Reduction of Order

This approach provides a method to solve a general homogeneous equation:

$$y'' + p(t)\cdot y' + q(t)\cdot y = 0 \qquad (*)$$

when one solution y_1 is known: $L[y_1]=0$

For the second ~~equation~~ solution we use the following:

$$y_2(t) = v(t)\cdot y_1(t)$$

where $v(t)$ is a yet to be determined function.

$$L[v\cdot y_1] = (vy_1)'' + p\cdot(vy_1)' + q\cdot(vy_1) =$$

$$= (v'\cdot y_1 + vy_1')' + p\cdot(v'\cdot y_1 + v\cdot y_1') + v\cdot q\cdot y_1 =$$

$$= v''\cdot y_1 + v'y_1' + v'y_1' + v\cdot y_1'' + v'\cdot p\cdot y_1 + v\cdot p\cdot y_1' + vqy_1 =$$

$$= v''\cdot y_1 + v'(2y_1' + p\cdot y_1) + v\cdot\underbrace{(y_1'' + p\cdot y_1' + q\cdot y_1)}_{L[y_1]=0} =$$

$$= v''\cdot y_1 + v'\cdot(2y_1' + p\cdot y_1)$$

Therefore, for $v\cdot y_1$ to be solution of $(*)$, v must satisfy:

$$v''\cdot y_1 + v'\cdot(2y_1' + p\cdot y_1) = 0$$

Let $w = v'$. Then:

$$w'\cdot y_1 + w\cdot(2y_1' + p\cdot y_1) = 0$$

which is a first order (separable) linear equation.

A solution is:

$$W(t) = \exp\left(\int (2y_1' + p \cdot y_1)\,dt \right) = \exp\left(2y_1(t) + \int p(t) y_1(t)\,dt \right)$$

And then, by integration:

$$v = \int w(t)\,dt = \int \exp\left(2y_1(t) + \int p(t) y_1(t)\,dt \right) dt$$

$$\Rightarrow \quad y_2(t) = v(t) \cdot y_1(t).$$

We analyze this procedure on an example.

<u>Example</u>

Find a second solution for:

$$t^2 y'' - 4ty' + 6y = 0$$

if $y_1(t) = t^2$ is a solution.

<u>Solution</u>:

We follow the reduction of order procedure:

$$y_2(t) = v(t) \cdot y_1(t) = v(t) \cdot t^2$$

$$y_2'(t) = v' \cdot t^2 + 2t \cdot v$$

$$y_2''(t) = v'' \cdot t^2 + v' \cdot 2t + 2v + 2tv' = v'' t^2 + 4t \cdot v' + 2v$$

$$t^2 y_2'' - 4t y_2' + 6y_2 = v'' t^4 + \cancel{4t^3 v'} + \underline{2t^2 v} - \cancel{4t^3 v'} - \underline{8t^2 v} + \underline{6t^2 v}$$

$$= v'' t^4$$

Hence; in order for y_2 to be a solution:

$$v'' \cdot t^4 = 0$$

or:

$$v'' = 0$$

By integration:

$v' =$ constant, for instance $v' = 1$.

By a second integration:

$$v = t$$

Thus:

$$y_2(t) = t^3$$

Let's check if they are independent:

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = 3t^4 - 2t^4 = t^4 \neq 0$$

(except at $t=0$).

Thus (y_1, y_2) form a fundamental set of solutions.