). The square \( C^2 \) is called Lipschitz bound (similar to the Bessel bound).

\( \mathcal{F} \) is called *bi-Lipschitz* with constants \( C_1, C_2 > 0 \) if for any \( f, \tilde{f} \in X \),

\[ C_1 \, d_X(f, \tilde{f}) \leq d_Y(\mathcal{F}(f), \mathcal{F}(\tilde{f})) \leq C_2 \, d_X(f, \tilde{f}) \]

The square \( C_1^2, C_2^2 \) are called *Lipschitz bounds* (similar to frame bounds).
Consider the typical neural network as a feature extractor component in a classification system:

\[ g = \mathcal{F}(f) = \mathcal{F}_M(...\mathcal{F}_1(f; W_1, \varphi_1);...; W_M, \varphi_M) \]

\[ \mathcal{F}_m(f; W_m, \varphi_m) = \varphi_m(W_m f) \]

\( W_m \) is a linear operator (matrix); \( \varphi_m \) is a Lip(1) scalar nonlinearity (e.g. Rectified Linear Unit).
Example from [SZSBEGF13] ('Intriguing properties ...'). ImageNet dataset [DDSLLF09] (10,184 categories; 8.9 mil.img.); **AlexNet architecture [KSH12]:**

![AlexNet Architecture Diagram](image)

**Figure:** From Krizhevsky et al. 2012 [KSH12]: AlexNet: 5 convolutive layers + 3 dense layers. Input size: 224x224x3 pixels. Output size: 1000.
Problem Formulation

Motivating Example: AlexNet

The authors of [SZSBEGF13] (‘Intriguing properties …’) found small variations of the input, almost imperceptible, that produced completely different classification decisions:

![Figure: From Szegedy et all 2013 [SZSBEGF13]: AlexNet: 6 different classes: original image, difference, and adversarial example – all classified as ’ostrich’](image)
Szegedy et al. 2013 [SZSBEGF13] computed the Lipschitz constants of each layer.

<table>
<thead>
<tr>
<th>Layer</th>
<th>Size</th>
<th>Sing. Val</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conv. 1</td>
<td>$3 \times 11 \times 11 \times 96$</td>
<td>20</td>
</tr>
<tr>
<td>Conv. 2</td>
<td>$96 \times 5 \times 5 \times 256$</td>
<td>10</td>
</tr>
<tr>
<td>Conv. 3</td>
<td>$256 \times 3 \times 3 \times 384$</td>
<td>7</td>
</tr>
<tr>
<td>Conv. 4</td>
<td>$384 \times 3 \times 3 \times 384$</td>
<td>7.3</td>
</tr>
<tr>
<td>Conv. 5</td>
<td>$384 \times 3 \times 3 \times 256$</td>
<td>11</td>
</tr>
<tr>
<td>Fully Conn.1</td>
<td>$9216(43264) \times 4096$</td>
<td>3.12</td>
</tr>
<tr>
<td>Fully Conn.2</td>
<td>$4096 \times 4096$</td>
<td>4</td>
</tr>
<tr>
<td>Fully Conn.3</td>
<td>$4096 \times 1000$</td>
<td>4</td>
</tr>
</tbody>
</table>

Overall Lipschitz constant:

$$\text{Lip} \leq 20 \times 10 \times 7 \times 7.3 \times 11 \times 3.12 \times 4 \times 4 = 5,612,006$$
Problem Formulation
Motivating Example: Scattering Network

Example of Scattering Network; definition and properties: [Mallat12]; this example from [BSZ17]:

Input: $f$; Outputs: $y = (y_{l,k})$. 
Problem Formulation
Motivating Example: Scattering Network

Remarks:
- Outputs from each layer
Problem Formulation
Motivating Example: Scattering Network

Remarks:
- Outputs from each layer
- Tree-like topology
Problem Formulation
Motivating Example: Scattering Network

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- Backpropagation/Chain rule: Lipschitz bound 40.
Problem Formulation
Motivating Example: Scattering Network

Remarks:
- Outputs from each layer
- Tree-like topology
- Backpropagation/Chain rule: Lipschitz bound 40.
- Mallat’s result predicts $Lip = 1$. 

