MRI Reconstruction via Fourier Frames on Interleaving Spirals
Project Proposal

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Talk Outline

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MRI Reconstruction

- Common inversion problem: given frequency information, reconstruct image in the spatial domain
- Sampling on interleaving spirals make for fast data acquisition (Bourgeois et al)

“Carolyn’s MRI”, by ClintJCL (Flickr)
The *Paley-Wiener space* $PW_E$ is defined as

$$PW_E = \{ \varphi \in L^2(\hat{\mathbb{R}}^d) : \text{supp } \varphi^\vee \subseteq E \},$$

where

- $\hat{\mathbb{R}}^d$ is the spectral equivalent of $\mathbb{R}^d$
- $E \subseteq \mathbb{R}^d$ is compact
- $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\hat{\mathbb{R}}^d)$ such that
  $$\mathcal{F}(f)(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \cdot \omega} \, dx$$
- $\varphi^\vee = \mathcal{F}^{-1}(\varphi)$
A frame is a sequence \( \{ x_n : n \in \mathbb{Z}^d \} \subseteq H \), a separable Hilbert space, for which there exist \( A, B > 0 \) such that

\[
\forall y \in H, \quad A \| y \|^2 \leq \sum_n |\langle y, x_n \rangle|^2 \leq B \| y \|^2.
\]

Let \( \Lambda \subseteq \hat{\mathbb{R}}^d \) be a sequence and let \( E \subseteq \mathbb{R}^d \) be compact. The sequence \( \{ e_\lambda : \lambda \in \Lambda \} \), where \( e_\lambda(x) = e^{-2\pi i x \cdot \lambda} \), defines a frame for \( PW_E \) if and only if there exist \( 0 < A \leq B < \infty \) such that

\[
\forall \varphi \in PW_E, \quad A \| \varphi \|_{L^2(\hat{\mathbb{R}}^d)} \leq \sum_{\lambda \in \Lambda} |\varphi(\lambda)|^2 \leq B \| \varphi \|_{L^2(\hat{\mathbb{R}}^d)}.
\]

We call such a sequence a Fourier frame for \( PW_E \) (Au-Yeung, Benedetto).
Let \( E = B(0, R) \subset \mathbb{R}^d \).

Let \( \Lambda \subseteq \mathbb{R}^d \) be uniformly discrete.

Let \( \text{dist}(\xi, \Lambda) = \inf_{\lambda \in \Lambda} \sqrt{\sum_{i=1}^{d} |\xi_i - \lambda_i|^2} \) denote the Euclidean distance between the point \( \xi \in \mathbb{R}^d \) and the set \( \Lambda \).

Define \( \rho = \rho(\Lambda) = \sup_{\xi \in \mathbb{R}^d} \text{dist}(\xi, \Lambda) \).

If \( R\rho < \frac{1}{4} \), then \( \Lambda \) is a Fourier frame for \( \text{PW}_{B(0, R)} \subseteq L^2(\mathbb{R}^d) \).

Every finite energy signal \( f \in L^2(E) \) can thus be represented as

\[
f(x) = \sum_{\lambda \in \Lambda} a_\lambda(f) e_\lambda 1_E
\]
Let \( \{A_k : k = 0, 1, ..., M - 1\} \subseteq \mathbb{R}^d \) denote a finite set of interleaving Archimedean spirals of the form

\[
A_k = \{ c\theta e^{2\pi i(\theta - (k/M))} : \theta \geq 0 \}.
\]

1. Choose \( \delta > 0 \) such that \( R \rho < 1/4 \).
2. For each \( k \), choose a uniformly discrete set \( \Lambda_k \) of points along \( A_k \) where the curve distance between consecutive points is less than \( 2\delta \), beginning within \( 2\delta \) of the origin.

The set \( \Lambda_R = \bigcup_{k=0}^{M-1} \Lambda_k \subseteq B = \bigcup_{k=1}^{M-1} A_k \) defines a Fourier frame for \( PW_{B(0,R)} \) (Benedetto, Wu).
Problem Overview

The goal of this project is to use nonuniform sampling on interleaving spirals to define a Fourier frame in $\hat{\mathbb{R}}^d$, from which we can reconstruct a low-resolution MRI image.

Images courtesy of U.S. Patent US5485086 A and “Carolyn’s MRI”, by ClintJCL (Flickr)
Computational Approach

\[ f(p) = f(x, y) = \sum_{b_j=0}^{N^2-1} f_{b_j} \mathbb{1}_{\square_{b_j}}(x, y) \]

\[ \hat{f}(\alpha) = \hat{f}(\lambda, \mu) = \sum_{b_j=0}^{N^2-1} f_{b_j} H_{b_j}(\lambda, \mu) \]

where

\[ H_{b_j}(\alpha) = H_{b_j}(\lambda, \mu) = \hat{1}_{\square_{b_j}}(\lambda, \mu) \]
Computational Approach

Choose $M \geq N^2$ points $\alpha_i = (\lambda_i, \mu_i)$ on the interleaving spirals to get

$$\hat{f}(\alpha_i) = \sum_{j=0}^{N^2-1} f_{bj} H_{bj}(\alpha_i).$$

(1)

Let

$$\hat{F} = [\hat{f}(\alpha_0) \, \hat{f}(\alpha_1) \ldots \hat{f}(\alpha_{M-1})]^T$$

and

$$F = [f_{b_0} \, f_{b_1} \ldots f_{b_{N^2-1}}]^T.$$ 

Let $H = [H_{bj}(\alpha_i)]_{i,j}$, and (1) becomes

$$\hat{F} = HF.$$ 

(2)
We will solve the least-squares problem

\[ F = (H^*H)^{-1} H^* \hat{F}, \]  

