The Alternating Direction Method of Multipliers

Customizable software solver package

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The Dual Problem

- Consider the following problem (*primal problem*):
$$\min_x(f(x)) \text{ subject to } Ax = b.$$
- Important components of this problem:
  1. The Lagrangian: $L(x, y) = f(x) + y^T(Ax - b)$
     - We refer to the original x variable as the *primal variable* and the y variable as the *dual variable*.
  2. Dual function: $g(y) = \inf_x(L(x, y))$
     - New function made purely out of the dual variable.
     - Gives a lower bound on the objective value.
  3. Dual problem: $\max_{y \geq 0}(g(y))$
     - The problem of finding the best lower bound.
- End goal: recover $x^* = \arg\min_x(L(x, y^*))$, where x^* and y^* are corresponding optimizers.

The Alternating Direction Method of Multipliers (ADMM)

- Robustness of the Method of Multipliers.
- Supports Dual Decomposition $\rightarrow$ parallel x-updates possible.
- Problem form: (where f and g are both convex)

$$\min \left(f(x) + g(z)\right) \text{ subject to } Ax + Bz = c,$$

- Objective is separable into two sets of variables.
- ADMM defines a special Augmented Lagrangian to enable decomposition: $(r = Ax + Bz - c,\ u = y/\rho)$

$$\begin{aligned} L_\rho(x, z, y) &= f(x) + g(z) + y^T(r) + \frac{\rho}{2}||r||_2^2 \\ &= f(x) + g(z) + (\rho/2)||r + u||_2^2 - const \\ &= L_\rho(x, z, u) \end{aligned}$$

ADMM Algorithm

- Repeat for $k = 0$ to specified n, or until convergence:
  1. $x^{(k+1)} := \arg\min_x (L_\rho(x, z^{(k)}, u^{(k)}))$
  2. $z^{(k+1)} := \arg\min_z (L_\rho(x^{(k+1)}, z, u^{(k)}))$
  3. $u^{(k+1)} := u^{(k)} + (Ax^{(k+1)} + Bz^{(k+1)} - c)$
- Recall the *proximal operator*: (with $v = Bz^{(k)} - c + u^{(k)}$)

$$\mathbf{prox}_{f,\rho}(v) := \arg\min_x (f(x) + (\rho/2)||Ax + v||_2^2)$$

- If $g(z) = \lambda||z||_1$, then $\mathbf{prox}_{g,\rho}(v)$ is computed by soft-thresholding: (with $v = Ax^{(k+1)} - c + u^{(k)}$)

$$z_i^{(k+1)} := sign(v_i)(|v_i| - \lambda)_+$$

In this project...

- Our goal is to make ADMM easier to use in practice.
- Maximizing ADMM's potential means tweaking parameters such as step size ρ, starting values for x and z, efficient proximal operators, etc., for specific problem.
- Want a comprehensive library for general ADMM use.
  - Generalized ADMM functionality (with customizable options).
  - Adaptive step-size selection.
  - Ready to go optimized functions for problems ADMM is most used for (with customizable options).
  - High performance computing capabilities (MPI).
  - Implementations in Python and Matlab.

Prior Progress

1. Created a fully customizable general ADMM function:
   - Convergence checking of proximal operators.
   - Multiple types of stopping conditions.
   - Over/under relaxation.
   - Complete run-time information.
   - Accelerated and Fast ADMM
2. Created library of solvers for problems ADMM is used for:
   - **Constrained Convex Optimization:** Linear and Quadratic Programming.
   - ℓ_1 **Norm Problems:** Least Absolute Deviations, Huber Fitting, and Basis Pursuit.
   - ℓ_1 **Regularization:** Linear SVMs, LASSO, TVM, Sparse Inverse Covariance Selection, Logistic Regression.

Prior Progress (continued)

3. Testing and validation software for ADMM and solvers:
   - **For ADMM:** general solver (simple quadratic model) to test on.
   - **For solvers:** tester functions. Set up random problems and solve them, knowing the "correct" solution.
   - **Batch tester** to run solvers over a problem size scaling function.
4. Adaptive Step Sizes:
   - Tried several interpolation + 1D least squares methods:
     1. On Ye and Huan's $w = \left[x^T, z^T, u^T\right]^T$ values.
     2. On Esser's ϕ and ψ based residual.
   - Step sizes tended to explode.

Further Progress

- File organization and setup routines:
  - With solvers, testers, and other files, about 30 programs.
  - Organized into subfolders containing solvers, testers, examples.
  - Nifty routine to automatically setup paths no matter what file is run.
- Code Restructuring:
  - Streamlined solver code.
  - Added different algorithms do some solvers.
  - Prepped all code for parallel implementation.
- Implemented local, parallel capabilities into ADMM to use all cores efficiently.

