Classification of Hand-Written Digits Using Scattering Convolutional Network - Mid-Year Report

Dongmian Zou
Email Address: zou@math.umd.edu
Department of Mathematics, University of Maryland, MD 20742

Advisor: Prof. Radu Balan
Email Address: rvbalan@math.umd.edu
Department of Mathematics, University of Maryland, MD 20742

Co-advisor: Dr. Maneesh Singh
Email Address: maneesh.singh@sri.com
SRI International, Princeton, NJ 08540

December 13, 2015

Abstract

In image classification it is advantageous to build classfiers based on feature extractors. It is demanded that the feature extractors unmask the identity of the image and remain stable with regard to small-scale deformations. A scattering convolutional network is such a feature extractor. In this project, we propose to use modern machine learning techniques to train the scattering convolutional network and test them on a standard database (MNIST) for the task of recognizing hand-written digits.

1 Background

Recently convolutional neural networks (ConvNets) have enjoyed conspicuous success in various pattern recognition tasks ([8], [9]). Although a clear understanding of why they perform so well is still absent, it is believed that it is the multi-layer structure that makes ConvNets outstanding ([3], [4]). As a feature extractor, a ConvNet is built to catch different features of an image in different layers. In [7], the authors apply wavelet theory to formulate a struture named Scattering Convolutional Network, which falls in the general categetory of ConvNets. The authors showed that the feature extractors they built are approximately invariant to translation and stable to small-scale deformation. The theory is extended to general semi-discrete frames in [11]. We are going to apply these theories to perform classification of hand-written digits.

1.1 Convolutional Networks

Convolutional networks (ConvNets) are artificial neural networks that "use convolution in place of general matrix multiplication in at least one place" ([1], Ch. 9). In one dimension, the convolution operation "$*$" is defined by

$$(x * h)(t) = \int x(s)h(t-s)ds \ .$$

In the above, despite the symmetric role of x and h, we usually call x the input, and h the filter. A digital image is a discretized two-dimensional object, whence we consider the convolution operator in two dimension with the discrete measure:

$$(X * H)(t_1, t_2) = \sum_{s_1} \sum_{s_2} X(s_1, s_2) H(t_1 - s_1, t_2 - s_2) \ .$$

In ConvNets, generally X is the input to the current layer and H is the filter which is commonly of much smaller size than X. The reason is both for computational efficiency and for sparse interactions between pixels ([1], Ch. 9). Since the size of the filter is small, it makes more sense to use the equivalent definition for the convolution, that is,

$$(X * H)(t_1, t_2) = \sum_{s_1} \sum_{s_2} X(t_1 - s_1, t_2 - s_2) H(s_1, s_2) \ . \tag{1}$$

Here t_1, t_2, s_1, s_2 are the pixel positions. Suppose $X \in \mathbb{R}^{D \times D}$ and $H \in \mathbb{R}^{d \times d}$ (we can replace $\mathbb{R}$ by $\mathbb{C}$ for some applications, but it does not affect our discussions here), in practice we would consider $D > d$. It is useful to think of X as a function

$X : \mathbb{Z}_D \times \mathbb{Z}_D \to \mathbb{R}$. Then it makes sense when $t_1 - s_1$ or $t_2 - s_2$ is a negative number in (1).

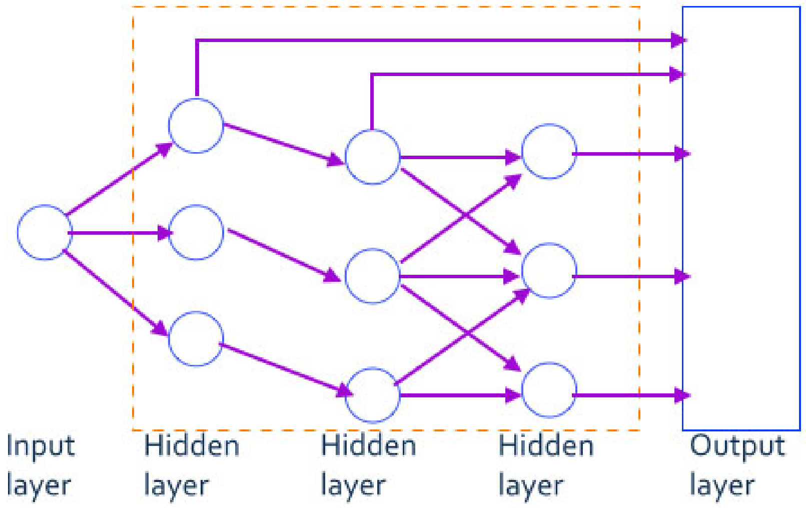


Figure 1: A typical ConvNet

Figure 1 illustrates a typical convolutional network. We see that between the input and the output it is natural to partition the neurons (operations) into several layers. In general, each layer of a ConvNet has three stages. The first stage is a convolution stage, where the input is convolved with filters; the second stage is a detector stage, where nonlinearity is applied to the output from the convolution stage; the last stage is a pooling stage, where local maximizing or averaging is applied to the output from the detector stage.

1.2 Scattering Convolutional Networks

Scattering Convolutional Networks are in the category of ConvNets with pre-defined filters ([7]). The filters are built by dilations of a wavelet. In the detector stage in each layer, the nonlinearity is chosen to be the absolute value function. In the pooling stage, a local averaging is done by convolution with a low-pass filter.