Problem Formulation

Problem 1

Given a deep network:

Estimate the Lipschitz constant, or bound:

$$Lip = \sup_{f \neq \tilde{f} \in L^2} \frac{\|y - \tilde{y}\|_2}{\|f - \tilde{f}\|_2}, \quad Bound = \sup_{f \neq \tilde{f} \in L^2} \frac{\|y - \tilde{y}\|_2^2}{\|f - \tilde{f}\|_2^2}.$$
Problem Formulation

Problem 1

Given a deep network:

Estimate the Lipschitz constant, or bound:

\[ \text{Lip} = \sup_{f \neq \tilde{f} \in L^2} \frac{\|y - \tilde{y}\|_2}{\|f - \tilde{f}\|_2}, \quad \text{Bound} = \sup_{f \neq \tilde{f} \in L^2} \frac{\|y - \tilde{y}\|^2_2}{\|f - \tilde{f}\|^2_2}. \]

Methods (Approaches):

1. **Standard Method**: Backpropagation, or chain-rule
2. **New Method**: Storage function based approach (dissipative systems)
3. **Numerical Method**: Simulations
Problem Formulation

Problem 2

Given a deep network:

![Diagram of a deep network with layers 1 to M, input f, and output y.]

Estimate the stability of the output to specific variations of the input:

1. Invariance to deformations: \( \tilde{f}(x) = f(x - \tau(x)) \), for some smooth \( \tau \).
2. Covariance to such deformations \( \tilde{f}(x) = f(x - \tau(x)) \), for smooth \( \tau \) and bandlimited signals \( f \);
3. Tail bounds when \( f \) has a known statistical distribution (e.g. normal with known spectral power)
A deep convolution network is composed of multiple layers:
ConvNet
One Layer

Each layer is composed of two or three sublayers: convolution, downsampling, detection/pooling/merge.
ConvNet: Sublayers
Linear Filters: Convolution and Pooling-to-Output Sublayer

\[ f^{(2)} = g * f^{(1)} , \quad f^{(2)}(x) = \int g(x - \xi) f^{(1)}(\xi) d\xi \]

where \( g \in \mathcal{B} = \{ g \in S', \hat{g} \in L^\infty(\mathbb{R}^d) \} \).

\((\mathcal{B}, *)\) is a Banach algebra with norm \( \| g \|_\mathcal{B} = \| \hat{g} \|_\infty \).
Notation: \( g \) for regular convolution filters, and \( \Phi \) for pooling-to-output filters.
**Problem Formulation**

**Deep Convolutional Neural Networks**

**Lipschitz Analysis**

**Numerical Results**

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**ConvNet: Sublayers**

**Downsampling Sublayer**

\[
\begin{align*}
f^{(1)}(x) & = f^{(1)}(Dx) \\
\|f^{(2)}\|^2_2 &= \int |f^{(2)}(x)|^2 dx = \frac{1}{|\text{det}(D)|} \int |f^{(1)}(x)|^2 dx = \frac{1}{D_0^d} \|f^{(1)}\|^2_2
\end{align*}
\]
ConvNet: Sublayers
Detection and Pooling Sublayer

We consider three types of detection/pooling/merge sublayers:

- **Type I, $\tau_1$:** Componentwise Addition: $z = \sum_{j=1}^{k} \sigma_j(y_j)$
- **Type II, $\tau_2$:** $p$-norm aggregation: $z = \left(\sum_{j=1}^{k} |\sigma_j(y_j)|^p\right)^{1/p}$
- **Type III, $\tau_3$:** Componentwise Multiplication: $z = \prod_{j=1}^{k} \sigma_j(y_j)$

Assumptions: (1) $\sigma_j$ are scalar Lipschitz functions with $\text{Lip}(\sigma_j) \leq 1$; (2) If $\sigma_j$ is connected to a multiplication block then $\|\sigma_j\|_{\infty} \leq 1$. 
ConvNet: Sublayers
MaxPooling and AveragePooling

MaxPooling can be implemented as follows:
ConvNet: Sublayers
MaxPooling and AveragePooling

MaxPooling can be implemented as follows:

AveragePooling can be implemented as follows:
ConvNet: Sublayers

Long Short-Term Memory

Long Short-Term Memory (LSTM) networks [HS97, GSKSS15].
By BiObserver - Own work, CC BY-SA 4.0,
https://commons.wikimedia.org/w/index.php?curid=43992484
**ConvNet: Layer $m$**

Components of the $m^{th}$ layer

- $N_{m,1}$
- $N_{m,2}$
- $N_{m,n_m}$

Detection / Merge / Pooling

- $N'_{m,1}$
- $N'_{m,2}$
- $N'_{m,n_m}$

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Lipschitz Analysis of CNN
ConvNet: Layer $m$

Topology coding of the $m^{th}$ layer

$n_m$ denotes the number of input nodes in the $m$-th layer:

$\mathcal{I}_m = \{N_{m,1}, N_{m,2}, \ldots, N_{m,n_m}\}$.

Filters:

1. pooling filter: $\phi_{m,n}$ for node $n$, in layer $m$;
2. convolution filter: $g_{m,n,k}$ for input node $n$ to output node $k$, in layer $m$;

For node $n$: $G_{m,n} = \{g_{m,n;1}, \ldots, g_{m,n;k_m,n}\}$.