Decomposition In ADMM

- Suppose function f is *separable* in $x = (x_1, \cdots, x_n)^T$; then:

$$f(x) = f_{n_1}(x_{n_1}) + \cdots + f_{n_m}(x_{n_m}), x = (x_{n_1}, \cdots, x_{n_m}), \sum_{i=1}^{m} n_i = n$$

- Can decompose the proximal operator for f.
- Thus, our x-minimization step in ADMM is split into m separate minimizations that can be carried out in parallel:

$$x_i^{(k+1)} := \mathbf{prox}_{f_{n_i},\rho}(x_{n_i}, z, u)$$

Parallelizing Updates

- Can use this observation to parallelize x-updates in ADMM.
- No reason this can't be done for g as well!
  - We often use simple z-updates (soft-thresholding, projections)
  - Can be updated component-wise, or block component-wise.
- For the u-update: $u^{(k+1)} := u^{(k)} + (Ax^{(k+1)} + Bz^{(k+1)} - c)$
  - Can compute $\hat{x} = Ax^{(k+1)}$ and $\hat{z} = Bz^{(k+1)}$ by similar parallel computation.
  - Clearly can update $u^{(k+1)}$ component-wise:

$$u_i^{(k+1)} := u_i^{(k+1)} + (\hat{x}_i + \hat{u}_i - c_i)$$

- Note that update chunks n_i can differ between x, z, and u.

Implementing Parallel Updates

- Interpret user provided proximal operators for f or g as component proximal operators:
  - Normally given function `proxf(x,z,u,rho)`.
  - Change to `proxf(x,z,u,i)` (Vector variables passed by reference)
- Instead of looping over i, distribute workload to workers (processors) in each update.
- User provides *slices* $(n_1, \cdots, n_m)$ for every update they wish to parallelize as acknowledgement to perform Parallel ADMM.
- In MATLAB, all this is easy to do for local parallel processes:
  - Use `parfor` loop over i on each parallel update (x or z).
  - Most matrix/vector operations already distributed among workers.

Example: LASSO Problem

- Standard LASSO formulation: $\min_x(1/2||Dx - s||_2^2 + \lambda||x||_1)$
- ADMM form: $\min(f(x) + g(z))$ subject to $x - z = 0$, where $f(x) = 1/2||Dx - s||_2^2$ and $g(z) = \lambda||z||_1$.
- $L_\rho(x, z, u) = f(x) + g(z) + (\rho/2)||x - z + u||_2^2 - const(u)$
- Proximal operator for f is x such that:

$$\nabla_x(L_\rho(x, z, u)) = D^T(Dx - s) + \rho(x - z + u) := 0$$

- Update step: $x := (D^T D + \rho I)^{-1}(D^T s + \rho(z - u))$
- Update for z can be parallel (soft-thresholding). What about the x update?

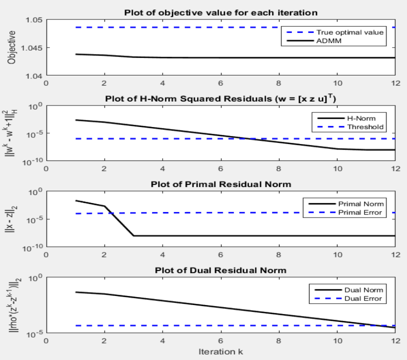
Parallel LASSO

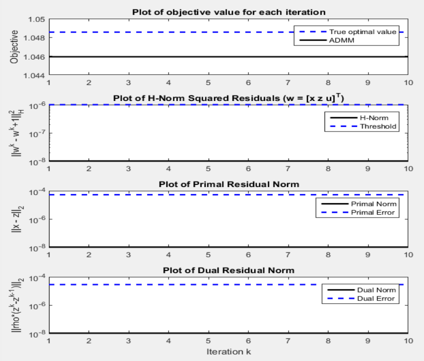
- Slice up rows of D and s into i chunks $\{D_{n_i}\}$ and $\{s_{n_i}\}$:

$$x_{n_i} := (D_{n_i}^T D_{n_i} + \rho I)^{-1}(D_{n_i}^T s_{n_i} + \rho(z_{n_i} - u_{n_i}))$$

- Consensus update $z = \mathbb{S}_{\lambda/(\rho N)}(\bar{x} + \bar{u})$.
- In both serial and parallel LASSO, cache Cholesky factorizations ($X = Y^T Y$) of matrix to invert and solve the system for updating.
- Parallel preprocessing:
  - You need to factor and store each chunk's decomposition.
  - Solution: Add parameter to `options` struct, `options.preprocess`, a function handle to local preprocessing function in user's program.