Let ψ be a wavelet in $L^2(\mathbb{R}^d)$. The dilation of ψ by λ, ψ_λ, is defined by

$$\psi_\lambda(t) = \lambda^d \psi(\lambda t) \ .$$

Consider a path $q = (\lambda_1, \lambda_2, \cdots, \lambda_m)$. The scattering propagator, $U[q]$, is defined by

$$U[q]x = |||x * \psi_{\lambda_1}| * \psi_{\lambda_2}| \cdots \psi_{\lambda_m}| \ ,$$

and the scattering transform, $S[q]$, is defined by

$$S[q]x = U[q]x * \phi_J \ ,$$

where $J > 0$ is some pre-determined scale. For an input signal x, in the m-th layer, a path $q = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ generates an output given by $S[q]x$. Figure 2 showes the process.

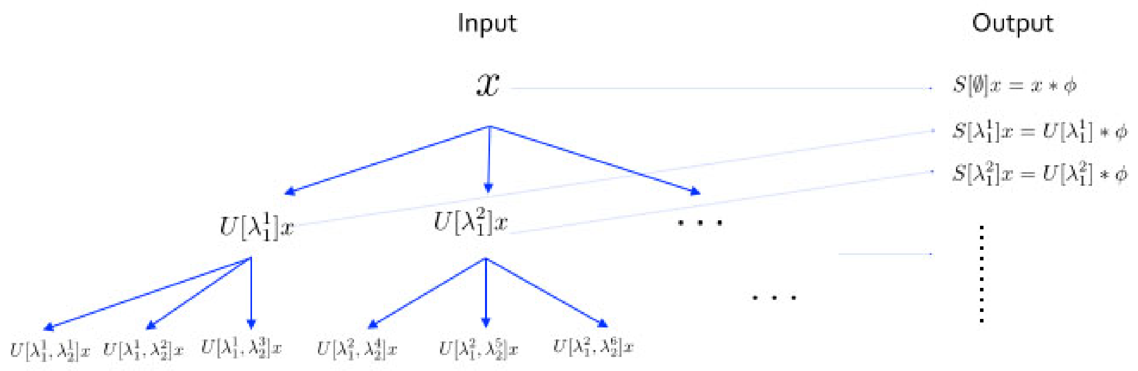

Input
Output
x
$S[\emptyset]x = x * \phi$
$S[\lambda_1^1]x = U[\lambda_1^1] * \phi$
$S[\lambda_1^2]x = U[\lambda_1^2] * \phi$
$U[\lambda_1^1]x$
$U[\lambda_1^2]x$
$U[\lambda_1^1, \lambda_2^1]x$
$U[\lambda_1^1, \lambda_2^2]x$
$U[\lambda_1^1, \lambda_2^3]x$
$U[\lambda_1^2, \lambda_2^4]x$
$U[\lambda_1^2, \lambda_2^5]x$
$U[\lambda_1^2, \lambda_2^6]x$

Figure 2: A typical scattering convolutional network

The Scattering Convolutional Networks are shown to be approximately tranlation invariant and stable to small-scale deformation. However, the stabily are guaranteed only if we choose a specifically designed wavelet to start with. In [11], the authors extend the theory to general semi-discrete frames. They use the same network structure but give a less strict condition on the filters.

1.3 Machine Learning Techniques

Our project is motivated by the above scattering network and its extension. Instead of fixing everything in the network beforehand, we want the computer to “tell” us what parameters to use for this structure. For instance, if we want to relate the input x and the output y by some function $y = f(x)$, then the process for determining the precise form of f is called the *training phase* ([2], Ch. 1). From the perspective of human, we “train” the system; while from the perspective of the machine, it “learns” from the given data set. For our project, we are given a set of training data, where each image has a “label” that tells us the de facto output.

The machine learning techniques we will use are not limited to but include the gradient descent optimization, the error backpropagation and the support vector machine.

1.3.1 Gradient Descent Optimization

In most machine learning tasks we would like to find the minimum of a *loss function* (also called a *cost function*). Suppose $l(\boldsymbol{\lambda})$ is the loss function with respect to the parameter $\boldsymbol{\lambda}$. The gradient descent is an iterative method for locating the minimum of l (see [1], Ch. 8; [2], Ch. 5). The updating step is given by

$$\boldsymbol{\lambda}^{(\nu+1)} = \boldsymbol{\lambda}^{(\nu)} - \eta \nabla l(\boldsymbol{\lambda}^{(\nu)}) \ . \tag{2}$$

where (ν) is the step number. At each step, $\boldsymbol{\lambda}$ travels towards the direction of the steepest descent. The scalar η is called the *learning rate* by the machine learning community. It is the same as the *stepsize* in standard optimization literature.

There are some alternative optimization methods, for example, conjugate gradients and quasi-Newton methods. They are in general more efficient than the gradient descent method. However, for training convolutional networks, we usually have a large amount of training data. Therefore, the loss function will be the sum of a large number of terms if we use deterministic methods. Taking this into consideration, deterministic methods are rarely used for network training. In practice, most training algorithms use stochastic approximation for the gradient. The conjugate gradients and quasi-Newton methods are not amenable to stochastic gradients. Nevertheless, it has been empirically demonstrated that the stochastic gradient descent method performs well for training ConvNets ([8], [9]). We will use both deterministic and stochastic methods and compare the results.

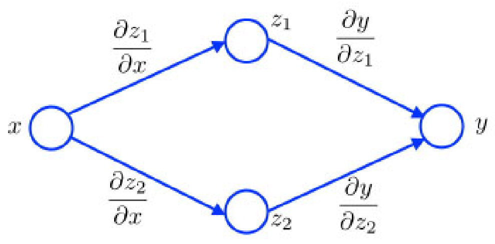
1.3.2 Error Backpropagation

The idea of *error backpropagation* is that the partial derivative of the loss function $l(\boldsymbol{\lambda})$ with respect to λ can be decomposed ([1], Ch. 6). It is based on the chain rule of taking derivatives.