(3)

where

- $H$ is the Bessel map
  \[ \ell^2(\{0, 1, \ldots, N^2 - 1\}) \rightarrow \ell^2(\{0, 1, \ldots, M - 1\}) \]
- $H^*$ is its adjoint
- $H^*H$ is the frame operator

We consider the following algorithms:

- Transpose reduction (direct approach with efficient storage)
- Conjugate gradient algorithm
Let $A = H^*H$ and $b = H^*\hat{f}$. Define $V_j = (H_0(\alpha_j), ..., H_{N^2-1}(\alpha_j))^*$ s.t.

$$H = \begin{pmatrix}
H_0(\alpha_0) & \cdots & H_{N^2-1}(\alpha_0) \\
H_0(\alpha_1) & \cdots & H_{N^2-1}(\alpha_1) \\
\vdots & \ddots & \vdots \\
H_0(\alpha_{M-1}) & \cdots & H_{N^2-1}(\alpha_{M-1}) \\
\end{pmatrix} = \begin{pmatrix}
V_0^T \\
V_1^T \\
\vdots \\
V_{M-1}^T \\
\end{pmatrix}.$$ 

Note that

$$A = H^*H = \sum_{k=0}^{M-1} \begin{pmatrix}
H_0(\alpha_k) & \cdots & H_0(\alpha_k) \\
H_1(\alpha_k) & \cdots & H_1(\alpha_k) \\
\vdots & \ddots & \vdots \\
H_{N^2-1}(\alpha_k) & \cdots & H_{N^2-1}(\alpha_k) \\
\end{pmatrix} \begin{pmatrix}
H_0(\alpha_k) \\
H_1(\alpha_k) \\
\vdots \\
H_{N^2-1}(\alpha_k) \\
\end{pmatrix}$$

$$= \sum_{k=0}^{M-1} \begin{pmatrix}
H_0(\alpha_k) \\
H_1(\alpha_k) \\
\vdots \\
H_{N^2-1}(\alpha_k) \\
\end{pmatrix} \begin{pmatrix}
H_0(\alpha_k) & H_1(\alpha_k) & \cdots & H_{N^2-1}(\alpha_k) \\
\end{pmatrix}$$

$$= \sum_{k=0}^{M-1} V_k V_k^*.$$
Transpose Reduction

\[
b = \mathbf{H}^* \hat{f} = \begin{pmatrix}
\sum_{k=0}^{M-1} H_0(\alpha_k) \hat{f}_k \\
\vdots \\
\sum_{k=0}^{M-1} H_{N^2-1}(\alpha_k) \hat{f}_k
\end{pmatrix} = \sum_{k=0}^{M-1} \hat{f}_k V_k.
\]

To construct \( A = \mathbf{H}^* \mathbf{H} \) and \( b = \mathbf{H}^* \hat{f} \):

1. Let \( V_j = (H_0(\alpha_0), \ldots, H_{N^2-1}(\alpha_0))^* \)
2. Set \( A = V_j V_j^* \) and \( b = \hat{f}_0 V_j \)
3. For \( j = 0 : M - 1 \)
   - Set \( V_j = (H_0(\alpha_j), \ldots, H_{N^2-1}(\alpha_j))^* \)
   - \( A = A + V_j V_j^* \)
   - \( b = b + \hat{f}_j V_j \)

Factor of \( N^2 / M \) less memory than direct approach with naive storage.
Conjugate Gradient Method

Let \( A = H^*H \) and \( b = H^*\hat{f} \). To solve \( Af = b \) for symmetric, positive definite \( A \):

1. Choose \( f_0 \). Let \( r_0 = b - Af_0 \). Set \( p_0 = r_0 \).
2. for \( n = 1 \) until convergence
   \[ \gamma = \frac{(r_n^T r_n)/((A p_n)^T p_n)}{r_n^T r_n} \]
   \[ f_{n+1} = f_n + \gamma p_n \]
   \[ r_{n+1} = r_n - \gamma A p_n \]
   if norm(\( r_{n+1} \)) < tol, break
   \[ \beta_n = \frac{(r_{n+1}^T r_{n+1})/(r_n^T r_n)}{r_n^T r_n} \]
   \[ p_{n+1} = r_{n+1} + \beta_n p_n \]

Generally has linear convergence, but the speed of convergence depends on the condition number of \( A \).
Validation and Testing

Validation

- Small problems (on the order of $64 \times 64$ pixels) can be solved directly.
- CG algorithm should follow the same convergence trajectory as Matlab’s version.

Testing will primarily consist of error analysis.

Error measures:

- Signal-to-noise ratio (SNR)
- Structural Similarity measure (SSIM) (Wang et al)

Software: Matlab. Hardware: Acer Aspire V5 (6GB RAM)
October 2015: Code the sampling routine to form the Fourier frame.

November 2015: Proof of concept on small problems.

December 2015: Code the transpose reduction algorithm and begin testing.

January 2016: Code the conjugate gradient algorithm.

February - March 2016: Error analysis/testing. Explore how much frequency information we need to adequately recover \( f \). Explore condition number of \( H^*H \) and how it affects the reconstruction.

April 2016: Finalize results.
Deliverables

- Synthetic data set
- Fourier frame sampling routine
- Transpose reduction routine
- Conjugate gradient routine
- Final report and error analysis


Questions?