A State-Space Model for a Nonlinear Time-Delayed Feedback Loop

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AMSC 663

October 14, 2008

Goal

To implement an alternative, discrete time model for coupled nonlinear (chaotic) time-delayed feedback loops.

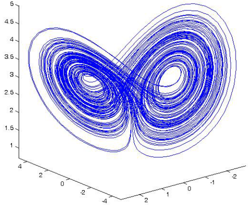
Introduction to Chaos

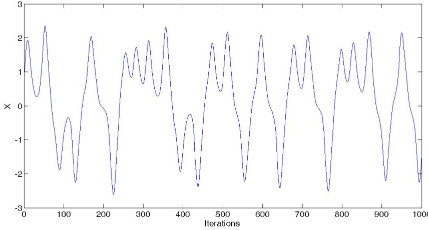
Properities of a Chaotic System

- sensitive to initial conditions
- topologically mixing
- periodic orbits are dense

Classic Example: The Lorenz System

$$\dot{x} = \sigma(y - x) \qquad \sigma = 10$$
$$\dot{y} = rx - y - 20xz \qquad r = 60$$
$$\dot{z} = 5xy - bz \qquad b = \frac{8}{3}$$



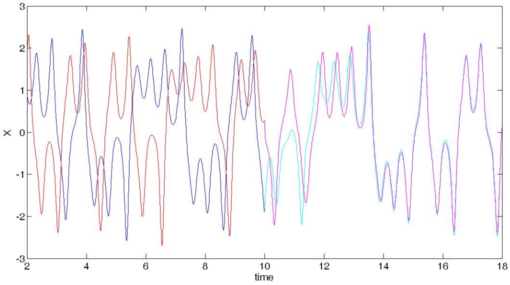


Lorenz Synchronization

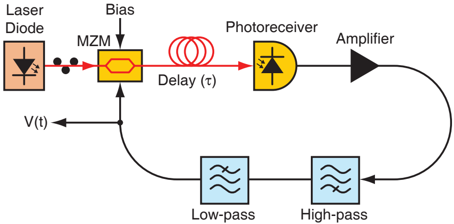
Coupled Lorenz Equations:

$$
\begin{aligned}
\dot{x}_1 &= \sigma(y_1 - x_1) + \gamma(x_2 - x_1) & \dot{x}_2 &= \sigma(y_2 - x_2) + \gamma(x_1 - x_2) \\
\dot{y}_1 &= rx_1 - y_1 - 20x_1 z_1 & \dot{y}_2 &= rx_2 - y_2 - 20x_2 z_2 \\
\dot{z}_1 &= 5x_1 y_1 - bz_1 & \dot{z}_2 &= 5x_2 y_2 - bz_2
\end{aligned}
$$

$$\sigma = 10 \qquad r = 60 \qquad b = \frac{8}{3}$$



System Overview

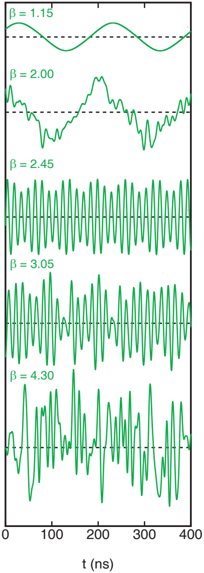


$$x(t) + \tau\dot{x}(t) + \frac{1}{\theta}\int_{t_o}^{t} x(s)ds = \beta cos^2[x(t - T) + \phi]$$

- $x(t)$ is normalized RF voltage
- τ is the low pass filter time constant
- θ is the high pass filter time constant
- T is the time delay in the loop
- β is the feedback strength
- ϕ is the phase offset in nonlinearity

[Kouomou, Thesis]

Modeling the system



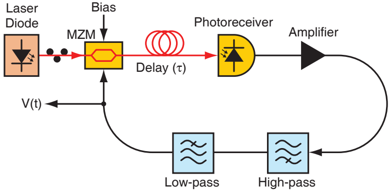
- Traditional numeric methods can be applied
- Generates a variety of dynamics

