Solving the steady state diffusion equation with uncertainty

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Abstract

- Goal: to efficiently solve a steady state diffusion equation with a random coefficient.
- Monte-Carlo methods are time intensive.
- Using principal components analysis (also known as the Karhunen-Loéve expansion) allows the random coefficient to be approximated with a finite sum of random variables.
- This expansion combined with a stochastic finite element method should reduce computation time.

Problem

The equation to be solved is

$$-\nabla \cdot (c(x,\omega)\nabla u) = f(x) \ , \tag{1}$$

where the diffusion coefficient is a random field.

- c takes the form $c = e^{a(x,\omega)}$ to ensure that it is positive for all x. This guarantees existence and uniqueness of the solution of Equation (1).
- Assume a bounded spatial domain $D \subset \mathbb{R}^2$.
- The boundary conditions are deterministic.

$$u(x,\omega) = g(x) \text{ on } \partial D_D$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial D_n \ .$$

Problem

- The solution is a function of the sample space from which quantities such as the moments or cumulative distribution functions can be found.
- Applications include modeling groundwater flow through a porous medium.

Background

- In general, the structure of the diffusion coefficient is unknown.
- Previous work has been done where the log of the diffusion coefficient, $a(x, \omega)$, is written as an infinite series expansion of random variables [1],[4].
- The random field can then be approximated by a finite number of terms in this expansion.
- This project will instead look at the series expansion of $c(x, \omega)$.

Approach

- Determine the covariance at each pair of points on the spatial domain.

$$C(x,y) = \int_\Omega (c(x,\omega) - \mu(x))(c(y,\omega) - \mu(y))dP(\omega)$$

$$C_{ij} = \frac{1}{n}\sum_{k=1}^{n}(c(x_i,\omega_k) - \hat{\mu}_i)(c(x_j,\omega_k) - \hat{\mu}_j)$$

- The mean, $\mu(x)$ is defined as

$$\mu(x) = \int_\Omega c(x,\omega)dP(\omega)$$

- Under the assumption that the random field is stationary, the mean and variance are constant at each point on the domain.

Approach

- Find the eigenpairs of

$$\mathcal{C}c(x) = \int_D C(x,y)c(y)dy = \lambda c(x)$$

- An expansion for the random field in terms of uncorrelated random variables is given as

$$c(x,\omega) = \mu(x) + \sum_{s=1}^{\infty} \sqrt{\lambda_s} c_s(x) \xi_s(\omega) \ .$$

- Keeping the first M terms provides an approximation for the random field.

Weak formulation

Find $u \in H^1(D) \times L^2(\Omega)$ such that

$$a(u, v) = l(v), \;\; \forall v \in H_0^1(D) \times L^2(\Omega) \; .$$

$$a(u, v) = \int_\Omega \int_D c(x, \omega) \nabla u(x, \omega) \cdot \nabla v(x, \omega) dx \; dP(\omega)$$

$$l(v) = \int_\Omega \int_D f(x) v(x, \omega) dx \; dP(\omega)$$

Stochastic collocation method

- The physical space, $H^1(D)$, and probability space, $L^2(\Omega)$, are discretized separately.
- Because the random field is represented as a finite expansion of random variables, consider $L^2(\Gamma)$.
- A number of points, known as collocation points are selected from Γ.
- The deterministic finite element method is used to discretize $H^1(D)$ and to find the solution at each collocation point.
- Lagrange interpolation is used to find an approximation of u for points not in the set of collocation points.

Stochastic Galerkin method

- The stochastic Galerkin method is similar to stochastic collocation, except the discretization is found for the entire space.
- The stochastic discretization comes from polynomials of the random variables, where increasing the degree of the polynomials improves the approximation.
- This produces a larger matrix that is to be solved using a Galerkin finite element method.
- However, certain aspects of the structure of this matrix and/or its sparsity can be used to reduce this computation time– see [2],[4].

Issues

- How best to discretize the problem in space?
- How do we find a probability density function for $\eta_s(\omega)$ and/or sample it?
- Which approach to use? Galerkin vs. Collocation method
- Will preconditioning be used? – for more about this see [2].
- How many terms to keep in the series? Does this compare to the results when $a(x, \omega)$ was expanded?

Implementation

- Computer: Desktop with 1.9 GB RAM
- Language: Matlab R2008b
- Some previous code may be used for the Galerkin method and preconditioning.

Validation

- One way to solve this problem is using Monte Carlo simulations.
- For each sample of $c(x, \omega)$, the resulting pde can be solved using a deterministic finite element method.
- The moments from the Monte Carlo method will be compared to the results of the stochastic finite element method.

Milestones

Stage 1: October-Late November

- Clearly define the problem (what assumptions will be made?)
- Build the covariance matrix
- Compute the eigennodes
- Write code which generates Monte-Carlo solutions

Stage 2: Late November-December

- Run the Monte-Carlo simulations
- Begin construction of the principal components analysis

Milestones

Stage 3: December- late February

- Complete construction of PCA
- Write solution method

Stage 4: March - April

- Run numerical method
- Analyze accuracy and validity of the method
- Draw conclusions

Deliverables

- Code that calculates the moments of the solution to equation (1) using a Monte-Carlo method
- Code that calculates the moments of the solution to equation (1) using a KL expansion and stochastic evaluation technique
- Comparison of the results for a varying number of terms in the KL expansion
- Comparison of computational cost between the two methods

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