The set of all convolution filters in layer $m$: $G_m = \bigcup_{n=1}^{n_m} G_{m,n}$. 

Radu Balan (UMD)  Lipschitz Analysis of CNN
ConvNet: Layer $m$

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For node $n$: $G_{m,n} = \{g_{m,n;1}, \cdots g_{m,n;k_m,n}\}$.

The set of all convolution filters in layer $m$: $G_m = \bigcup_{n=1}^{n_m} G_{m,n}$.

$\mathcal{O}_m = \{N'_{m,1}, N'_{m,2}, \cdots, N'_{m,n'_m}\}$ the set of output nodes of the $m$-th layer.

Note that $n'_m = n_{m+1}$ and there is a one-one correspondence between $\mathcal{O}_m$ and $\mathcal{I}_{m+1}$.

The output nodes automatically partitions $G_m$ into $n'_m$ disjoint subsets $G_m = \bigcup_{n'_1}^{n'_m} G_{m,n'}$, where $G'_{m,n'}$ is the set of filters merged into $N'_{m,n'}$. 

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Lipschitz Analysis of CNN
ConvNet: Layer $m$

Topology coding of the $m^{th}$ layer

For each filter $g_{m,n;k}$, we define an associated multiplier $l_{m,n;k}$ in the following way: suppose $g_{m,n;k} \in G'_{m,n'}$, let $K = |G'_{m,n'}|$ denote the cardinality of $G'_{m,n'}$. Then

$$ l_{m,n;k} = \begin{cases} 
K & \text{, if } g_{m,n;k} \in \tau_1 \cup \tau_3 \\
K^{\max\{0,2/p-1\}} & \text{, if } g_{m,n;k} \in \tau_2
\end{cases} \quad (2.1) $$
ConvNet: Layer $m$

Topology coding of the $m^{th}$ layer
**ConvNet: Layer $m$**

Topology coding of the $m^{th}$ layer

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$G_{m,3}$

$n_m' = n_{m+1}$
**ConvNet: Layer \( m \)**

Topology coding of the \( m^{th} \) layer

In

\[ G'_{m,2} \]

\[ n'_m = n_{m+1} \]

Out
Layer Analysis

Bessel Bounds

In each layer $m$ and for each input node $n$ we define three types of Bessel bounds:

- 1st type Bessel bound:

$$B_{m,n}^{(1)} = \| \hat{\phi}_{m,n} \|^2 + \sum_{k=1}^{k_{m,n}} l_{m,n;k} D_{m,n;k}^{-d} \| \hat{g}_{m,n;k} \|^2 \|_{\infty}$$  \hspace{1cm} (3.2)

- 2nd type Bessel bound:

$$B_{m,n}^{(2)} = \| \sum_{k=1}^{k_{m,n}} l_{m,n;k} D_{m,n;k}^{-d} \| \hat{g}_{m,n;k} \|^2 \|_{\infty}$$  \hspace{1cm} (3.3)

- 3rd type (or generating) bound:

$$B_{m,n}^{(3)} = \| \hat{\phi}_{m,n} \|_{\infty}^2 .$$  \hspace{1cm} (3.4)
Layer Analysis
Bessel Bounds

Next we define the layer $m$ Bessel bounds:

1\textsuperscript{st} type Bessel bound  \[ B_m^{(1)} = \max_{1 \leq n \leq n_m} B_{m,n}^{(1)} \] (3.5)

2\textsuperscript{nd} type Bessel bound  \[ B_m^{(2)} = \max_{1 \leq n \leq n_m} B_{m,n}^{(2)} \] (3.6)

3\textsuperscript{rd} type (generating) Bessel bound  \[ B_m^{(3)} = \max_{1 \leq n \leq n_m} B_{m,n}^{(3)} \] (3.7)

Remark. These bounds characterize semi-discrete Bessel systems.
Lipschitz Analysis
First Result

Theorem

[BSZ17] Consider a Convolutional Neural Network with $M$ layers as described before, where all scalar nonlinear functions are Lipschitz with $\text{Lip}(\varphi_{m,n,n'}) \leq 1$. Additionally, those $\varphi_{m,n,n'}$ that aggregate into a multiplicative block satisfy $\|\varphi_{m,n,n'}\|_{\infty} \leq 1$. Let the $m$-th layer 1st type Bessel bound be

$$B_m^{(1)} = \max_{1 \leq n \leq n_m} \left\| \hat{\phi}_{m,n} \right\|^2 + \sum_{k=1}^{k_{m,n}} l_{m,n;k} D_{m,n;k}^{-d} \left\| \hat{g}_{m,n;k} \right\|^2_{\infty}.$$