- Problems
  - Step-Size Concerns
  - Speed (3+ Min per run)
  - Complicated filters difficult to model

Comparing Differentials vs. State-Space

Differential Form

$$x(t) + \tau\dot{x}(t) + \frac{1}{\theta}\int_{t_o}^{t} x(s)ds = y(t)$$



Discrete State-Space Form

$$\mathbf{u}[n+1] = \mathbf{A}\mathbf{u}[n] + \mathbf{B}y[n]$$
$$x[n] = \mathbf{C}\mathbf{u}[n] + Dy[n]$$

Some Details

Discrete State-Space Form

$$\mathbf{u}[n+1] = \mathbf{A}\mathbf{u}[n] + \mathbf{B}y[n]$$
$$x[n] = \mathbf{C}\mathbf{u}[n] + Dy[n]$$

- Choose canonical form from z-space transform of discrete filter
- Include nonlinear delayed feedback ($y[n] \rightarrow f(x[n-k])$)

This gives us for matrices:

A=1.4939 -0.4972 **B**=1 **C**= 0 -0.2514
1 0 0 D=0

Simplified Model

With all of the previous simplifications we are left with:

$$\mathbf{u}[n+1] = \mathbf{A}\mathbf{u}[n] + \mathbf{B}f(x[n-k])$$
$$x[n] = \mathbf{C}\mathbf{u}[n]$$
$$\Downarrow$$
$$\mathbf{u}[n+1] = \mathbf{A}\mathbf{u}[n] + \mathbf{B}f(\mathbf{C}\mathbf{u}[n-k])$$
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Where we have reintroduced the nonlinearity from the Mach-Zehnder.

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Coupled State-Space

Finally we build a coupled system similar to the coupled Lorenz System (with abstraction for easy reading)

$$\mathbf{u}_1[n+1] = \mathbf{A}\mathbf{u}_1[n] + \mathbf{B}f(\mathbf{C}\mathbf{u}_1[n-k]) + \gamma\mathbf{B}(f(\mathbf{C}\mathbf{u}_2[n-k]) - f(\mathbf{C}\mathbf{u}_1[n-k]))$$

$$\mathbf{u}_2[n+1] = \mathbf{A}\mathbf{u}_2[n] + \mathbf{B}f(\mathbf{C}\mathbf{u}_2[n-k]) + \gamma\mathbf{B}(f(\mathbf{C}\mathbf{u}_1[n-k]) - f(\mathbf{C}\mathbf{u}_2[n-k]))$$

$$\Downarrow$$

$$\mathbf{u}_1[n+1] = \mathbf{A}\mathbf{u}_1[n] + \mathbf{B}((1-\gamma)f(\mathbf{C}\mathbf{u}_1[n-k]) + \gamma f(\mathbf{C}\mathbf{u}_2[n-k]))$$
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Implementation

Considerations

- Quick Graphical Output
- Vector Operations
- In-built filter functions

Decision: Matlab (w/ C)

- Major Concern: Quantization

Validation

- Stage 1: Single Loop
  - Characteristic Curves from Kouomuo [1]
- Stage 2: Coupled Lorenz
  - Conditions by Anishchenko [4]
- Stage 3: Coupled Mach-Zehnders
  - Open Loop: Argysis [5]
  - Symmetric 50/50 coupling: Piel [6]

Use of Code

- Previously Explored
  - 50/50 Coupling
  - Synchronization for Coupling vs. Feedback Strength
- To be Explored
  - Synchronization for:
    - Coupling vs. Delay
    - Coupling vs. Optical Bias
  - Noise and Quantization
  - Non-Symmetric Coupling
  - Variations in Other parameters

Milestones

- Implementation and Verification of individual simulations — November 1^{st} week
- Implementation and Verification of final simulation — December 1^{st}
- Generation of new results — Feburary 1^{st}
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November 1^{st} week

December 1^{st}

Feburary 1^{st}

To be Determined

References

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