Figure 3 epitomizes a simple neural network. The output y is connected to the input x through two intermediate outputs z_1 and z_2. Hence we have

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial z_1}\frac{\partial z_1}{\partial x} + \frac{\partial y}{\partial z_2}\frac{\partial z_2}{\partial x} \ .$$

It is as if we had a network in the opposite direction for the partial derivatives. In general, we compute the partial derivatives between each connected neurons, and



$\frac{\partial z_1}{\partial x}$
z_1
$\frac{\partial y}{\partial z_1}$
x
y
$\frac{\partial z_2}{\partial x}$
z_2
$\frac{\partial y}{\partial z_2}$

Figure 3: A simple illustration of backpropagation

then propagate backward along the network to get the derivative of the loss function with respect to the parameters that we want to train.

1.3.3 Support vector machine

The support vector machine (SVM) is a decision machine designed for two-class classification problems ([2], Ch. 7). A typical trained SVM has two parameters: the weights w and bias b. Suppose the two classes are given by $\{\pm 1\}$, then given an input y, the sign of $\langle w, y\rangle + b$ determines whether y falls into the class $\{-1\}$ or $\{+1\}$.

We would like to train the SVM parameters $\{w, b\}$. The idea of the SVM is to maximize the margin between the decision boundaries. It turns out that we need to maximize $\|w\|^{-1}$, that is, to minimize $\|w\|$. However, at the same time, we want to mildly penalize points that go to the wrong side. A standard way to do this is the following: suppose the inputs provided by the training data are $\{y_n\}_{n=1}^N$ with labels $\{a_n\}_{n=1}^N$ where each label $a_n \in \{\pm 1\}$, we want to solve

$$\min_{w,b} \quad \frac{1}{2}\|w\|^2 + C\sum_{n=1}^{N} l(y_n, a_n; w, b) \ ,$$

where C is a pre-determined parameter and l is the hinge-loss function defined by

$$l(y, a; w, b) = \max(0, 1 - a(b + \langle w, y\rangle)) \ ,$$

where $a \in \{\pm 1\}$ is the label corresponding to y. In the above, the parameter C seeks a balance between the two factors: maximizing the distance of the two classes (minimizing $\|w\|^2/2$) and minimizing the loss caused by points that are wrongly classified ($\sum_n l(y_n, a_n; w, b)$).

2 Approach

2.1 Network Structure

In this project, our task is the classification of hand-written digits. Our goal is to build a classifier that identifies the digit contained in our input image. Our classifier is composed of two parts. The first is a scattering convolutional network that extracts the features of the images and outputs a feature vector. The second is an SVM that does the classification job on the feature vector. In our project, the structure of the classifier is fixed, while some parameters in the classifier are to be either trained using machine learning techniques, or selected from a few possible choices.

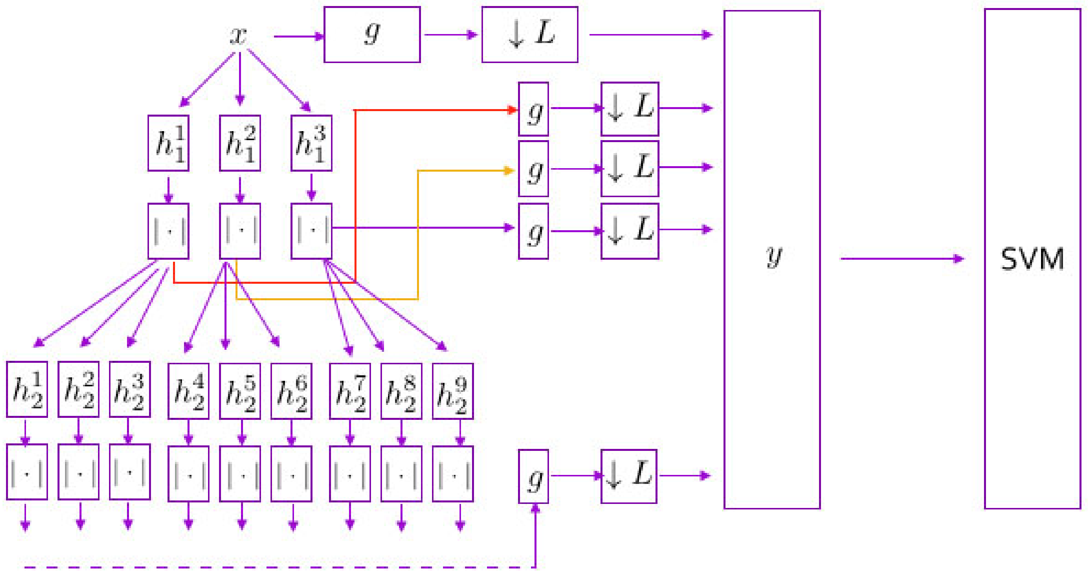


Figure 4: The structure of our scattering convolutional network

Figure 4 shows the detailed structure. The scattering network is a four-layer neural network that consists of one input layer, two convolutional layers and one output layer. In each convolutional layer, h_k^j is the filter to be trained; "$|\cdot|$" is the operation of taking absolute value pointwise; g is a fixed low-pass filter that does local averaging. A downsampling is done before we send the feature vector y to the SVM (this guarantees that the output of the scattering network is appoximately of the same size as the input). We select g and L from several possible candidates after cross-validation.

As we have discussed in Section 1.3, once we have determined all the parameters of the network in Figure 4, we can use it for the classification task. Suppose we have an image x and want to determine the digit it represents. We use x as an input and let it propagate through the network. It first goes through the scattering network to generate the feature vector y, and then the SVM determines which class x belongs to by evaluating $\langle w, y\rangle + b$.

