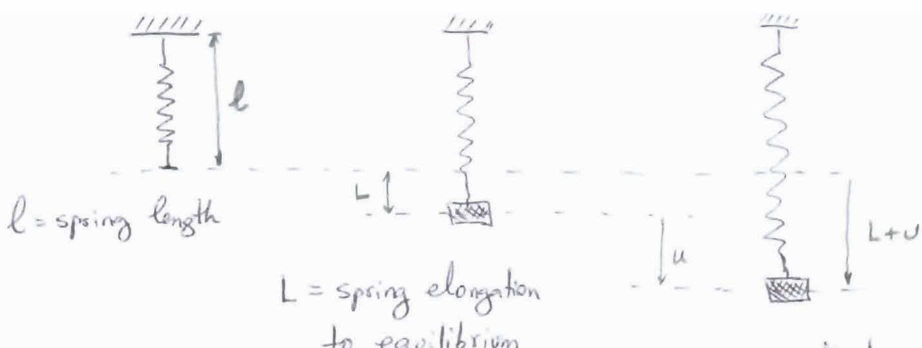
VIBRATIONS

(sections 3.7, 3.8)

<u>Motivation</u>: Mechanical (and Electrical) Vibrations are described by the solution of an initial value problem of the form:

$$\begin{cases} a y'' + b y' + c y = g(t) \\ y(0) = y_0 , \quad y'(0) = y_0' \end{cases}$$

<u>Physical System</u>: Vertical motion of a mass m on a spring



u = displacement of the mass from its equilibrium position.

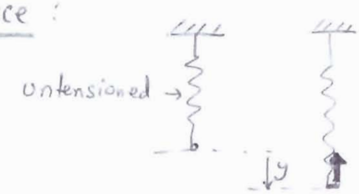
$L+u(t)$: spring elongation at time t.

What forces act on m:

1. The <u>gravitational force</u>:

$w = mg$ $\qquad$ g: gravitational acceleration

2. The <u>spring force</u>:



$F_s = -ky$ $\qquad$ k: spring constant

$(y = L + u)$

3. The damping or resistive force, F_d

Due to: (i) resistance from the air or other medium in which the mass moves
(ii) internal energy dissipation due to extension or compression of the spring
(iii) friction between the mass and the guide that constraints its motion to one dimension.

We assume: $F_d = -\gamma \cdot \text{velocity}$, $\text{velocity} = \frac{d}{dt}(L+u) = \frac{du}{dt}$

γ: the damping constant

Note: This is an approximation. Different scenarios may have completely different damping forces.

4. An applied external force: $F(t)$

Convention: $F(t) > 0$ if directed downward
$F(t) < 0$ if directed upward

Newton's law of motion:

$$m \cdot \text{acceleration} = w + F_s + F_d + F$$

$$m \cdot \text{acceleration} = m \cdot g - k \cdot \text{elongation} - \gamma \cdot \text{velocity} + F(t)$$

$$\text{acceleration} = \frac{d^2(\text{elongation})}{dt^2} = \frac{d^2(L+u)}{dt^2} = \frac{d^2u}{dt^2} = u''$$

$$\text{velocity} = \frac{d(\text{elongation})}{dt} = \frac{d(L+u)}{dt} = \frac{du}{dt} = u'$$

$$\Rightarrow \quad m \cdot u'' = m \cdot g - k(L+u) - \gamma \cdot u' + F(t)$$

L: spring elongation to equilibrium (no external force): $mg - k \cdot L = 0$.

$$\Rightarrow \quad mu'' = -ku - \gamma u' + F(t)$$

Add initial conditions: $u(0) = u_0$, $u'(0) = v_0$

$$\begin{cases} mu'' + \gamma u' + ku = F(t) \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases} \qquad (1)$$

2^{nd} order nonhomogeneous linear equation with constant coefficients.

