Second Order Linear Equations: Nonhomogeneous Equations

(Sections 3.5, 3.6)

Brief Summary:

1. General Results
2. Method of Undetermined Coefficients
3. Variation of Parameters

1. General Results

We want to solve:

$$y'' + p(t)\cdot y' + q(t)\cdot y = g(t) \qquad (1)$$

where p, q, g are continuous functions on an open interval I.

We are interested in the structure of the solutions.

The following result solves this issue:

Theorem [3.5.2] The general solution of the nonhomogeneous equation (1) can be written in the form:

$$y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where y_1, y_2 are a fundamental set of solutions of the corresponding homogeneous equation $L[y] = y'' + p(t)\cdot y' + q(t)\cdot y = 0$, c_1 and c_2 are arbitrary constants, and Y is some specific solution of the nonhomogeneous equation (1).

$\square$

Why ?

(1) Assume first that y_1, y_2 satisfy $L[y_1] = L[y_2] = 0$

and Y is a specific solution of $L[Y] = g$.

Then:

$$L[c_1 y_1 + c_2 y_2 + Y] = L[c_1 y_1] + L[c_2 y_2] + L[Y] =$$

$$= c_1 \underbrace{L[y_1]}_{0} + c_2 \underbrace{L[y_2]}_{0} + \underbrace{L[Y]}_{g} = g.$$

Hence $c_1 y_1 + c_2 y_2 + Y$ is a solution of (1).

(2) Assume that $\Phi(t)$ is a solution of (1), and Y is a specific solution as in hypothesis. Then:

$$L[\Phi - Y] = L[\phi] - L[Y] = g - g = 0$$

Hence $\phi - Y$ is a solution of the homogeneous equation $L[y] = 0$.

Since (y_1, y_2) is a fondamental set of solutions of $L[y] = 0$

it follows:

$$\phi - Y = c_1 y_1 + c_2 y_2$$

for some (real) constants c_1, c_2. Thus:

$$\phi = c_1 y_1 + c_2 y_2 + Y.$$

The "morale" of this result is the following recipe:

To solve $y'' + p(t)\cdot y' + q(t)\cdot y = g(t)$ we must do three things.

(1) Find the general solution $y_c(t) = c_1 y_1(t) + c_2 y_2(t)$ of the corresponding homogeneous equation $y'' + p(t)\cdot y' + q(t)\cdot y = 0$.

(2) Find some single solution $Y(t)$ of the nonhomogeneous equation

(3) Add together $y_c(t) + Y(t)$.

→ We know how to solve (1) – the general solution of the homogeneous equation.

⇒ We will concentrate on finding one particular solution of the nonhomogeneous equation.

Two methods:

(A) Method of Undetermined Coefficients

- It is straight forward to execute
- Works for homogeneous equations with constant coefficients and nonhomogeneous terms restricted to a relatively small class of functions

(B) Variation of Parameters

- It is computationally more difficult
- It is a general method: works for all nonhomogeneous equations.

(A) Method of Undetermined Coefficients

This method applies to equations of the form:

$$ay'' + by' + cy = g(t)$$

with $g(t)$ a combination of $e^{r \cdot t}$, $\cos(\alpha t)$, $\sin(\alpha t)$, $d_0 t^n + d_1 t^{n-1} + \dots + d_{n-1} t + \dots$

Example 1 Find a particular solution of:

$$y'' - 3y' - 4y = 3e^{2t} \qquad (*)$$

Solution.

The right-hand side term $3e^{2t}$ suggests to look for a solution of the form $A e^{2t}$. Substitute into (the) equation:

$$A\, 2 \cdot 2 e^{2t} - 3A \cdot 2e^{2t} - 4Ae^{2t} = 3e^{2t}$$

$$-6Ae^{2t} = 3e^{2t}$$

$$\Rightarrow \quad A = -\frac{1}{2}$$

Thus:

$$Y(t) = -\frac{1}{2} e^{2t}$$

is a particular solution.