LASSO Serial vs. Parallel: 2^{11} Rows, 2^3 Columns





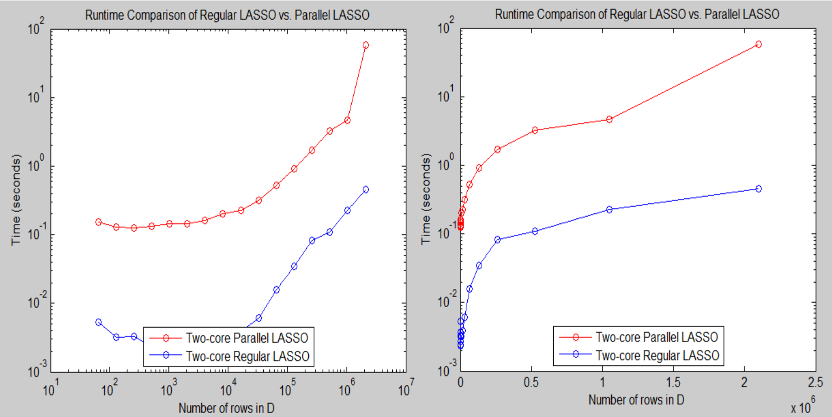
SERIAL

```
LASSO TEST RESULTS ---
True objective value: 1.0486
ADMM's objective value for x: 1.0431
ADMM's (x,z) objective value: 1.0431
Relative error in x objectives: 0.0051858
Number of iteration steps: 12
ADMM Runtime: 0.13614 seconds.
Solver Runtime: 0.70649 seconds.
```

PARALLEL

```
LASSO TEST RESULTS ---
True objective value: 1.0486
ADMM's objective value for x: 1.0464
ADMM's (x,z) objective value: 1.0459
Relative error in x objectives: 0.0021078
Number of iteration steps: 1
ADMM Runtime: 0.69316 seconds.
Solver Runtime: 0.85629 seconds.
```

LASSO Serial vs. Parallel Thorough Test



Transpose Reduction

- Want more efficient parallel x-update for skinny matrix D, which is typical.
- Note: $1/2||Dx - s||_2^2 = 1/2x^T(D^TD)x - x^TD^Ts + 1/2||s||_2^2$
- Now, a central server needs only D^TD and D^Tb. For tall, large D, D^TD has much fewer entries.
- Note that: $D^TD = \sum_i D_i^TD_i$ and $D^Tb = \sum_i D_i^Tb_i$.
- Now each server need only compute local components and aggregate on a central server.
- Once D^TD and D^Tb are computed, solve with ADMM.

Unwrapped ADMM

- Problem statement: $min_z(g(z))$ subject to $z = Dx$.
- ADMM form: $\min(f(x) + g(z))$ subject to $Dx - z = 0$, where $f(x) = 0$, and $g(z)$ is the same.
- Define the pseudoinverse of D as $D^+ = (D^T D)^{-1} D^T$
- As $f(x) = 0$, proximal operator for f is simply x such that:

$$\nabla_x(\rho/2||Dx - z + u||_2^2) = D^T(Dx - z + u) := 0$$

  which is simply $x = (D^T D)^{-1} D^T (z - u) = D^+(z - u)$.
- Cache D^+. For separable function g, can parallelize z update.
- Can we parallelize x update?

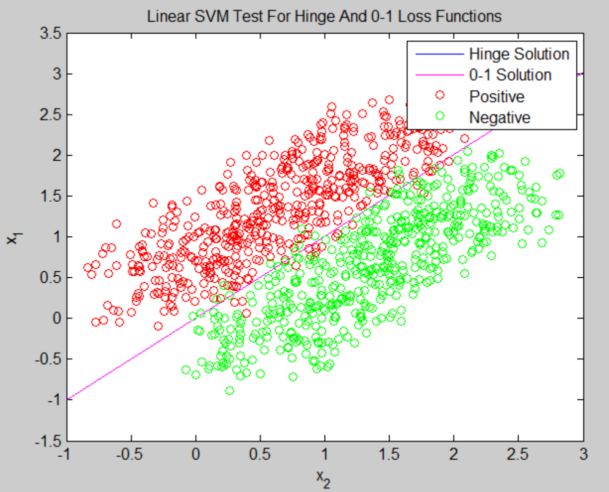
Unwrapped ADMM With Transpose Reduction

- Abuse Transpose Reduction: slice D into $D = [D_1^T \cdots D_N^T]^T$.
- Then update $x = D^+(z-u) = W \sum_i D_i^T(z_i - u_i)$, where $W = (\sum_i D_i^T D_i)^{-1}$.
- Note that for skinny D, W is very small and linear system solve much cheaper!
- In distributed setting, can:
  1. Store $D_i^T D_i$, D_i^T, z_i and u_i on each machine.
  2. Have central server compute and cache W.
  3. Central server adds up $d_i = D_i^T(z_i - u_i)$ into sum d and computes Wd.
- In local, parallel settings, can:
  1. Do everything distributed does, but locally.
  2. Compute summations in parallel.