Cases to study:

$\boxed{A}$ Free vibrations: $F(t) = 0$

- (A.1) Undamped Free Vibrations: $\gamma = 0$
- (A.2) Damped Free Vibrations: $\gamma > 0$

$\boxed{B}$ Forced Vibrations: $F(t) \neq 0$

- (B.1) Forced Vibrations with Damping: $\gamma > 0$
- (B.2) Forced Vibrations with no damping: $\gamma = 0$ (RESONANCE

A. FREE VIBRATIONS : $F(t) = 0$

$$\begin{cases} mu'' + \gamma u' + k \cdot u = 0 \\ u(0) = u_0, \quad u'(0) = u_0' \end{cases}$$

A.1 Undamped Free Vibrations : $\gamma = 0$

$$\begin{cases} mu'' + ku = 0 \\ u(0) = u_0, \quad u'(0) = u_0' \end{cases} \longrightarrow u'' + \underbrace{\frac{k}{m}}_{\omega_0^2} u = 0 \quad : \quad u'' + \omega_0^2 \cdot u = 0 \qquad (*)$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Solve $(*)$: characteristic equation : $r^2 + \omega_0^2 = 0$

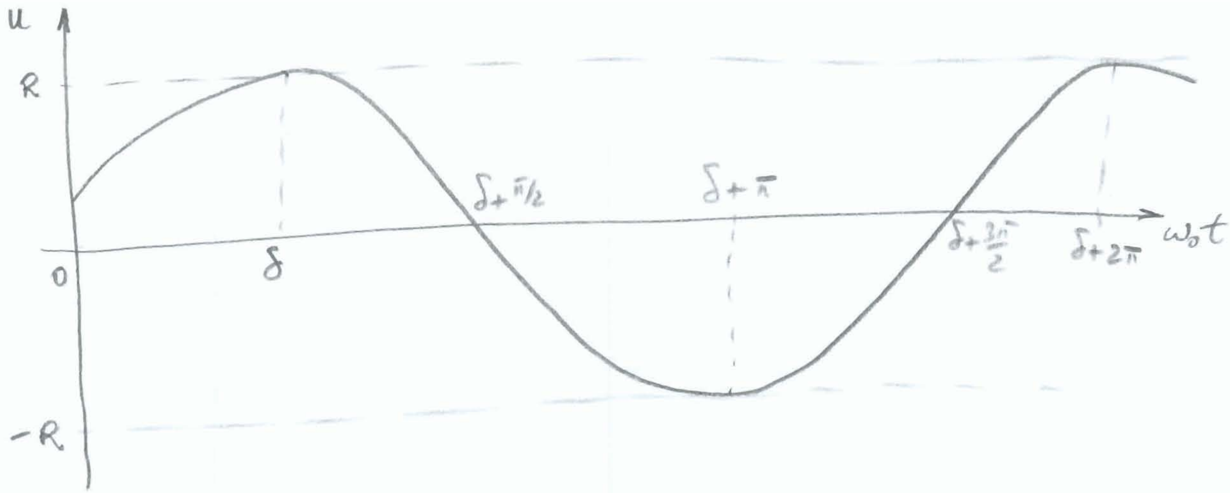
$$r_{1,2} = \pm i \omega_0$$

Hence the general solution is: $u(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t) = R \cos(\omega_0 t - \delta)$

(A, B) or (R, δ) are obtained from initial conditions:

$$u'(t) = -\omega_0 A \sin(\omega_0 t) + \omega_0 B \cos(\omega_0 t) \Rightarrow \begin{cases} u(0) = A = u_0 \\ u'(0) = \omega_0 B = v_0 \end{cases} \Rightarrow \begin{cases} A = u_0 \\ B = \frac{v_0}{\omega_0} \end{cases}$$

and: $R = \sqrt{A^2 + B^2}$, $\cos(\delta) = \frac{A}{\sqrt{A^2+B^2}}$, $\sin(\delta) = \frac{B}{\sqrt{A^2+B^2}}$



Terminology:

$R=\sqrt{A^2+B^2}$ Amplitude : maximum displacement from equilibrium.

δ Phase

$\omega_0=\sqrt{\frac{k}{m}}$ Natural frequency of the vibration.

$u(t)$ is periodic : $u(t+T) = u(t)$ period of motion

where : $\omega_0 T = 2\pi$: $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$

A2 Damped Free Vibrations

$m u'' + \gamma u' + k u = 0$ $m, \gamma, k > 0$

Characteristic equation: $mr^2 + \gamma \cdot r + k = 0$

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

i) $\gamma^2 - 4mk > 0$ (overdamped)

$\Rightarrow u(t) = Ae^{r_1 t} + Be^{r_2 t}$ Note: $r_1, r_2 < 0$

ii) $\gamma^2 - 4mk = 0$ $\gamma = 2\sqrt{mk}$ (critical damping).