Remark: The general solution of $(*)$ is obtained by solving first the homogeneous equation: $y'' - 3y' - 4y = 0$

charact. eqn.: $r^2 - 3r - 4 = 0$

$$r_{1,2} = \frac{3 \pm \sqrt{9+16}}{2} \quad \begin{cases} r_1 = 4 \\ r_2 = -1 \end{cases}$$

Hence:

$$y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2} e^{2t}$$

Example 2 Find a particular solution of

$$y'' - 3y' - 4y = 2\sin(t)$$

Solution

The right-hand side suggests to look for a solution of the form:

$$Y(t) = A\sin(t) + B\cos(t)$$

Substitute:

$$\begin{array}{r|l} -4 & Y = A\sin(t) + B\cos(t) \\ -3 & Y' = A\cos(t) - B\sin(t) \\ & Y'' = -A\sin(t) - B\cos(t) \end{array}$$

$$-4Y - 3Y' + Y'' = (-4A + 3B - A)\sin(t) + (-4B - 3A - B)\cos(t) \overset{\text{must be}}{=} 2\sin(t)$$

$$\Rightarrow \begin{cases} -4A + 3B - A = 2 \\ -4B - 3A - B = 0 \end{cases} \qquad \begin{cases} 3B - 5A = 2 \\ -5B - 3A = 0 \end{cases} \longrightarrow B = -\frac{3}{5}A$$

$$\rightarrow -\frac{9}{5}A - 5A = 2 \qquad -\frac{34}{5}A = 2 \qquad A = -\frac{10}{34} = -\frac{5}{17}$$

$$B = \frac{3}{17}$$

$$\Rightarrow \quad Y(t) = -\frac{5}{17}\sin(t) + \frac{3}{17}\cos(t)$$

Remark The general solution of $y'' - 3y' - 4y = 2\sin(t)$ is:

$$y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{5}{17}\sin(t) + \frac{3}{17}\cos(t).$$

Example 3 Find a particular solution of

$$y'' - 3y' - 4y' = 4t^2 - 1$$

Solution

We seek a solution of the form:

$$Y(t) = At^2 + Bt + C$$

Substitute:

$$\begin{array}{r|l} -4 & Y(t) = At^2 + Bt + C \\ -3 & Y'(t) = 2At + B \\ & Y''(t) = 2A \end{array}$$

$$-4Y - 3Y' + Y'' = -4A\,t^2 + (-4B - 6A)t - 4C - 3B + 2A \overset{\text{must be}}{=} 4t^2 - 1$$

$$\begin{cases} -4A = 4 \\ -4B - 6A = 0 \\ -4C - 3B + 2A = -1 \end{cases} \qquad \begin{aligned} A &= -1 \\ B &= \frac{3}{2} \\ C &= \frac{1 - 2 - \frac{9}{2}}{4} = -\frac{11}{8} \end{aligned}$$

$$\Rightarrow \quad Y(t) = -t^2 + \frac{3}{2}t - \frac{11}{8}$$

Remark: The general solution is:

$$y(t) = c_1 e^{4t} + c_2 e^{-t} - t^2 + \frac{3}{2}t - \frac{11}{8}.$$

Example 4

$$y'' - 3y' - 4y = -8e^{t}\cos(2t)$$

Solution

Look for a solution of the form:

$$Y(t) = Ae^{t}\cos(2t) + Be^{t}\sin(2t)$$

Substitute:

$$\begin{array}{r|l}
-4 & Y = Ae^{t}\cos(2t) + Be^{t}\sin(2t) \\
-3 & Y' = Ae^{t}\cos(2t) - 2Ae^{t}\sin(2t) + Be^{t}\sin(2t) + 2Be^{t}\cos(2t) \\
1 & Y'' = (A+2B)e^{t}\cos(2t) - 2(A+2B)e^{t}\sin(2t) + (B-2A)e^{t}\sin(2t) + \\
 & \qquad + 2(B-2A)e^{t}\cos(2t)
\end{array}$$

$$-4Y - 3Y' + Y'' = \left[(A+2B) + 2(B-2A) - 3(A+2B) - 4A\right]e^{t}\cos(2t) +$$

$$+ \left[(B-2A) - 2(A+2B) - 3(B-2A) - 4B\right]e^{t}\sin(2t) \overset{\text{must be}}{=} -8e^{t}\cos(2t)$$

$$\begin{cases} (A+2B) + 2(B-2A) - 3(A+2B) - 4A = -8 \\ (B-2A) - 2(A+2B) - 3(B-2A) - 4B = 0 \end{cases} \qquad \begin{cases} -10A - 2B = -8 \\ -10B + 2A = 0 \end{cases} \quad \begin{array}{l} -52B = -8 \\ \uparrow \\ \rightarrow A = 5B \end{array}$$

$$B = \frac{8}{52} = \frac{2}{13} \rightarrow A = \frac{10}{13}$$

$$\Rightarrow \quad Y(t) = \frac{10}{13}e^{t}\cos(2t) + \frac{2}{13}e^{t}\sin(t).$$

