# IMAGE DEBLURRING - COMPUTATION OF CONFIDENCE INTERVALS 

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NOVEMBER 30, 2010

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## Introduction: The image

- An image is an array of pixels
- For a grayscale image, these values are in the interval $[0,255]$
$0=$ black
$255=$ white



## Introduction: The problem

- Notation:

| Symbol | Size | Explanation |
| :---: | :---: | :--- |
| $K$ | $m \times n$ | Matrix defined through the Point Spread <br> function (PSF) in the case of a linear prob- <br> lem |
| $X$ |  | Original Clear Image |
| $x$ | $n \times 1$ | Vector containing the values corresponding <br> to the pixels of the image $X$ |
| $B$ |  | The blurred image we measure |
| $b$ | $m \times 1$ | Vector which contains the values of the pix- <br> els of the blurred image $B$ |
| $e$ | $m \times 1$ | Noise Vector |

- $b=K x+e$


## Introduction: Confidence intervals

Two different types of confidence intervals are of interest.

## Definition

One-at-a-time confidence intervals bound each $\varphi_{k}$ indvidually with probability $\alpha$ ( $\alpha \%$ confidence).

$$
\operatorname{Pr}\left\{I_{k} \leq \varphi_{k} \leq u_{k}\right\}=\alpha, k=1,2, \ldots, p
$$

## Definition

Simultaneous confidence intervals which bound all the $\varphi_{k}$ simultaneously with a probability greater than $\alpha$.

$$
\operatorname{Pr}\left\{I_{k} \leq \varphi_{k} \leq u_{k}, k=1,2, \ldots, p\right\} \geq \alpha
$$

## Introduction: Confidence intervals

O'Leary and Rust have proven that:

## Theorem

Supposing that the noise is normally distributed with mean zero and standard deviation a matrix $S$ nonsingular and symmetric, and also assuming that $x$ is nonnegative and less than 255 , then the probability that $\varphi_{k}=w_{k}^{\top} x$ is contained in the interval $\left[I_{k}, u_{k}\right]$ is greater than or equal to $\alpha$ where

$$
I_{k}=\min \left\{w_{k}^{T} x:\|K x-b\|_{s} \leq \mu, 0 \leq x \leq 255\right\}
$$

and

$$
u_{k}=\max \left\{w_{k}^{\top} x:\|K x-b\|_{s} \leq \mu, 0 \leq x \leq 255\right\}
$$

$\operatorname{rank}(K)=q, \int_{0}^{\gamma^{2}} \chi_{q}^{2}(\rho) d \rho=\alpha, r_{0}=\min _{x \geq 0}\|K x-b\|_{S}^{2}, \mu^{2}=r_{0}+\gamma^{2}$ and $\chi_{q}^{2}$ is the probability density function for the chi-squared distribution with $q$ degrees of freedom.

## Approach: Point Spread Function, (PSF)

- The blurring matrix $K$ is defined by the Point Spread Functions.
- A Point Spread Function is the function that blurs an image of a single white point.
- Example: image of single white point $5 \times 5$

- If the blur is spatially invariant(affects only neighboring pixels), then having measured only one column of the blurring matrix $K$ is enough to determine the whole matrix.


## Approach: Gaussian Point Spread Function

The Point Spread Functions are constructed as Gaussian of size $5 \times 5$ :

$$
\operatorname{PSF}(k, I)=\exp \left(-\frac{1}{2} \frac{\left(k-c_{1}\right)^{2}}{s_{1}^{2}}-\frac{1}{2} \frac{\left(I-c_{2}\right)^{2}}{s_{2}^{2}}\right)
$$

for $k, l=1 \ldots 5$, where $s_{1}=s_{2}=3$ and $c_{1}$ and $c_{2}$ are the coordinates of the center of the Point Spread Function.

## Approach: Construction of the blurring matrix

Image $5 \times 5$, PSF $3 \times 3$

PSF array: $\quad \begin{array}{lll}\times & \times & \times \\ & \times & \times \\ & \times\end{array}$
For the pixels $(1,1)$ and $(2,1)$, the PSFs will be of the form:

| $\times$ | $\times$ | 0 | 0 | 0 | $\times$ | $\times$ | 0 | 0 | 0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\times$ | $\times$ | 0 | 0 | 0 | $\times$ | $\times$ | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | $\times$ | $\times$ | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Reshaping these matrices into vectors column by column, we get the first two columns of the blurring matrix K.

## Approach: The blurring Matrix K

If we do this for all the pixels of the image we will end up having the block-tridiagonal blurring matrix $K$ shown in the figure below.


Figure: Blurring Matrix $K$ for an image of size $5 \times 5$ with Point Spread Functions of size $3 \times 3$.

## Approach: Including noise

- Blurred image without noise $K x$, where $K$ is the blurring matrix and $x$ the vector of the initial image.
- In order to use the theory of the confidence intervals, we need the standard deviation matrix $S$ of the noise symmetric and non-singular. To construct the noise, we use a random vector, e, with elements with mean 0 and standard deviation a specified number sdv. The standard deviation matrix in this case is the identity matrix multiplied by the number sdv, apparently symmetric and invertible.
- The noisy blurred image thus corresponds to the sum $b=K x+e$.