$u(t) = (A+Bt)\,e^{r_0 t}$, $r_0 = -\frac{\gamma}{2m} < 0$

$= (A+Bt)\,e^{-\frac{\gamma t}{2m}}$

iii) $\gamma^2 - 4mk < 0$

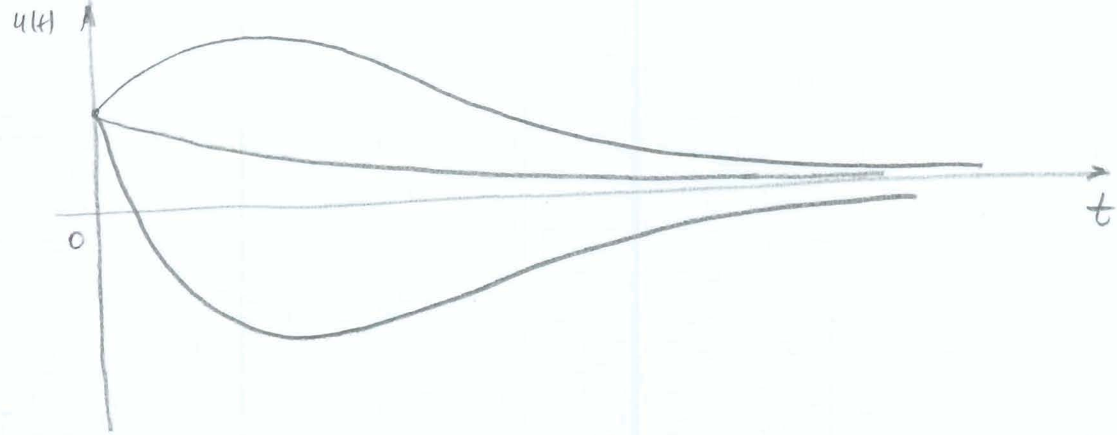
$r_{1,2} = r_0 \pm i\mu \Rightarrow u(t) = e^{-\frac{\gamma t}{2m}} \cdot \left(A\cos(\mu t) + B\sin(\mu t)\right)$

Let's analyze each case:

First note, in all cases: $\lim_{t\to\infty} u(t) = 0$

Cases (i) and (ii) [overdamped, or critically damped] are similar.

Typical behavior:



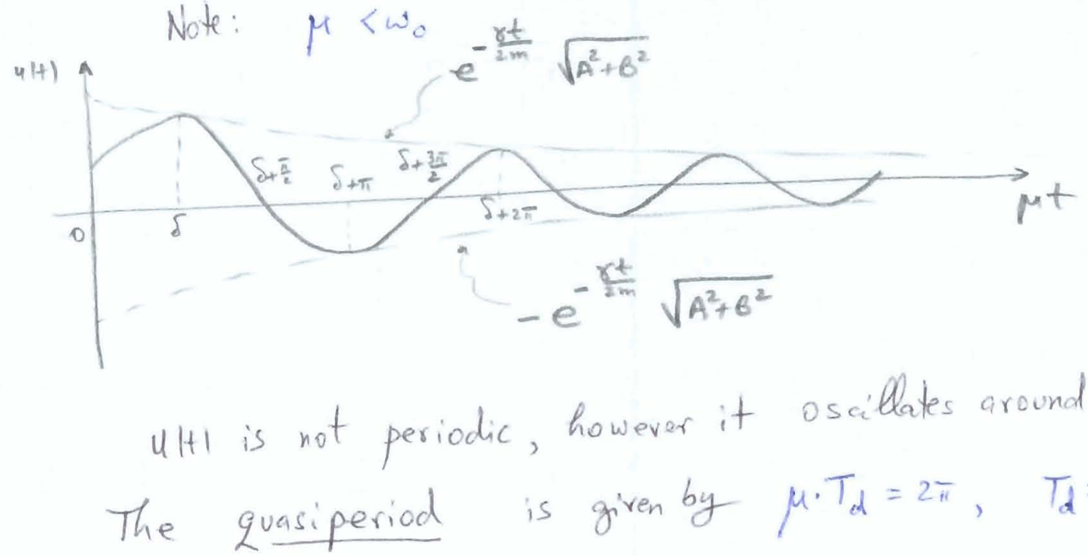
Characteristic: No crossing, or at most one crossing of $\vec{0t}$ axis.

