GPUs and Einstein's Equations

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December 2, 2010

Outline

Project Motivation

- LIGO is depending on reliable simulations to identify black hole detections in very noisy data.
- Black hole simulations are computationally intensive.
- GPUs are a reasonably flexible and efficient for large scale computations.
- Using GPUs may reduce computation time and cost.
- This project will focus on building up code that will be the groundwork for simulating black holes on GPUs.

Project Summary

Implement a spectral method PDE solver for Einstein's equations:

1. Matlab (Fall 2010)
2. C (Spring 2011)
3. C/CUDA (Spring 2011)

Verify each stage of the code using a known solution special case of a spherically symmetric black hole.

This Semester

- Learn about spectral collocation methods.
- Gain a basic understanding of the problem at hand: solving the special case of Einstein's equations for a spherically symmetric black hole.
- Write Matlab code to solve Einstein's equations using spectral methods in space and Runge-Kutta in time.
- Verify code against known analytic solution by showing spectral convergence to the true solution.

Collocation Points and Differentiation Matrix

Chebyshev Collocation Points (Degree N)

$x_i = \cos(\pi \cdot i/N), \qquad i \in \{0, \ldots, N\}, \; x_i \in [-1, 1].$

Differentiation Matrix

D is the $(N+1) \times (N+1)$ differentiation matrix relevant for the collocation points $x_i \in [-1, 1]$.

$$u'(x_i) \approx Du(x_i)$$

Spectral Convergence of Derivative

Want $u'(r_i)$ for $r_i \in [1.9, 11.9]$.

- $r_i = x_i \cdot ((r_{max} - r_{min})/2) + ((r_{max} + r_{min})/2)$.
- $u'(r_i) = (du/dx) \cdot (dx/dr) \approx Du \cdot 2/(r_{max} - r_{min})$.

Spectral Convergence

$u'(r_i)$ converges exponentially in N to $Du(r_i) \cdot 2/(r_{max} - r_{min})$.

Matlab Demo.

Evolution in Time

Fourth-Order Runge-Kutta (RK4)

$$y' = f(t, y), \qquad y(t_0) = y_0$$

$$t_{n+1} = t_n + h$$

$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1)$$

$$k_3 = f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_2)$$

$$k_4 = f(t_n + h, y_n + hk_3)$$

$$y_{n+1} = y_n + \tfrac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$

The function f is the RHS computation for Einstein's equations.

The Solution

- We are solving 6 coupled hyperbolic equations that are 1st order in space and time.
- Since the we have a spherically symmetric black hole, the coordinates are 1d in space (r is the radius) instead of 3d in space.
- There are 6 variables $g_{rr}, g_T, K_{rr}, K_T, f_{rrr}, f_{rT}$, that describe a spherically symmetric metric on a Lorentzian manifold.

RHS Computations

- Given initial values of main variables $g_{rr}, g_T, K_{rr}, K_T, f_{rrr}, f_{rT}$, approximate the time derivative of the main variables.
- The time derivatives $\dot{g_{rr}}, \ldots, \dot{f_{rT}}$ depend on $g'_{rr}, \ldots, f'_{rT}$.
- Use spectral method (differentiation matrix) to approximate the spacial derivatives $g'_{rr}, \ldots, f'_{rT}$.
- For our special case, $\dot{g_{rr}}, \ldots, \dot{f_{rT}}$ should be 0. This is how we will verify the code later.

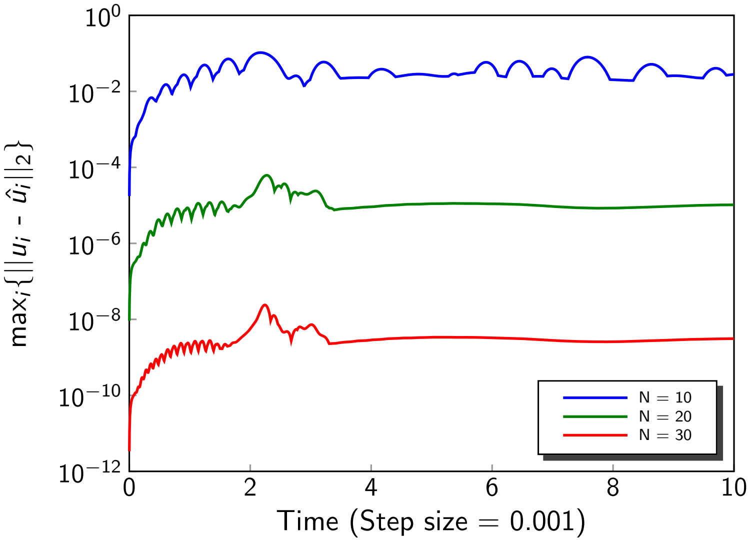
Boundary Conditions

- Inner boundary is inside the black hole, so no boundary conditions need to be imposed explicitly.
- After each step within RK4, we adjust the outer boundary using the initial conditions.

Comparing to Analytic Solution

1. Choose a series of degrees N of Chebyshev polynomials (e.g., $N \in \{10, 20, 30\}$).
2. Choose a fixed time step size (e.g. 0.001).
3. Evolve Einstien's equations T time steps.
4. At each time, report the 6-component solution as an $(N + 1) \times 6$ matrix.
5. Let u be the know solution and $\hat{u}$ be the approximation. Use $||u_i(\mathbf{x}) - \hat{u}_i(\mathbf{x})||_2$ as the measure of the error in the i^{th} component; $\mathbf{x}$ is the $N + 1$ long vector of collocation points.
6. Verify that the error decreases exponentially in N, and that the error does not diverge after many time steps.

Error in Solution to Einstein's Equations



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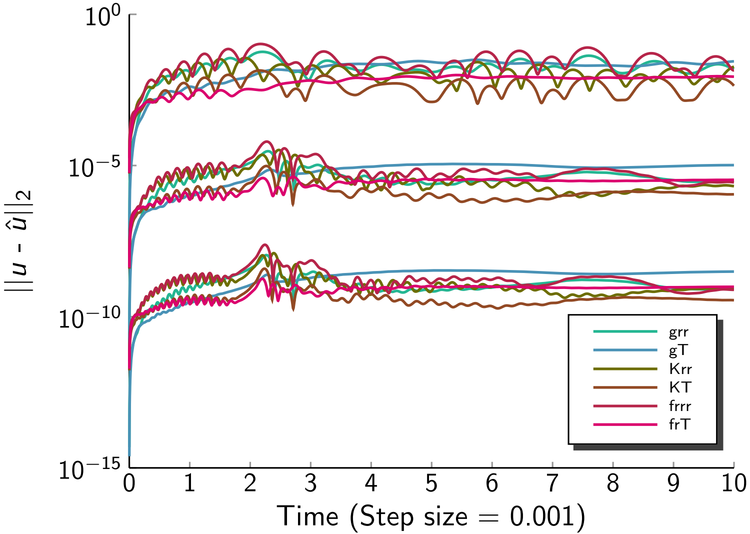

References

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