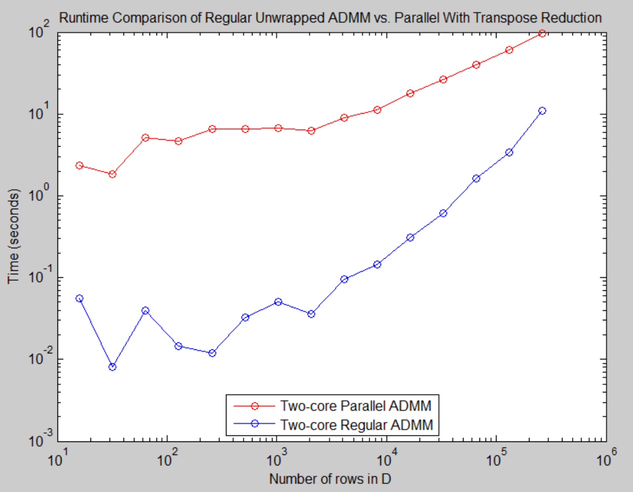
Example: Linear SVMs

- General Form: $\min(1/2||x||^2 + Ch(Dx))$, C a regularization parameter. D is training data, with ℓ the training labels.
- "Hinge loss" function: $h(z) = \sum_{k=1}^{M} \max(1 - \ell_k z_k, 0)$.
- Unwrapped ADMM can solve this problem, even for 0-1 loss.
- For hinge loss: $z^{k+1} = Dx + u + \ell \max(\min(1 - v, C/\rho), 0)$
- For 0-1 loss: $z^{k+1} = \ell \mathbb{I}(v \geq 1 \text{ or } v < (1 - \sqrt{2C/\rho}))$
- Here, $v = \ell(Dx + u)$
- Can use both parallel and serial Unwrapped ADMM, as z update is a component-wise computation.
- Perform parallel sums and preprocessing using `options.preprocess`.

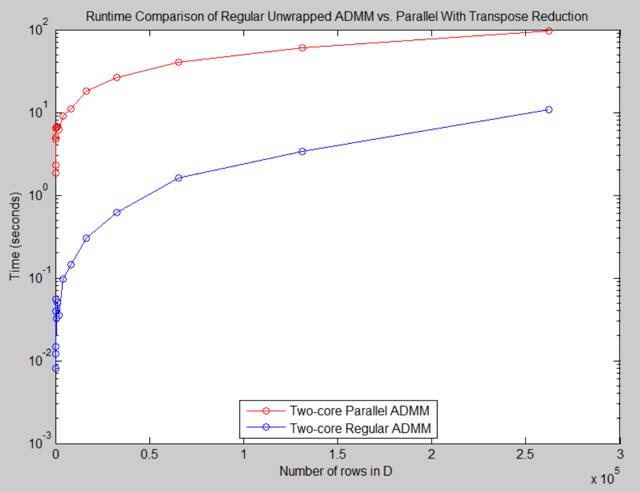
Solution: Serial vs. Parallel SVM for 2^9 Rows



Serial vs. Parallel: Thorough Test



Serial vs. Parallel: Thorough Test



Final Stretch

- Need to finish code restructuring on about 4 solvers.
- Need to add parallel versions a few more solvers.
- Test and validate everything using testers.
- Documentation.
- Write final report.

High Performance Computing

- We have an efficient parallel implementation of ADMM. Can take full advantage of all cores on a machine.
- Would like a distributed version:
  - Parallel ADMM allows for distributed computing.
  - Distribute to many machines, then use all cores with clever `parfor` usage.
  - Optimize my solvers for big data.
- Looking into `MatlabMPI` to do this.
  - Distributed computing via MPI-like behavior.
  - Potential to completely automate ADMM usage for big data.

Adaptive Step-Sizes

- Previous attempts at adaptive step-sizes had issue of blowup of stepsizes.
- Restarting at detection of blowups negates this. Tends to improve convergence regardless of starting stepsize.
- Drawback: no theoretical support for this.
- **Future work:** adaptive stepsizes with strong theoretical support and better results.

A Better Software Library

- Create general solvers and group more specific ones under them:
  - Streamlines code.
  - General solvers more useful for users.
- More solvers:
  - Need solvers for consensus and sharing problems.
  - Need more distributed solvers for big data.
- More user friendliness, examples, and documentation. Want this to be a base for future ADMM research.

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Thank you! Any questions?