Case (iii).

$$u(t) = e^{-\frac{\gamma t}{2m}} \left(A \cos(\mu t) + B \sin(\mu t) \right) = e^{-\frac{\gamma t}{2m}} \sqrt{A^2+B^2} \cos(\mu t - \delta)$$

$$\mu = \frac{\sqrt{4mk-\gamma^2}}{2m} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} = \sqrt{\frac{k}{m}\left(1-\frac{\gamma^2}{4mk}\right)} = \omega_0 \sqrt{1-\frac{\gamma^2}{4mk}}$$

Note: $\mu < \omega_0$



u(t) is not periodic, however it oscillates around zero.

The quasiperiod is given by $\mu \cdot T_d = 2\pi$, $T_d = \frac{2\pi}{\mu}$

Note: $\mu < \omega_0$, $T_d = \frac{2\pi}{\mu} > \frac{2\pi}{\omega_0} = T$.

B. Forced Vibrations

$$m u'' + \gamma u' + k u = F(t) \qquad (*)$$

We consider only the case: $F(t) = F_0 \cos(\omega t)$

B.1. Forced Vibrations with Damping $(\gamma > 0)$.

Homogeneous Equation: $m u'' + \gamma u' + k u = 0$

$$u_c(t) = c_1 u_1(t) + c_2 u_2(t)$$

where (u_1, u_2) is one of $(e^{r_1 t}, e^{r_2 t})$, $(e^{r_0 t}, t e^{r_0 t})$, $(e^{r_0 t}\cos(\mu t), e^{r_0 t}\sin(\mu t))$.

A particular solution of the nonhomogeneous equation:

$$u_p(t) = A\cos(\mu t) + B\sin(\mu t) = R\cdot\cos(\mu t - \delta)$$

Terminology:

1) $u_c(t) = c_1 u_1(t) + c_2 u_2(t)$ is called the transient solution.

Note: $\lim_{t\to\infty} u_c(t) = 0$.

2) $u_p(t) = R\cos(\omega t - \delta)$ is called the steady state solution, or the forced response.

(c_1, c_2) are obtained from initial value conditions (after computing R, δ)

(R, δ) are obtained as follows:

We need to plug u_p into $(*)$ and obtain R, δ.

To simplify computations, we use the following trick:

$$u_p = R\cos(\omega t - \delta) = \text{Real}\left[R e^{i(\omega t - \delta)}\right]$$

$$F(t) = F_0\cos(\omega t) = \text{Real}\left[F_0 e^{i\omega t}\right]$$

$$\text{Real}\left[\left(m\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + k\right) R e^{i(\omega t - \delta)}\right] = \text{Real}\left[F_0 e^{i\omega t}\right].$$

$$\text{Real}\left[(-m\omega^2 + \gamma i\omega + k) R e^{i(\omega t - \delta)}\right] = \text{Real}\left[F_0 e^{i\omega t}\right]$$

$$(k - m\omega^2 + i\gamma\omega) R e^{i(\omega t - \delta)} = F_0 e^{i\omega t}$$

$$(k - m\omega^2 + i\gamma\omega) R e^{-i\delta} = F_0 .$$

Since : $k - m\omega^2 + i\gamma\omega = \sqrt{(k-m\omega^2)^2 + \gamma^2\omega^2} \cdot e^{i\varphi}$

$$\cos\varphi = \frac{k - m\omega^2}{\sqrt{(k-m\omega^2)^2 + \gamma^2\omega^2}} , \quad \sin\varphi = \frac{\gamma\omega}{\sqrt{(k-m\omega^2)^2 + \gamma^2\omega^2}}$$

$$\Rightarrow \quad R = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + \gamma^2\omega^2}} , \qquad \delta = \varphi$$

Let's analyze how R depends on ω.