Example 5

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin(t) - 8e^{t}\cos(2t)$$

Solution

Let's solve first:

$$y'' - 3y' - 4y = 3e^{2t}$$

$$y'' - 3y' - 4y = 2\sin(t)$$

$$y'' - 3y' - 4y = -8e^{t}\cos(2t).$$

See examples 1, 2, 4 :

$$Y_1(t) = -\frac{1}{2}e^{2t}$$

$$Y_2(t) = \frac{3}{17}\cos(t) - \frac{5}{17}\sin(t)$$

$$Y_3(t) = \frac{10}{13}e^{t}\cos(2t) + \frac{2}{13}e^{t}\sin(2t).$$

$$Y(t) = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos(t) - \frac{5}{17}\sin(t) + \frac{10}{13}e^{t}\cos(2t) + \frac{2}{13}e^{t}\sin(2t)$$

Example 6

Find a particular solution of

$$y'' - 3y' - 4y = 2e^{-t}$$

Solution

Let's apply the same approach and search for a solution of the form:

$$Y(t) = Ae^{-t} \qquad (\square)$$

Substitute into equation:

$$\begin{array}{r|l} -4 & Y = Ae^{-t} \\ -3 & Y' = -Ae^{-t} \\ 1 & Y'' = Ae^{-t} \end{array}$$

$$Y'' - 3Y' - 4Y = \underbrace{(A + 3A - 4A)}_{0} e^{-t} \overset{\text{must be}}{=} 2e^{-t}$$

$$0 = 2e^{-t} \Rightarrow \text{Not possible!}$$

The reason for this difficulty is the fact that e^{-t} is a solution of the homogeneous equation.

Then, we replace the form $(\square)$ by:

$$Y(t) = Ate^{-t}$$

Substitute:

$$\begin{array}{r|l} -4 & Y = Ate^{-t} \\ -3 & Y' = Ae^{-t} - Ate^{-t} \\ 1 & Y'' = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t} \end{array}$$

$$Y'' - 3Y' - 4Y = (-2A - 3A)e^{-t} + At\underbrace{(1 + 3 - 4)}_{0}e^{-t} \overset{\text{must be}}{=} 2e^{-t}$$

$$-5Ae^{-t} = 2e^{-t}$$

$$\Rightarrow A = -\frac{2}{5}$$

Thus: $Y(t) = -\frac{2}{5}te^{-t}$

Let's summarize the algorithm to find the general solution of

$$ay'' + by' + cy = g(t)$$

1. Find the general solution of the homogeneous equation:

$$ay'' + by' + cy = 0$$

2. Check that $g(t)$ is a linear combination of terms of the form:

$$P_n(t)\, e^{a_i t}, \quad P_n(t)\, e^{a_i t} \cos(\alpha t), \quad P_n(t)\, e^{a_i t} \sin(\alpha t)$$

If this is not the case, use the method of variation of parameters.

3. If $g(t) = g_1(t) + g_2(t) + \dots + g_m(t)$

then solve each subproblem separately:

$$ay'' + by' + cy = g_1(t)$$

$$\vdots$$

$$ay'' + by' + cy = g_m(t)$$

4. In each subproblem i, $1 \le i \le m$, consider a solution of the form: $Y_i(t) = \text{Polynomial} \cdot e^{a_i t}$, or $Y_i(t) = \text{Polynomial}_1 \cdot e^{a_i t} \cos(\alpha t) + \text{Polynomial}_2 \cdot e^{a_i t} \sin(\alpha t)$

where the polynomial has the same degree as ~~the~~ $P_n(t)$ that shows up in $g_i(t)$.

<u>Note</u>: If this Y_i turns out to be a solution of the homogeneous equation, the multiply Y_i by t, or t^2, to remove the duplication.