In our project, the two-dimensional filters, h_k^j, are parametrized as dilations of the tensor products of two one-dimensional wavelets. We use the same pre-defined wavelet ψ for both. That is,

$$h_k^j(t_1,t_2) = \psi_{\lambda_{k,1}^j} \otimes \psi_{\lambda_{k,2}^j}(t_1,t_2) = \lambda_{k,1}^j \lambda_{k,2}^j \psi(\lambda_{k,1}^j t_1)\psi(\lambda_{k,2}^j t_2)\ .$$

2.2 Optimization Problem

Let $\boldsymbol{\lambda}$ denote the vector composed of all the λ's in the h's. The training process deals with the unknown parameters $\boldsymbol{\lambda}; w, b$. The trained $\boldsymbol{\lambda}; w, b$ are optimizers of the minimization problem of the SVM (see Section 1.3.3). Note that the feature vector y (which is the input of the SVM) now contains the unknown parameters $\boldsymbol{\lambda}$. Therefore, according the structure in Figure 4, the optimization problem for our project is as follows.

$$\min_{\boldsymbol{\lambda}; w, b} \quad \frac{1}{2}\|w\|^2 + C\sum_{n=1}^{N} l(y_n, a_n; w, b)\ , \tag{3}$$

where

$$l(y,a;w,b) = \max(0, 1 - a(b + \langle w, y\rangle))\ ,$$

and y is the grouping of the following
(recall that "$*$" is in the sense of Equation (1))

$$\begin{aligned} y_0 &= x * g\ ; \\ y_1^j &= \left|x * h_1^j\right| * g\ , 1 \le j \le 3\ ; \\ y_2^j &= \left|\left|x * h_1^{\lceil j/3\rceil}\right| * h_2^j\right| * g\ , 1 \le j \le 9\ , \end{aligned}$$

where the two-dimensional filters h_k^j is parametrized as

$$h_k^j(t_1,t_2) = \lambda_{k,1}^j \lambda_{k,2}^j \psi(\lambda_{k,1}^j t_1)\psi(\lambda_{k,2}^j t_2)\ .$$

The above training problem works for two-class classification, which is adapted to multi-class classification as we describe in the next section. In the optimization objective function (3), N can be either the total amount of training data (if we use a deterministic method), or the number of samples (if we use a stochastic method). The input of the network is N pairs of (x_n, a_n) where x_n's are images of 28×28 pixels (according to the database) and a_n's are the corresponding labels indicating which class x is in (normalized so that $a_n \in \{\pm 1\}$). The unknown parameters are the λ's in the filters and the w and b in the SVM. The optimization problem with respect to $\{\boldsymbol{\lambda}, w, b\}$ is a non-convex problem and difficult to solve. Nevertheless, if we fix $\boldsymbol{\lambda}$, then training the SVM is a convex optimization problem. Inspired by this fact, we will follow a two-step procedure. Specifically, we first fix $\boldsymbol{\lambda}$ and train w and b (which is convex), and then fix w and b to train $\boldsymbol{\lambda}$ (which is easier than the original problem and we use gradient descent for the minimization); and then iterate it until our stop criterion is met. We use a software called libSVM to train w and b, as described in Section 3.

The following diagram summarizes our algorithm.

Algorithm 1: The algorithm for network training

```
Start with learning rate $\eta$, regularization parameter $C$ ;
randomly generate $\boldsymbol{\lambda}$;
while stop criterion not met do
    sample $N$ examples $\{x_1, x_2, \cdots, x_N\}$ from the training set;
    get the corresponding labels $\{a_1, a_2, \cdots, a_N\}$;
    propagate forward to get $\{y_1, y_2, \cdots, y_N\}$;
    call libSVM with input $\{y_1, y_2, \cdots, y_N\}$, $\{a_1, a_2, \cdots, a_N\}$ and $C$;
    update $w, b \leftarrow$ output of libSVM;
    set $\boldsymbol{r} = 0$;
    for $n = 1$ to $N$ do
        compute $\nabla_{\boldsymbol{\lambda}} l(\boldsymbol{\lambda}; x_n)$;
        $\boldsymbol{r} \leftarrow \boldsymbol{r} + \nabla_{\boldsymbol{\lambda}} l(\boldsymbol{\lambda}; x_n)$;
    update $\boldsymbol{\lambda} \leftarrow \boldsymbol{\lambda} - \eta \boldsymbol{r}$ ;
    adapt $\eta$ accordingly.
```

2.3 Multi-Class Classification

Now we discuss the multi-class variation of the above method. An SVM is inherently a binary classifier. If we want to build a multi-class classifier based on SVM methods, we have to train multiple SVM's. Broadly speaking, there are two approaches: "one-against-all" and "one-against-one" ([6]).

1. One-against-all: We have 10 classes $\{0, 1, \cdots, 9\}$. We build 10 classifiers F_0, F_1, $\cdots$, F_9. For $p \in \{0, 1, \cdots, 9\}$, we use the label 1 if the input image is the number p and -1 if the input image is not the number p. Note that each classifier has parameters $\{\boldsymbol{\lambda}_p, w_p, b_p\}$. Our decision rule is

$$\text{output} \;=\; \arg\max_{p} \quad \langle w_p, y_p \rangle + b_p \ ,$$

where y_p is the feature vector obtained from the convolutional network in F_p.

2. One-against-one: We build a classifier for each pair of classes (45 classifiers in total). In this case, our decision rule is given by a voting strategy: for each pair of classes, we use the associated classifier to choose one class, and then the vote of that class is added by one. Our decision rule is that we take the class that has the largest number of votes (if two classes have the same number of votes, then take an arbitrary one).

Note that in the above, we build multiple classifiers with different filters in the convolutional network. If we want to apply the above principle but only use one feature extractor, then we need to formulate an optimization problem for the multi-class classification. The way to do this is not unique ([6]). One formulation based on (not exactly) "one-against-all" is:

$$\min_{\substack{\boldsymbol{\lambda}; w_p, b_p \\ p=0,\cdots,9}} \quad \frac{1}{2}\sum_{p=0}^{9} \|w_p\|^2 + C\sum_{n=1}^{N}\sum_{p\neq a_n} l(y_n; w_p, b_p) \ ,$$

where $a_n \in \{0, 1, \cdots, 9\}$,

$$l(y_n; w_p, w_p) = \max\left(0, 2 + \langle w_p, y_n\rangle + b_p - \langle w_{a_n}, y_n\rangle - b_{a_n}\right) \ ,$$

and y_n's are the same as given in Equation (3). The decision rule is the same as described in "one-against-all".

Optimization problem for "one-against-one" is rarely seen in literature. Nevertheless, if we want to use only one feature extractor for "one-against-one", we can consider the following problem:

$$\min_{\substack{\boldsymbol{\lambda}; w_q, b_q \\ q=1,\cdots,45}} \quad \frac{1}{2}\sum_{q=1}^{45} \|w_q\|^2 + C\sum_{n=1}^{N}\sum_{q: a_n\in A_q} l(y_n, a_n; w_q, b_q) \ ,$$

where $a_n \in \{0, 1, \cdots, 9\}$, A_q is the set composed of the two classes of the q-th pair, and

$$l(y_n, a_n; w_q, b_q) = \max(0, 1 - c_{n,q}(b_q + \langle w_q, y_n \rangle)) \ ,$$

where $c_{n,q} \in \{\pm 1\}$ depends on the labeling of the q-th SVM. Since w and b's are to be retrieved from the libSVM (which uses the "one-against-one" principle by default), we only need the partial derivatives of the objective function, which is simply the sum of the derivatives of the loss functions we considered in Section 2.2.

3 Implementation

3.1 Hardware and Software

We implement the algorithm on the personal laptop with

- CPU: 2GHz Intel Core i7
- Memory: 8 GB 1600 MHz DDR3
- OS: OS X El Capitan Version 10.11

The software we use is MATLAB R2015b. All the experiments are run in the terminal window without a GUI.

3.2 Database

The database we use is the standard MNIST database. MNIST is a publicly available database (http://yann.lecun.com/exdb/mnist/) of hand-written digits. The images are already preprocessed and formatted: they are of 28×28 pixels; each pixel has an value ranging from 0 to 255 (we normalize the pixel values to be in $[0, 1]$ when processing). The database provides 60,000 training images and 10,000 testing images. Examples of the images are given in Figure 5.

In the training examples, we do not have an equal number of images for each digit. We take 5,400 images for each digit for training purpose. For cross-validation purposes, we further divide the 5,400 images into 3,600 images for training, and 1,800 for testing.

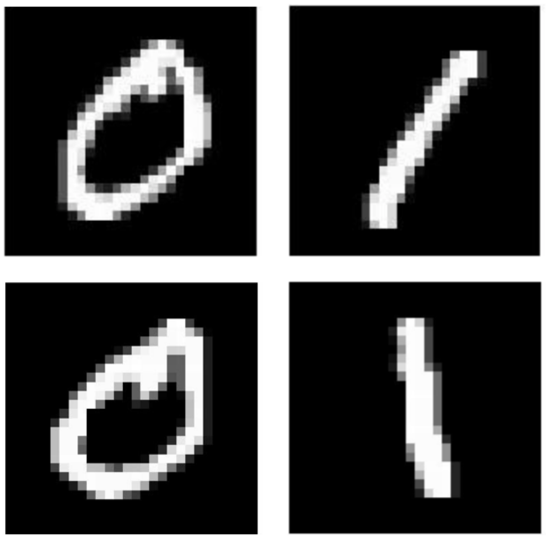


Figure 5: Examples of training images in the MNIST Database

3.3 Implementation Details

Our implementation is based on Algorithm 1 described in Section 2. We run both a stochastic version and a deterministic version. In the stochastic method, the number of samples, N, is taken to be a small number (We take $N = 10$ in this project); in the deterministic version, N is just the total number of images available for training (i.e. for two-class classification problem, $N = 2 \times 3600 = 7200$). We repeat several steps until $\boldsymbol{\lambda}$ converges or the number of iterations exceeds some value. Those steps are described in details as follows.

3.3.1 Forward Propagation

We randomly generate the starting value of $\boldsymbol{\lambda}$. We make sure that (1) the starting value is positive; (2) the starting value for each branch is apart from each other. For each training image x_n, we need to get a corresponding feature vector y_n. This is done by forward propagation through the network. In our implementation, the operation of convolution is done following a variant version of Formula (1). The detail is shown in Figure 6. In this way, the output $X * H$ is of the same size as X.

Note that there are 12 filters in our structure. Therefore, including the original signal x_n (which has size 28×28), the feature vector y_n has size $28 \times 28 \times 13$ before we downsample. Roughly speaking, the downsampled y_n has size $28 \times 28 \times 13/L^2$.

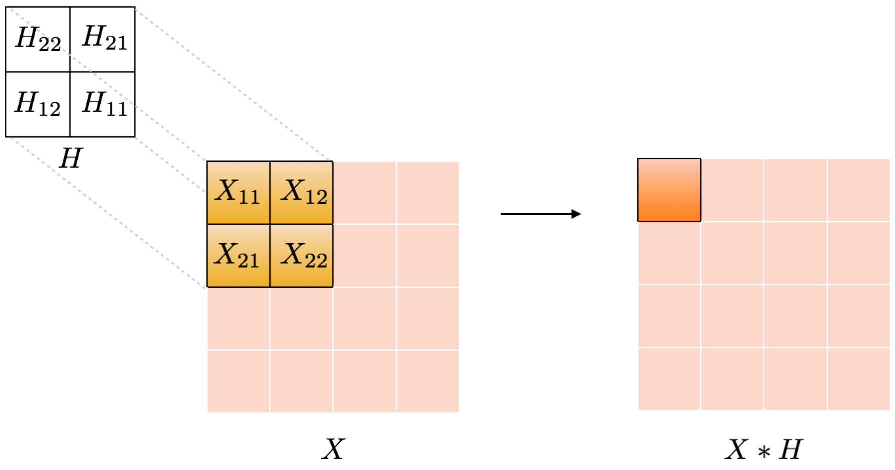


Figure 6: Convolution of X and H. Here the filter H is a small patch. We rotate it by 180° and align the upper left corner with the pixel in X that we want to compute the convolution value. For instance, as shown in this figure, $(X * H)_{11} = X_{11}H_{22} + X_{12}H_{21} + X_{21}H_{12} + X_{22}H_{11}$.

We can consider it as a column vector as we send it to the SVM.

3.3.2 LibSVM

Once we have the feature vector y_n's, we use libSVM to get w and b. LibSVM is a publicly available (https://www.csie.ntu.edu.tw/˜cjlin/libsvm/) software for training SVM's.

The main program of libSVM is written in C. Therefore, before using it we need to generate "mex" files for MATLAB to run libSVM. For training purposes, we call the file "svmtrain.mexmaci64". The input are two matrices: first, a matrix whose columns are the y_n's; second, a row vector whose entries are the label a_n's. The output is a strcture from which we retrieve the optimizers w and b. Here w is a vector whose size is the same as y_n (the same for each n), b is a scalar.

3.3.3 Back Propagation

After we update w and b from the SVM, we train the parameter $\boldsymbol{\lambda}$ from the filter h_k^j's in the convolution layer. We have 12 filters and thus 24 parameters to train. With fixed w and b, the minimization problem (3) is turned into

$$\min_{\boldsymbol{\lambda}} \quad \sum_{n=1}^{N} l(\boldsymbol{\lambda}; x_n) \ ,$$

where we use

$$l(\boldsymbol{\lambda}; x_n) := l(y_n, a_n; w, b), \ \forall n$$

to emphasize the fact that l depends on the unknown parameters $\boldsymbol{\lambda}$ and the training data x_n's. The updaing step follows Equation (2).

We now compute the gradient $\nabla_{\boldsymbol{\lambda}} l(\boldsymbol{\lambda}; x)$ for a fixed x. Note that the loss function $l(y, t; w, b)$ is not a smooth function with respect to y. We use a smooth function in place. Our choice is a C^1 function L defined by

$$L(y, a; w, b) = \begin{cases} 0.5 - a(b + \langle w, y \rangle) & , \text{if} \quad a(b + \langle w, y \rangle) \leq 0; \\ 0.5(1 - a(b + \langle w, y \rangle))^2 & , \text{if} \quad 0 < a(b + \langle w, y \rangle) \leq 1; \\ 0 & , \text{otherwise.} \end{cases}$$

Its partial derivative with respect to y is given by

$$\nabla_y L(y, a; w, b) = \begin{cases} -aw & , \text{if} \quad a(b + \langle w, y \rangle) \leq 1; \\ a(a(b + \langle w, y \rangle) - 1)w & , \text{if} \quad 0 < a(b + \langle w, y \rangle) \leq 1; \\ 0 & , \text{otherwise.} \end{cases}$$

Also, we need to use some smooth function F in place of $|\cdot|$. Our choice is a C^∞ function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(t) = (|t|^2 + \epsilon^2)^{1/2}$$

for some small ϵ. We use F (resp. F') for the operation of applying F (resp. F') pointwise as well (the same treatment as for $|\cdot|$). We have

$$F'(t) = \frac{t}{(|t|^2 + \epsilon^2)^{1/2}} \ .$$

Moreover, we use $\Psi(\lambda)$ defined by

$$\Psi(\lambda)(t) := \lambda \psi(\lambda t)$$

to emphasize the dependence on the variable λ. Note that

$$\frac{\partial L}{\partial \lambda_{k,i}^j} = \left\langle \nabla_{y_k^j} L, \frac{\partial y_k^j}{\partial \lambda_{k,i}^j} \right\rangle \ .$$

Now we give expressions for $\partial y^j_k / \partial \lambda^j_{k,i}$. We use $\odot$ to denote pointwise multiplication of two vectors or matrices of the same size (which is an abuse of notation, but we do not plan to use $\odot$ elsewhere). Since i takes value in $\{1, 2\}$, we use i' to denote the complement of i in $\{1, 2\}$. We have

$$\frac{\partial y^j_1}{\partial \lambda^j_{1,i}} = \left[F'\left(x * (\Psi(\lambda^j_{1,i}) \otimes \Psi(\lambda^j_{1,i'}))\right) \odot \left(x * (\Psi'(\lambda^j_{1,i}) \otimes \Psi(\lambda^j_{1,i'}))\right)\right] * g \ ;$$

$$\begin{aligned} \frac{\partial y^{3j-\iota}_2}{\partial \lambda^j_{1,i}} = \Bigg\{ & F'\left(F\left(x * (\Psi(\lambda^j_{1,i}) \otimes \Psi(\lambda^j_{1,i'}))\right) * \left(\Psi(\lambda^{3j-\iota}_{2,i}) \otimes \Psi(\lambda^{3j-\iota}_{2,i'})\right)\right) \odot \\ & \Bigg[\left[F'\left(x * (\Psi(\lambda^j_{1,i}) \otimes \Psi(\lambda^j_{1,i'}))\right) \odot \left(x * (\Psi'(\lambda^j_{1,i}) \otimes \Psi(\lambda^j_{1,i'}))\right)\right] \\ & * \left(\Psi(\lambda^{3j-\iota}_{2,i}) \otimes \Psi(\lambda^{3j-\iota}_{2,i'})\right)\Bigg]\Bigg\} * g \ , \quad \text{for } \iota = 1, 2, 3; \end{aligned}$$

