Application of Moment Expansion Method to Option Square Root Model

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Motivation

- Black-Scholes Model successfully explain stock option price
- Equity price follows a Geometric Brownian Motion
- Assumption: Log return is normal distribution with constant volatility
- Reality: Log return is NOT normal distribution, volatility is NOT constant

Comparison Between Heston Model and Black-Scholes

	Volatility	**Log Return Distribution**
Black-Scholes	Constant	Normal
Heston Model	Stochastic	Not Normal

Methods to Solve Heston Model

- Closed Form Exact Solution (Heston, 1993)
- Fast Fourier Transform (Carr and Madan, 1999): Characteristic Function Needed
- Moment Expansion (This Project, 2009)
  - can work for Stochastic Volatility Models (no exact solution, Characteristic Function hard to get)
- Other methods

What I did in this project?

- Get Moments of Log Return in Heston Model.
- Apply Gram-Charlier Expansion Approximation
- Compare the Approximation with Exact Solution
- Discuss Convergence of this Method

Heston Model

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t dW_t^s$$

$$d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} dW_t^\nu$$

- dW_t^s, dW_t^ν Brownian Motion with Correlation ρ
- S_t Stock Price at Time t
- ν_t Variance at Time t
- r Rate of Return
- θ Average Variance
- κ Mean Reversion Rate
- σ Volatility of Volatility

Moment Expansion Method

- Use Backward Equation to get any order of moments
- Use Gram-Charlier to seek an approximate distribution
  - Normal distribution + series approximation related to moments and Hermite Polynomials
- Replace normal distribution by the approximate distribution in option price formula

Gram-Charlier Expansion

$$g(z) = n(z)(1 + \sum_{i=3} \frac{\mu_i - norm_i}{i!} H_i(z))$$

- $z = \frac{ln(S_t/S_0) - (r - \sigma^2/2)t}{\sigma\sqrt{t}}$
- g(z) Approximate Distribution of Log Return
- n(z) Probability Density Function of Standard Normal
- μ_i Moments of Desired Distribution
- $norm_i$ Moments of Standard Normal Distribution
- $H_i(z)$ Hermite Polynomial

Option Price

$$C = e^{-rT} E(S_T - K)^+$$
$$= e^{-rT} \int_{-\infty}^{\infty} (e^{lnS_0 + (r - \frac{\sigma^2}{2})t + \sigma\sqrt{T}z} - K)^+ n(z)dz$$

- Replace n(z) by g(z)
- $Call(GC) = Call(BS) + \sum_{i=3} Q_i(\mu_i - norm_i)$
- Q_i Coefficient part involving integral of Hermite Polynomial

Moments Computing

- Analytical : Up to 4th order (Mathematica By Heston)
- Numerical : Matrix Exponential Method
- They are the same

Validation

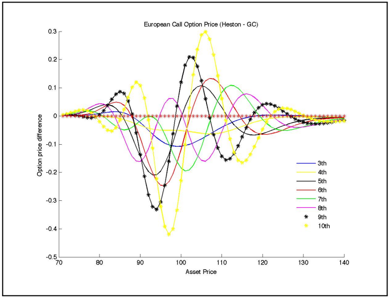
$$dS_t = rS_t dt + \sqrt{\nu_t} S_t dW_t^s$$

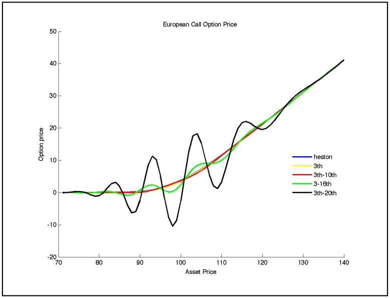
$$d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} dW_t^\nu$$

- Make Volatility as a constant
- $\sigma = 0$ and $\theta = \nu_t$
- Moments from Heston Model = Moments of Standard Normal
- Call Option Price by Gram Charlie = Call Option Price by Black-Scholes

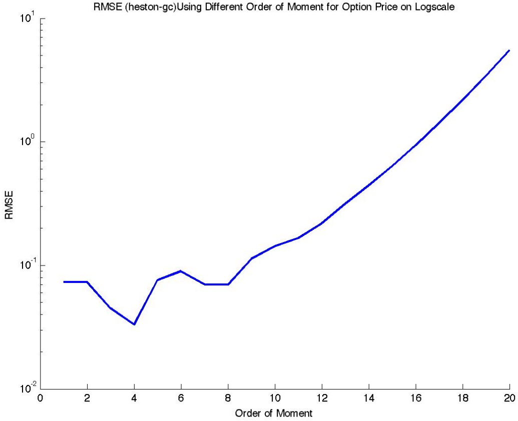
- Numerical Results make an agreement with above conditions

Results


European Call Option Price (Heston - GC)
Option price difference
Asset Price
3th
4th
5th
6th
7th
8th
9th
10th


European Call Option Price
Option price
Asset Price
heston
3th
3th-10th
3-16th
3th-20th

RMSE

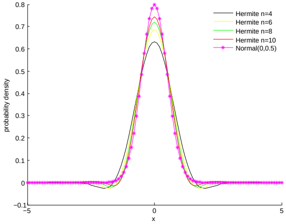


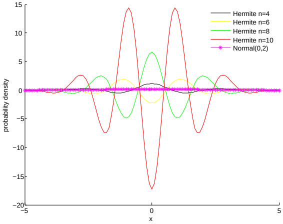
For Gram-Charlier, 4th order might be good

Convergence of Gram-Charlier Expansion

- Poor Convergence Properties (Cramer 1957)
- Souce of Divergence: g(x) must fall to 0 faster than $e^{-\frac{x^2}{4}}$
- Cramer's Condition for Convergence:
  $\int_{-\infty}^{\infty} e^{\frac{x^2}{4}} g(x) dx < \infty$

Examples





$\sigma = 0.5$ $\sigma = 2$

Convergence **Divergence**

Convergence of g(x)

- PDF of Log Return in Heston Model (Dragulescu and Yakovenko, 2002)
- Properties of PDF
  - Fall to Zero Slower than $e^{-\frac{x^2}{4}}$
  - Cramers Condition can not hold

Summary

- Moment Expansion Method is applied to Stochastic Volatility Model (Heston Model)
- Up to certain order of moments, adding higher moments can not increase accuracy of the approximation
- Convergence condition is disscussed

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References

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