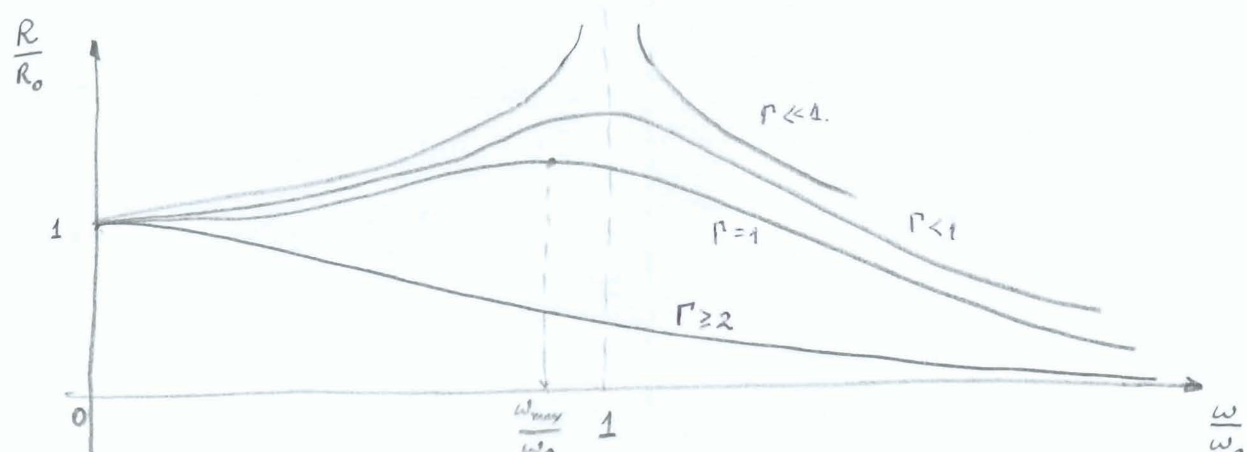
Recall: $\omega_0^2 = \frac{k}{m}$

$$R = \frac{F_0}{\sqrt{(m\omega_0^2 - m\omega^2)^2 + \gamma^2\omega^2}} = \frac{F_0}{m\omega_0^2} \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{\gamma^2}{m^2\omega_0^2} \frac{\omega^2}{\omega_0^2}}} = \frac{R_0}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma \frac{\omega^2}{\omega_0^2}}}$$

$$R_0 = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$$

$$\Gamma = \frac{\gamma^2}{m^2\omega_0^2} = \frac{\gamma^2}{mk}$$

$$\frac{R}{R_0} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0^2}}}$$



Maximum is achieved at: $\varphi'(x) = 0$, where $\varphi(x) = \dfrac{1}{\sqrt{(1-x^2)^2 + \Gamma x^2}}$

$$\varphi'(x) = -\frac{1}{2}\,\frac{-2x\cdot 2(1-x^2) + 2x\Gamma}{\left[(1-x^2)^2 + \Gamma x^2\right]^{3/2}} = 0 \quad : \quad -2(1-x^2) + \Gamma = 0$$

At maximum: $x^2 = 1 - \frac{\Gamma}{2} \Rightarrow \qquad \underline{\omega_{max} = \omega_0 \sqrt{1 - \frac{\Gamma}{2}}}$

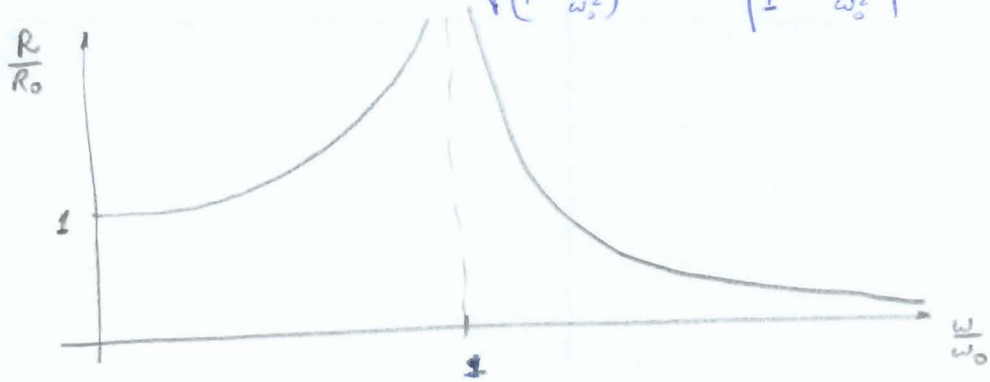
Note, the curve has a local maximum only if $\Gamma < 2$.