$$\begin{aligned} \frac{\partial y^j_2}{\partial \lambda^j_{2,i}} = & \Bigg[F'\Bigg(F\Big(x * (\Psi(\lambda^{\lceil j/3\rceil}_{1,i}) \otimes \Psi(\lambda^{\lceil j/3\rceil}_{1,i'}))\Big) * \left(\Psi(\lambda^j_{2,i}) \otimes \Psi(\lambda^j_{2,i'})\right)\Bigg) \odot \\ & \Bigg(F\Big(\Big(x * (\Psi(\lambda^{\lceil j/3\rceil}_{1,i}) \otimes \Psi(\lambda^{\lceil j/3\rceil}_{1,i'}))\Big)\Big) * \left(\Psi'(\lambda^j_{2,i}) \otimes \Psi(\lambda^j_{2,i'})\right)\Bigg)\Bigg] * g \ . \end{aligned}$$

The above expression is simply given by the chain rule. We can compute $\partial L/\partial \lambda^j_{k,i}$ by backpropagation as described in Section 1.3. A sketched illustration is given in Figure 7.

The updating step is $\boldsymbol{\lambda}^{(\nu+1)} = \boldsymbol{\lambda}^{(\nu)} - \eta \sum_{n=1}^{N} \nabla_{\boldsymbol{\lambda}} l(\boldsymbol{\lambda}^{(\nu)}; x_n)$. Note that this formula does not guarantee that we still get a positive $\boldsymbol{\lambda}$. If any component of $\boldsymbol{\lambda}$ is non-positive, we replace η by a smaller one so that $\boldsymbol{\lambda}$ stays positive.

3.4 Cross-Validation

For parameter selection, recall that for each digit we split the 5,400 examples further into 3,600 for training and 1,800 for testing. We use cross-validation to determine the downsampling factor L and the low-pass filter g.

For the downsampling factor, since the size of the feature vector y_n is 13 times that of the input image x_n, we consider to select the downsampling factor from 3×3 (3 for row, and 3 for column), 4×4 and 5×5. For simplicity, we represent them as $L = 3, 4, 5$, respectively.

For the low-pass filter, we consider to select it between a standard sinc function (we denote this choice by LP) and a simple local averaging kernel (constant in each

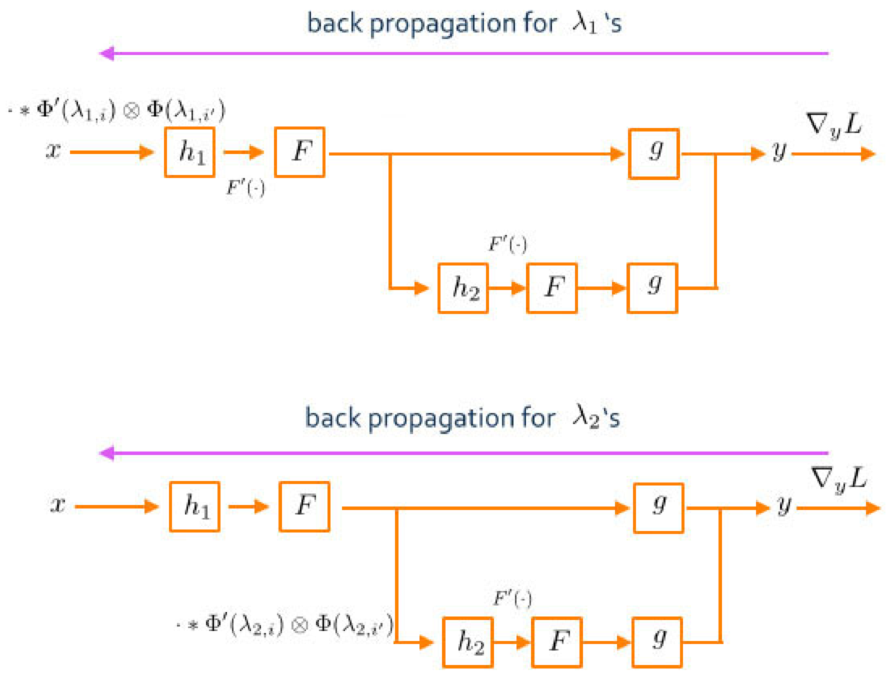

back propagation for λ_1's
$\cdot * \Phi'(\lambda_{1,i}) \otimes \Phi(\lambda_{1,i'})$
x
h_1
$F'(\cdot)$
F
g
y
$\nabla_y L$
$F'(\cdot)$
h_2
F
g
back propagation for λ_2's
x
h_1
F
g
y
$\nabla_y L$
$F'(\cdot)$
$\cdot * \Phi'(\lambda_{2,i}) \otimes \Phi(\lambda_{2,i'})$
h_2
F
g

Figure 7: Illustration of the backpropagation process ("$\cdot$" denotes the input from the last node). To compute the derivative, propagate backward, take the product of all marked values. Take the sum when two branches merge.

entry, which is equivalent to taking average value of neighboring pixels) (we denote this choice by AV). In the implementation, g is taken to be of size 4-by-4.

We run both a stochastic method and a deterministic method. For the stochastic method ($N = 10$), the result is demonstrated in Figure 8. We see that (LP) takes much longer time than (AV) and the error rate grows dramatically with the downsampling factor. On the other hand, the downsampling factor does not cause a big difference in the performance of (AV). We believe that "$L = 3$ / AV" is a reasonable choice for the stochastic method.