## Approach: Computation of $\mu^{2}$

- By the theorem of Rust and O'Leary, we need to compute:

$$
I_{k}=\min \left\{w_{k}^{T} x:\|K x-b\|_{s} \leq \mu, 0 \leq x \leq 255\right\}
$$

and

$$
u_{k}=\max \left\{w_{k}^{T} x:\|K x-b\|_{s} \leq \mu, 0 \leq x \leq 255\right\}
$$

where $\int_{0}^{\gamma^{2}} \chi_{q}^{2}(\rho) d \rho=\alpha, r_{0}=\min _{0 \leq x \leq 255}\|K x-b\|_{S}^{2}, \mu^{2}=r_{0}+\gamma^{2}$ and $\chi_{q}^{2}$ is the probability density function for the chi-squared distribution with $q$ degrees of freedom, $q=\operatorname{rank}(K)$.

- To compute the minimum of the norm $r_{0}$, we first transform the $S$-norm to the 2 -norm that matlab can handle. Thus,

$$
\begin{gathered}
\|K x-b\|_{S}^{2}=(K x-b)^{T} S^{-2}(K x-b)=\left(S^{-1}(K x-b)^{T}\right)\left(S^{-1}(K x-b)\right)= \\
=\left\|S^{-1} K x-S^{-1} b\right\|^{2}
\end{gathered}
$$

## Approach: Computation of $\mu^{2}$

- $q=\operatorname{rank}(K)$ can be easily computed using the SVD decomposition of the blurring matrix.
- Using $q$ and the desired probability that defines the confidence intervals, $\gamma^{2}$ can also be computed by Matlab or by interpolation of the values of the tables of the $\chi^{2}$ distribution. (For the purposes of this project, we use $\alpha=0.95$.)
- In addition, we can compute the minimum of a norm:
$r_{0}=\min _{0 \leq x \leq 255}\|K x-b\|_{S}^{2}$ using a least squares method with inequality constraints or by quadratic programming.
- Finally we add those to get $\mu^{2}=r_{0}+\gamma^{2}$.


## Approach: Confidence Intervals

O'Leary and Rust have proven the following theorem:

## Theorem

The values $I_{k}$ and $u_{k}$ are defined by the two extreme roots of $L(\varphi)-\mu^{2}=0$ where $L(\varphi)=\min _{x}\left\{\|K x-b\|_{S}^{2}: 0 \leq x \leq 255, w_{k}^{T} x=\varphi\right\}$

If we choose $w$ to be the columns of the identity matrix, then for each one of these, we get the value of one pixel. The lower and upper bounds of the confidence intervals will then give us the lower and the upper limits of the value of each one of the pixels with the probability that we used to compute the $\mu$.

## Databases

- Grayscale images of various sizes.

For validation, better to use small images.
The maximum size of the images that can be used by the code are determined by the memory that matlab can handle.

- An example of a test image is the firework image (left) which was cropped (right) and used at the initial steps of validating the code.


Figure: original $3648 \times 2736$ image, cropped $16 \times 16$ image

## Validation

- Blurring matrix $\rightarrow$ Small image and spy command of matlab to visualize it and see if it affects the correct pixels.


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- $\mu^{2}$
- $\mathrm{q}=\operatorname{rank}(\mathrm{K}) \rightarrow$ use of a matrix with known rank
- $\gamma^{2} \rightarrow$ chi2inv command of matlab given the rank of the blurring matrix and a probability $\alpha$. (easily verified using tables of the $\chi^{2}$ distribution).
- $r_{0}=\min _{0 \leq x \leq 255}\|K x-b\|_{S}^{2} \rightarrow$ use of a vector slightly modified by the product of the matrix we have and the solution vector that we want.


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- $r_{0}=\min _{0 \leq x \leq 255}\|K x-b\|_{S}^{2} \rightarrow$ use of a vector slightly modified by the product of the matrix we have and the solution vector that we want.
- Bounds of the Confidence intervals $\rightarrow$ Known initial image. Expect to find confidence intervals that contain the true value of the pixels of the image approximately in $100 \alpha \%$ of the cases.


## Relative size of PSF. Size of image change

Following is the figure of the original $16 \times 16$ cropped fireworks image (left) with its clear blurred image (middle) and the noisy blurred image (right). The blurring occurred using the Gaussian Point Spread Function of size $5 \times 5$.


## Relative size of PSF. Size of image change

When we use the following $7 \times 7$ image (left) which was blurred using a Gaussian PSF function of size $5 \times 5$ (middle) and noise was also added to it (right), the new distorted image doesn't remind the initial one.


## Relative size of PSF. Size of image change

The restored image (right) is a very good approximation of the original one (left).


Size of PSF change. The $2 \times 2$ domino image PSF size $1 \times 1$


Size of PSF change. The $2 \times 2$ domino image PSF size $2 \times 2$


Size of PSF change. The $2 \times 2$ domino image PSF size $3 \times 3$


## Next Steps

(1) Revise the code
(2) Run the code for bigger images
(3) Develop a parallel code
(9) Use sub-images and sub-matrices

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## Thank you