5. Add together the general solution of the homogeneous equation with $Y_1, \dots, Y_m$.

The Method of Undetermined Coefficients for Higher Order Linear Differential Equations

(Section 4.3)

2.4.11

The nonhomogeneous n^{th} order linear equation with constant coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = g(t)$$

1) When:

$$g(t) = A_0 t^m + A_1 t^{m-1} + \dots + A_{m-1} t + A_m$$

try:

$$Y(t) = B_0 t^m + B_1 t^{m-1} + \dots + B_{m-1} t + B_m$$

2) For:

$$g(t) = e^{rt}$$

try:

$$Y(t) = A e^{rt}$$

3) For:

$$g(t) = \sin(\alpha t) \quad \text{or} \quad g(t) = \cos(\alpha t)$$

try

$$g(t) = A \sin(\alpha t) + B \cos(\alpha t)$$

When g solves the homogeneous equation, than multiply Y with t^s, for s larger than the multiplicity of that zero of the characteristic polynomial

Example Find the general solution of

$$y''' - 3y'' + 3y' - y = 4e^t \qquad (*)$$

Solution

1. Find the general solution of the homogeneous equation:

$$y''' - 3y'' + 3y' - y = 0$$

Characteristic equation:

$$r^3 - 3r^2 + 3r - 1 = 0.$$

$$(r-1)^3 = 0 \Rightarrow r_1 = r_2 = r_3 = 1$$

$$y_1(t) = e^t, \quad y_2(t) = te^t, \quad y_3(t) = t^2 e^t$$

2. Since the right-hand side $g(t) = 4e^t$ includes a solution of the homogeneous equation, we need to multiply g by t^s, here $s = 3$

$$Y(t) = A t^3 e^t$$

Substitute in $(*)$:

$$\begin{array}{r|l}
-1 & Y(t) = At^3 e^t \\
3 & Y'(t) = 3At^2 e^t + At^3 e^t \\
-3 & Y''(t) = 6Ate^t + 6At^2 e^t + At^3 e^t \\
1 & Y'''(t) = 6Ae^t + 18Ate^t + 9At^2 e^t + At^3 e^t
\end{array}$$

$$Y''' - 3Y'' + 3Y' - Y = 6Ae^t + \underbrace{(18At - 18At)}_{0} e^t + \underbrace{(9At^2 - 18At^2 + 9At^2)}_{0} e^t + \underbrace{(At^3 - 3At^3 + 3At^3 - At^3)}_{0} e^t$$

$$= 6Ae^t \underset{\substack{\uparrow \\ \text{must be}}}{=} 4e^t$$

$$\rightarrow A = \frac{2}{3} \Rightarrow Y(t) = \frac{2}{3} t^3 e^t$$

The general solution is then:

$$y(t) = c_1 e^t + c_2 te^t + c_3 t^2 e^t + \frac{2}{3} t^3 e^t$$

[B] Variation of Parameters

This is a general method that applies to all nonhomogeneous linear equations. However it is computationally more involved.

Example

$$y'' + 4y = 3\csc(t) \qquad (*)$$

$$\left(\csc(t) = \frac{1}{\sin(t)} \right)$$

Solution

Since $\frac{1}{\sin(t)}$ is not in the class treated by the method of undetermined coefficients, we need to use the variation of parameters

1) Solve the homogeneous equation:

$$y'' + 4y = 0$$

Charact. eqn. : $r^2 + 4 = 0 \quad \Rightarrow \quad r_{1,2} = \pm 2i$

$$\Rightarrow \quad y_1(t) = \cos(2t)$$
$$y_2(t) = \sin(2t)$$

2) We seek a solution of the form:

$$y(t) = u_1(t) \cdot \cos(2t) + u_2(t) \cdot \sin(2t)$$

~~Substitute into equation (*)~~

Compute:

$$y_p'(t) = -2\,u_1(t)\sin(2t) + u_1'(t)\cos(2t) + 2\,u_2(t)\cos(2t) + u_2'\cdot\sin(2t)$$

We choose to impose the following constraint:

$$u_1'\cdot\cos(2t) + u_2'\cdot\sin(2t) = 0. \qquad (1)$$

Then:

$$y_p' = -2u_1\sin(2t) + 2u_2\cdot\cos(2t)$$

$$\Rightarrow \quad y'' = -4u_1\cos(2t) - 2u_1'\sin(2t) - 4u_2\sin(2t) + 2u_2'\cdot\cos(2t)$$

$$\Rightarrow \quad y'' + 4y = -4u_1\cos(2t) - 2u_1'\sin(2t) - 4u_2\sin(2t) + 2u_2'\cos(2t)$$
$$+4u_1\cos(2t) \qquad\qquad +4u_2\sin(2t)$$
$$= -2u_1'\cdot\sin(2t) + 2u_2'\cos(2t) \underset{\text{has to be}}{=} 3\csc(t) \qquad (2)$$

Collecting (1) & (2):

$$\begin{cases} u_1'\cdot\cos(2t) + u_2'\cdot\sin(2t) = 0 \\ -2u_1'\sin(2t) + 2u_2'\cos(2t) = 3\csc(t) \end{cases}$$

Solve for u_1', u_2'

$$2u_2'\cdot\left(\sin(2t)^2 + \cos(2t)^2\right) = 3\csc(t)\cos(2t) \Rightarrow u_2' = \frac{3}{2}\csc(t)\cdot\cos(2t)$$

$$u_2'(t) = \frac{3}{2}\csc(t)\cdot\left(1 - 2\sin^2(t)\right) = \frac{3}{2}\csc(t) - 3\sin(t)$$

$$2u_1'\left(\cos^2(2t) + \sin^2(2t)\right) = -3\csc(t)\sin(2t) \Rightarrow u_1' = -\frac{3}{2}\csc(t)\sin(2t)$$

$$u_1'(t) = -\frac{3}{2}\csc(t)\cdot 2\sin(t)\cos(t) = -3\cos(t)$$

Now integrate:

$$u_1(t) = \int -3\cos(t)\,dt = -3\sin(t) + c_1$$

$$u_2(t) = \int \left[\frac{3}{2}\csc(t) - 3\sin(t)\right] dt = \frac{3}{2}\ln\left|\csc(t) - \cot(t)\right| + 3\cos(t) + c_2$$

Thus we obtain:

$$y(t) = -3\sin t \cdot \cos(2t) + \frac{3}{2}\ln\left|\csc(t) - \cot(t)\right| \cdot \sin(2t) + 3\cos(t)\cdot\sin(2t) +$$
$$+ c_1\cos(2t) + c_2\sin(2t).$$

o.

$$y(t) = 3\sin t + \frac{3}{2}\ln\left|\csc(t) - \cot(t)\right| \cdot \sin(2t) + c_1\cos(2t) + c_2\sin(2t)$$

Remark

$$\int \csc(t)\,dt = \int \frac{1}{\sin(t)}\,dt \overset{*}{=}$$

$$x = \tan\left(\frac{t}{2}\right) \rightarrow \sin(t) = \frac{2x}{1+x^2}$$

$$dx = \frac{1}{2}\frac{1}{\cos^2\left(\frac{t}{2}\right)}dt \rightarrow dt = \frac{2}{1+x^2}dx$$

$$\overset{*}{=} \int \left(\frac{2x}{1+x^2}\right)^{-1}\cdot\frac{2}{1+x^2}\,dx = \int\frac{dx}{x} = \ln|x| = \ln\left|\tan\left(\frac{t}{2}\right)\right|$$

$$= \ln\left|\frac{2\sin^2\left(\frac{t}{2}\right)}{2\sin\left(\frac{t}{2}\right)\cdot\cos\left(\frac{t}{2}\right)}\right| = \ln\left|\frac{1-\cos(t)}{\sin(t)}\right| = \ln\left|\csc(t) - \cot(t)\right|.$$

Let's apply the same approach to ~~the~~ an arbitrary nonhomogeneous linear equation:

$$y'' + p(t)\cdot y' + q(t)\cdot y = g(t) \qquad (*)$$

Step 1. Solve the homogeneous equation:

$$y'' + p(t)\cdot y' + q(t)\cdot y = 0$$

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

Step 2 Look for a solution of the form:

$$y(t) = u_1(t)\, y_1(t) + u_2(t)\cdot y_2(t)$$

Compute the first derivative (formally):

$$y'(t) = u_1(t)\, y_1'(t) + u_1'(t)\, y_1(t) + u_2(t) y_2'(t) + u_2'(t) y_2(t)$$

Impose the following constraint!

$$u_1'(t)\, y_1(t) + u_2'(t)\, y_2(t) = 0 \qquad (1)$$

$$\Rightarrow \quad y' = u_1 \cdot y_1' + u_2 \cdot y_2'$$

Step 3

Substitute into $(*)$:

$$u_1 y_1'' + u_1' \cdot y_1' + u_2 \cdot y_2'' + u_2' \cdot y_2' + u_1\, p(t) y_1' + u_2 \cdot p(t) y_2' + u_1\, q(t) y_1 + u_2\, q(t) y_2 = g(t)$$

$$u_1 \cdot \underbrace{(y_1'' + p(t) y_1' + q(t) y_1)}_{0} + u_2 \cdot \underbrace{(y_2'' + p(t) y_2' + q(t) y_2)}_{0} + u_1' \cdot y_1' + u_2' \cdot y_2' = g(t)$$

$$u_1' \cdot y_1'(t) + u_2' \cdot y_2'(t) = g(t) \qquad (2)$$

Step 4. Solve (1) & (2):

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g \end{cases} \qquad \text{or} \qquad \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \cdot \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

Explicitly:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{\det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} \cdot \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix} = \frac{1}{W(y_1, y_2)} \begin{bmatrix} -y_2 \cdot g \\ y_1 \cdot g \end{bmatrix}$$

$$\begin{cases} u_1' = -\dfrac{y_2 \cdot g}{W(y_1, y_2)} \\[2ex] u_2' = \dfrac{y_1 \cdot g}{W(y_1, y_2)} \end{cases}$$

Step 5 Integrate to find u_1, u_2:

$$u_1(t) = -\int \frac{y_2(t) \cdot g(t)}{W(y_1, y_2)(t)}\, dt \quad +$$

$$u_2(t) = \int \frac{y_1(t)\, g(t)}{W(y_1, y_2)(t)}\, dt$$

Step 6 Compute a particular solution as:

$$Y(t) = -y_1(t) \cdot \int_{t_0}^{t} \frac{y_2(s)\, g(s)}{W(y_1, y_2)(s)}\, ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)\, g(s)}{W(y_1, y_2)(s)}\, ds$$

where $t_0 \in I$. The general solution is:

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t) + Y(t)$$

(see Theorem 3.6.1)

The Method of Variation of Parameters for Higher Order Nonhomogeneous Linear Differential Equations

(section 4.4)

The nonhomogeneous linear diff. eqn.:

$L[y] = y^{(n)} + p_1(t)\,y^{(n-1)} + \dots + p_{n-1}(t)\,y' + p_n(t)\,y = g(t)$

We follow the same algorithm as for the 2nd order equation.

Step 1. Find a fundamental set of solutions of the homogeneous equation:

$y_1(t),\ y_2(t), \dots, y_n(t)$

Step 2 Construct a function:

$$Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t) + \dots + u_n(t)\,y_n(t) \qquad (*)$$

Step 3. Impose on all derivatives $Y', \dots, Y^{(n-1)}$ that no derivative of $u_1, \dots, u_n$ to appear:

$Y' = u_1 y_1' + \dots + u_n y_n' + u_1' y_1 + \dots + u_n' y_n \Rightarrow$

Choose $u_1' y_1 + \dots + u_n' y_n = 0$ (1) $\Rightarrow Y' = u_1 y_1' + \dots + u_n y_n'$

$Y'' = u_1 y_1'' + \dots + u_n y_n'' + u_1' y_1' + \dots + u_n' y_n'$

Choose $u_1' y_1' + \dots + u_n' y_n' = 0$ (2) $\Rightarrow Y'' = u_1 y_1'' + \dots + u_n y_n''$

...

$Y^{(n-1)} = u_1 y_1^{(n-1)} + \dots + u_n y_n^{(n-1)} + u_1' y_1^{(n-2)} + \dots + u_n' y_n^{(n-2)}$

choose: $u_1' y_1^{(n-2)} + \dots + u_n' y_n^{(n-2)} = 0$ (3) $\Rightarrow Y^{(n-1)} = u_1 y_1^{(n-1)} + \dots + u_n y_n^{(n-1)}$

Finally:

$Y^{(n)} = u_1 y_1^{(n)} + \dots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + \dots + u_n' y_n^{(n-1)}$

Step 4 compute $L[Y]$

$$L[Y] = Y^{(n)} + p_1 Y^{(n-1)} + \dots + p_{n-1} Y' + p_n Y = \underbrace{u_1 L[y_1] + \dots + u_n L[y_n]}_{0} + u_1' y_1^{(n-1)} + \dots + u_n' y_n^{(n-1)}$$

$\Rightarrow \quad u_1' y_1^{(n-1)} + \dots + u_n' y_n^{(n-1)} = g \qquad (4)$

Step 5 Solve the $n\times n$ linear system (1), (2), (3), (4):

$$\begin{cases} u_1' \cdot y_1 + \dots + u_n' \cdot y_n = 0 \\ u_1' y_1' + \dots + u_n' y_n' = 0 \\ \dots \\ u_1' y_1^{(n-2)} + \dots + u_n' y_n^{(n-2)} = 0 \\ u_1' y_1^{(n-1)} + \dots + u_n' y_n^{(n-1)} = g. \end{cases}$$

or:

$$\begin{bmatrix} y_1 & \dots & y_n \\ y_1' & \dots & y_n' \\ \vdots & & \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix} \cdot \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(t) \end{bmatrix}$$

Note the system can be solve since the $n\times n$ matrix is invertible: its determinant is the Wronskian $W(y_1, y_2, \dots, y_n) \neq 0$.

Step 6 Integrate $u_1', \dots, u_n'$ and replace into (*).

<u>Example</u> Find a particular solution of:

$$y''' - y'' - y' + y = g(t)$$

in ~~t~~erms of an integral.

<u>Solution</u>

① Find a fundamental set of solutions:

$$P(r) = r^3 - r^2 - r + 1 = 0$$

$$P(1) = 0 \rightarrow \quad P(r) = (r-1)\cdot(r^2-1) = (r-1)^2(r+1)$$

$$r_1 = r_2 = 1, \quad r_3 = -1.$$

$$y_1(t) = e^t, \quad y_2(t) = te^t, \quad y_3(t) = e^{-t}$$

② Construct:

$$1| \quad Y = u_1 y_1 + u_2 y_2 + u_3 y_3 = u_1 \cdot e^t + u_2\, te^t + u_3 \cdot e^{-t}$$

③ Compute derivatives Y', Y'', Y'''

$$-1 \quad Y' = u_1 e^t + u_2(e^t + te^t) - u_3 e^{-t} + \underbrace{(u_1' e^t + u_2' te^t + u_3' e^{-t})}_{\text{want: } 0}$$

$$-1 \quad Y'' = u_1 e^t + u_2(2e^t + te^t) + u_3 e^{-t} + \underbrace{(u_1' e^t + u_2'(e^t + te^t) - u_3' e^{-t})}_{\text{want } 0}$$

$$1 \quad Y''' = u_1 e^t + u_2(3e^t + te^t) - u_3 e^{-t} + (u_1' e^t + u_2'(2e^t + te^t) + u_3' e^{-t})$$

④

$$Y''' - Y'' - Y' + Y = u_1\cancel{(e^t - e^t - e^t + e^t)} + u_2\cancel{(3e^t + te^t - 2e^t - te^t - e^t - te^t + te^t)} +$$

$$+ u_3\cancel{(-e^{-t} - e^{-t} + e^{-t} + e^{-t})} + u_1' e^t + u_2'(2e^t + te^t) + u_3' e^{-t} \underset{\text{must be}}{=} g(t)$$

5. Solve the linear system:

$$\begin{bmatrix} e^{t} & te^{t} & e^{-t} \\ e^{t} & (1+t)e^{t} & -e^{-t} \\ e^{t} & (2+t)e^{t} & e^{-t} \end{bmatrix} \cdot \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ g(t) \end{bmatrix}$$

$$u_1' = \frac{\begin{vmatrix} 0 & te^{t} & e^{-t} \\ 0 & (1+t)e^{t} & -e^{-t} \\ g & (2+t)e^{t} & e^{-t} \end{vmatrix}}{\begin{vmatrix} e^{t} & te^{t} & e^{-t} \\ e^{t} & (1+t)e^{t} & -e^{-t} \\ e^{t} & (2+t)e^{t} & e^{-t} \end{vmatrix}} = \frac{g(t) \cdot \begin{vmatrix} te^{t} & e^{-t} \\ (1+t)e^{t} & -e^{-t} \end{vmatrix}}{W(y_1, y_2, y_3)} = -\frac{g(t) \cdot (2t+1)}{W(y_1, y_2, y_3)}$$

$$u_2' = \frac{\begin{vmatrix} e^{t} & 0 & e^{-t} \\ e^{t} & 0 & -e^{-t} \\ e^{t} & g & e^{t} \end{vmatrix}}{\begin{vmatrix} e^{t} & te^{t} & e^{-t} \\ e^{t} & (1+t)e^{t} & -e^{-t} \\ e^{t} & (2+t)e^{t} & e^{t} \end{vmatrix}} = \frac{-g(t) \begin{vmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{vmatrix}}{W(y_1, y_2, y_3)} = \frac{2\, g(t)}{W(y_1, y_2, y_3)}$$

$$u_3' = \frac{\begin{vmatrix} e^{t} & te^{t} & 0 \\ e^{t} & (1+t)e^{t} & 0 \\ e^{t} & (2+t)e^{t} & g \end{vmatrix}}{\begin{vmatrix} e^{t} & te^{t} & e^{-t} \\ e^{t} & (1+t)e^{t} & -e^{-t} \\ e^{t} & (2+t)e^{t} & e^{-t} \end{vmatrix}} = \frac{g(t) \begin{vmatrix} e^{t} & te^{t} \\ e^{t} & (1+t)e^{t} \end{vmatrix}}{W(y_1, y_2, y_3)} = \frac{g(t)\, e^{2t}}{W(y_1, y_2, y_3)}$$

The Wronskian is:

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^{t} & te^{t} & e^{-t} \\ e^{t} & (1+t)e^{t} & -e^{-t} \\ e^{t} & (2+t)e^{t} & e^{-t} \end{vmatrix} = e^{t+t-t} \cdot \begin{vmatrix} 1 & t & 1 \\ 1 & 1+t & -1 \\ 1 & 2+t & 1 \end{vmatrix} =$$

$$= e^{t} \cdot \left[\begin{vmatrix} 1+t & -1 \\ 2+t & 1 \end{vmatrix} - t \cdot \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1+t \\ 1 & 2+t \end{vmatrix} \right] = e^{t} \cdot [3 + 2t - 2t + 1] = 4e^{t}$$

6. Integrate:

$$\begin{cases} u_1(t) = -\frac{1}{4} \int e^{-t} (2t+1)\, g(t)\, dt \\ u_2(t) = \frac{1}{2} \int e^{-t} g(t)\, dt \\ u_3(t) = \frac{1}{4} \int e^{t} g(t)\, dt \end{cases} \Rightarrow Y = u_1 e^{t} + u_2 t e^{t} + u_3 e^{-t}$$

Explicitely:

$$Y(t) = -\frac{1}{4}\, e^{t} \int_{t_0}^{t} e^{-s}\,(2s+1)\,g(s)\,ds + \frac{1}{2}\, t e^{t} \int_{t_0}^{t} e^{-s} g(s)\,ds + \frac{1}{4}\, e^{-t} \int_{t_0}^{t} e^{s} g(s)\,ds$$

$$= -\frac{1}{2} \int_{t_0}^{t} e^{t-s} \cdot s\, g(s)\,ds + \frac{1}{2} \int_{t_0}^{t} e^{t-s} \cdot t\, g(s)\,ds - \frac{1}{4} \int_{t_0}^{t} e^{t-s} g(s)\,ds + \frac{1}{4} \int_{t_0}^{t} e^{s-t} g(s)\,ds$$

$$= \int_{t_0}^{t} \underbrace{\left[\frac{1}{2} e^{t-s} \cdot (t-s) - \frac{1}{4} e^{t-s} + \frac{1}{4} e^{-(t-s)}\right]}_{K(t-s)} g(s)\,ds = \underbrace{\int_{t_0}^{t} K(t-s)\, g(s)\,ds}_{\text{convolution of } K \text{ with } g.}$$