On the other hand, for the deterministic method, we do not implement (LP) since it is inefficient. Figure 9 shows the results for (AV). For each L, the running time is much longer than the stochastic method. Nevertheless, the error rate is small and in this case we have much smaller sum of loss function. There is no significant difference in performance for each L, and "$L = 3$ / AV" still appears to be a reasonable choice.

	L = 3 / LP	L = 4 / LP	L = 5 / LP	L = 3 / AV	L = 4 / AV	L = 5 / AV
Running time (hour)	14.5	10.2	8.5	5.5	5.2	5.3
Error rate (%) for “0”	1.67	7.44	13.5	0.72	3.17	3.56
Error rate (%) for “1”	0.39	2.11	0	0.11	0	0.11
Sum of loss for “0”	399.3	555.1	976.8	253.6	456.6	494.3
Sum of loss for “1”	202.1	454.5	336.7	135.7	122.9	63.8

Figure 8: Testing results for the stochastic method. For instance, the column starting with “$L = 3$ / LP” means that the training process takes 14.5 hours; the frequency of “reading 0 incorrectly” is 1.67 %; the frequency of “reading 1 incorrectly” is 0.39 %; $\sum_{\{n:a_n=0\}} l(y_n, a_n; w, b) = 399.3$; $\sum_{\{n:a_n=1\}} l(y_n, a_n; w, b) = 202.1$.

	L = 3 / AV	L = 4 / AV	L = 5 / AV
Running time (hour)	22.4	21.6	22.4
Error rate (%) for “0”	0.17	0.28	0.28
Error rate (%) for “1”	0.17	0	0.28
Sum of loss for “0”	8.6	10.6	15.7
Sum of loss for “1”	8.5	4.1	18.0

Figure 9: Testing results for the deterministic method

4 Validation

We will use the MatConvNet Toolbox to validate our implementation. MatConvNet is a publicly available (http://www.vlfeat.org/matconvnet/) MATLAB toolbox for convolutional neural networks. It provides routines for implementing linear convolution, back propagation, gradient descent, etc.

The main elements of MatConvNet are *computational blocks* and *wrappers*. Computational blocks are used for computing fundamental building blocks of a network (e.g. the small blocks in Figure 4). They can do linear convolution, local averaging, etc. A wrapper is used for specifying the structure of the network (in our case, a chain of blocks).

We will use the built-in functions of MatConvNet and train our model discussed in Section 2. We need to specify the filters and the structure, and some parameters that we use. We also need to modify the loss function so that we can use libSVM together with MatConvNet. Ideally, with the same algorithm, this should work in a similar manner as our previous implementation and provide comparable results.

5 Testing

We will use the testing images in the MNIST database for testing our trained network. Our measure will still be the percentage of errors and the sum of loss functions. We will run libSVM independently (i.e. using SVM for classification without a feature extractor) and compare the results. We are first going to do it for the 2-class task. Time permitting, we are going to compare the multi-class results.

6 Project Schedule and Milestones

- September - October 2015: Define the project. Investigate in existing literature. Design the algorithm. (finished)
- November 2015: Write codes for training the convolutional filters. (finished)
- December 2015: Write codes for two-class classfication. (finished)
- December 2015: Write codes for multi-class classfication. (In progress)
- February 2016: Complete validation. (In progress)
- March 2016: Complete testing. (To do)
- April 2016: Wrap up the project. (To do)

7 Delivarables

At the end of the project, we should be able to deliver:

- the datasets
- the toolboxes
- the MATLAB codes
- the trained network
- the results
- the proposal, reports, presentation slides, etc.

References

[1] Y. Bengio, I. J. Goodfellow and A. Courville, *Deep Learning*, Book in preparation for MIT press, 2015.

[2] C. M. Bishop, *Pattern Recognition and Machine Learning*, New York: Springer, 2006.

[3] J. Bruna et al., *A Theoretical Argument for Complex-Valued Convolutional Networks*, submitted, 2015.

[4] J. Bruna and S. Mallat, *Invariant Scattering Convolution Networks*, Pattern Analysis and Machine Intelligence, IEEE Transactions 35(8) (2013), 1872–1886.

[5] C-C. Chang and C-J. Lin, *LIBSVM : a library for support vector machines*. ACM Transactions on Intelligent Systems and Technology 27(2) (2011), 1–27. Software available at http://www.csie.ntu.edu.tw/˜cjlin/libsvm.

[6] C-W. Hsu and C-J. Lin, *A Comparison of Methods for Multiclass Support Vector Machines*, Neural Networks, IEEE Transactions 13(2) (2002), 415–425.

[7] S. Mallat, *Group Invariant Scattering*, Comm. Pure Appl. Math., 65 (2012): 1331-1398.

[8] A. Krizhevsky, I. Sutskever, G. E. Hinton, *ImageNet Classification with Deep Convolutional Neural Networks*, Advances in Neural Information Processing Systems 2 (2012): 1097-1105.

[9] C. Szegedy et al, *Going Deeper with Convolutions*, Open Access Version available at http://www.cv-foundation.org/openaccess/content_cvpr_2015/papers/Szegedy_Going_Deeper_With_2015_CVPR_paper.pdf, retrieved October 9, 2015.

[10] A. Vedaldi and K. Lenc, *MatConvNet - Convolutional Neural Networks for MATLAB*, Proc. of the ACM Int. Conf. on Multimedia, 2015.

[11] T. Wiatowski and H. Bölcskei, *Deep Convolutional Neural Networks Based on Semi-Discrete Frames*, Proc. of IEEE International Symposium on Information Theory (ISIT), Hong Kong, China, pp. 1212–1216, June 2015.