Then the Lipschitz bound of the entire CNN is upper bounded by $\prod_{m=1}^{M} \max(1, B_m^{(1)})$. Specifically, for any $f, \tilde{f} \in L^2(\mathbb{R}^d)$:

$$\left\| \mathcal{F}(f) - \mathcal{F}(\tilde{f}) \right\|^2_2 \leq \left( \prod_{m=1}^{M} \max(1, B_m^{(1)}) \right) \left\| f - \tilde{f} \right\|^2_2.$$
Lipschitz Analysis
Second Result

Theorem

Consider a Convolutional Neural Network with $M$ layers as described before, where all scalar nonlinearities satisfy the same conditions as in the previous result. For layer $m$, let $B^{(1)}_m$, $B^{(2)}_m$, and $B^{(3)}_m$ denote the three Bessel bounds defined earlier. Denote by $L$ the optimal solution of the following linear program:

$$
\Gamma = \max_{y_1, \ldots, y_M, z_1, \ldots, z_M \geq 0} \sum_{m=1}^{M} z_m \\
\text{s.t. } y_0 = 1 \\
y_m + z_m \leq B^{(1)}_m y_{m-1}, \quad 1 \leq m \leq M \\
y_m \leq B^{(2)}_m y_{m-1}, \quad 1 \leq m \leq M \\
z_m \leq B^{(3)}_m y_{m-1}, \quad 1 \leq m \leq M
$$

(3.8)
Theorem

Then the Lipschitz bound satisfies $\text{Lip}(\mathcal{F})^2 \leq \Gamma$. Specifically, for any $f, \tilde{f} \in L^2(\mathbb{R}^d)$:

$$\|\mathcal{F}(f) - \mathcal{F}(\tilde{f})\|_2^2 \leq \Gamma \|f - \tilde{f}\|_2^2,$$
Example 1: Scattering Network

The Lipschitz constant:

- Backpropagation/Chain rule: Lipschitz bound 40 (hence \( Lip \leq 6.3 \)).
Example 1: Scattering Network

The Lipschitz constant:

- Backpropagation/Chain rule:
  Lipschitz bound 40 (hence $Lip \leq 6.3$).

- Using our main theorem,
  $Lip \leq 1$, but Mallat’s result:
  $Lip = 1$.

Filters have been choosen as in a dyadic wavelet decomposition. Thus
$B_m^{(1)} = B_m^{(2)} = B_m^{(3)} = 1, 1 \leq m \leq 4$. 
Example 2: A General Convolutive Neural Network
Example 2: A General Convolutive Neural Network

Set $p = 2$ and:

$$F(\omega) = \exp\left(\frac{4\omega^2 + 4\omega + 1}{4\omega^2 + 4\omega}\right)\chi(-1,-1/2)(\omega) + \chi(-1/2,1/2)(\omega) + \exp\left(\frac{4\omega^2 - 4\omega + 1}{4\omega^2 - 4\omega}\right)\chi(1/2,1)(\omega).$$

$$\hat{\phi}_1(\omega) = F(\omega)$$
$$\hat{g}_{1,j}(\omega) = F(\omega + 2j - 1/2) + F(\omega - 2j + 1/2), \ j = 1, 2, 3, 4$$

$$\hat{\phi}_2(\omega) = \exp\left(\frac{4\omega^2 + 12\omega + 9}{4\omega^2 + 12\omega + 8}\right)\chi(-2,-3/2)(\omega) +$$
$$\chi(-3/2,3/2)(\omega) + \exp\left(\frac{4\omega^2 - 12\omega + 9}{4\omega^2 - 12\omega + 8}\right)\chi(3/2,2)(\omega)$$

$$\hat{g}_{2,j}(\omega) = F(\omega + 2j) + F(\omega - 2j), \ j = 1, 2, 3$$
$$\hat{g}_{2,4}(\omega) = F(\omega + 2) + F(\omega - 2)$$
$$\hat{g}_{2,5}(\omega) = F(\omega + 5) + F(\omega - 5)$$

$$\hat{\phi}_3(\omega) = \exp\left(\frac{4\omega^2 + 20\omega + 25}{4\omega^2 + 20\omega + 24}\right)\chi(-3,-5/2)(\omega) +$$
$$\chi(-5/2,5/2)(\omega) + \exp\left(\frac{4\omega^2 - 20\omega + 25}{4\omega^2 - 20\omega + 25}\right)\chi(5/2,3)(\omega).$$
Example 2: A General Convolutive Neural Network

Bessel Bounds: \( B_m^{(1)} = 2e^{-1/3} = 1.43 \), \( B_m^{(2)} = B_m^{(3)} = 1 \).

The Lipschitz bound:

- Using backpropagation/chain-rule: \( Lip^2 \leq 5 \).
- Using Theorem 1: \( Lip^2 \leq 2.9430 \).
- Using Theorem 2 (linear program): \( Lip^2 \leq 2.2992 \).
References