For small γ, Γ : $\omega_{max} \simeq \omega_0$ and:

$$R_{max} \simeq R(\omega = \omega_0) = \frac{R_0}{\sqrt{\Gamma}} = \frac{F_0}{m\omega_0^2}\,\frac{1}{\sqrt{\frac{\gamma^2}{mk}}} = \frac{F_0}{\omega_0 \gamma}$$

Note: $R_{max} \sim \frac{1}{\gamma}$ As $\gamma \downarrow 0$, $R_{max} \uparrow \infty$. <u>RESONANCE!</u>

At limit: $\gamma = 0$: $\dfrac{R(\omega)}{R_0} = \dfrac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2}} = \dfrac{1}{\left|1 - \frac{\omega^2}{\omega_0^2}\right|}$



B.2. Forced Vibrations Without Damping

$$m u'' + k u = F_0 \cos(\omega t) \qquad (**)$$

1. If $\omega \neq \omega_0$, then the general solution is:

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + R\cos(\omega t - \delta)$$

We computed: $R = \dfrac{R_0}{1 - \frac{\omega^2}{\omega_0^2}}$, $\delta = 0$ $\left(\sin\delta = \dfrac{\gamma\omega}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}} = 0 \right)$.

$$R_0 = \frac{F_0}{m\omega_0^2} \Rightarrow R = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

$$\Rightarrow \quad u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2-\omega^2)} \cos(\omega t)$$

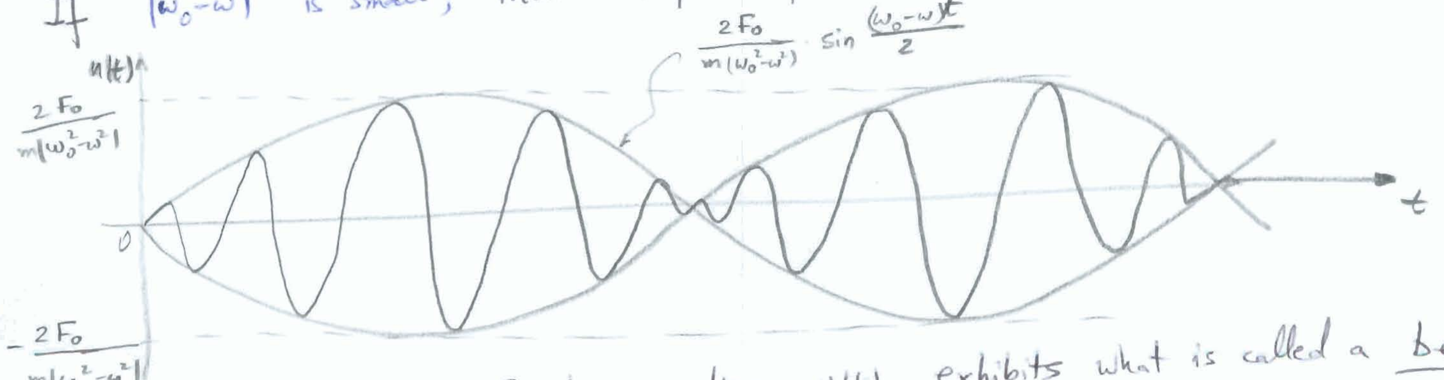
Assume the following initial condition: $u(0)=0, \quad u'(0)=0.$

$$0 = c_1 + \frac{F_0}{m(\omega_0^2-\omega^2)} \quad \Rightarrow c_1 = -\frac{F_0}{m(\omega_0^2-\omega^2)}$$

$$0 = c_2\, \omega_0 \quad \Rightarrow c_2 = 0$$

$$u(t) = \frac{F_0}{m(\omega_0^2-\omega^2)} \cdot \left[\cos(\omega t) - \cos(\omega_0 t)\right] = \frac{2F_0}{m(\omega_0^2-\omega^2)} \sin\frac{(\omega_0-\omega)t}{2} \cdot \sin\frac{(\omega+\omega_0)t}{2}.$$

If $|\omega_0-\omega|$ is small, then the plot of u looks like:



Such a motion, $u(t)$, exhibits what is called a <u>beat.</u>

The amplitude of $\sin\frac{(\omega+\omega_0)t}{2}$ is modulated by $\frac{2F_0}{m(\omega_0^2-\omega^2)} \cdot \sin\frac{(\omega_0-\omega)t}{2}$

Such a variation is called <u>amplitude modulation</u>.

2. At $\omega = \omega_0$, the general solution is:

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + R\, t \cos(\omega_0 t - \delta)$$

Insert in equation $(**)$ and obtain:

$$R = \frac{F_0}{2m\omega_0}, \quad \delta = \frac{\pi}{2}$$

Thus:

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$$

In this case $|u(t)|$ is unbounded: