Classification of Differential Equations.

(Section 1.3)

Example 1

$$\frac{dy}{dx} = -y + 5$$

is an ordinary differential equations.

It is first order : because $\frac{dy}{dx}$

It is linear : because $-y$

Hence it is a linear, first order differential equation.

A Solution of this equation:

$$y(x) = 5 + e^{-x}$$

Because:

$$\frac{dy}{dx} = -e^{-x}$$

$$-y+5 = -e^{-x}$$

Hence: $\frac{dy}{dx} = -y + 5$

Example 2

$$\underbrace{y'''}_{\text{Third order.}} + 2e^{t}y'' + \underbrace{y \cdot y'}_{\text{nonlinear.}} = t^{4}$$

Third order. !

nonlinear. !

It is a nonlinear 3^{rd} order differential equation.

In general:

$$F(t, y, y', \ldots, y^{(n)}) = 0$$

is a differential equation of order n.

Order: it is the order of the highest derivative that appears.

Assumption: We will always assume we can solve for $y^{(n)}$ (the n^{th} derivative of y) as function of the other derivatives.

$\Downarrow$

The STANDARD FORM:

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$$

In Example 2:

$$y''' + 2e^t y'' + y \cdot y' = t^4$$

$n = 3$

$$\left.\begin{array}{l} n=3 \\ f(t,y,y',y'') = t^4 - 2e^t y'' - y \cdot y' \end{array}\right\} \Rightarrow \underline{y^{(n)} = f(t,y,y',y'')}$$

Definitions

Consider the general n^{th} order D.E.:

$$F(t, y, y', \dots, y^{(n)}) = 0.$$

If F is linear in $(y, y', \dots, y^{(n)})$ then the diff. equation is said LINEAR. Otherwise it is called NONLINEAR.

In Example 1: $F(\cancel{t}, y, y') = \underbrace{y' + y}_{\text{linear}} - 5$

In Example 2.: $F(t, y, y', y'', y''') = y''' + 2e^t \cdot y'' + \underbrace{y \cdot y'}_{\text{nonlinear}} - t^4$

The n^{th} order linear ODE:

$$a_0(t) y^{(n)} + a_1(t) y^{(n-1)} + \dots + a_{n-1}(t) y' + a_n(t) \cdot y = g(t)$$

Ordinary versus Partial Differential Equations

(ODE's vs. PDE's).

A PDE involves a function that depends on several independent variables.

Examples 3

body temperature

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \qquad \text{(Heat Equation)}$$

wave amplitude

$$a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \qquad \text{(Wave Equation)}$$

α, a : physical constant

$u = u(x,t)$: unknown function on 2 variables.

Systems of Differential Equations

Example 4.

$$\begin{cases} \dfrac{dx}{dt} = ax - \alpha x \cdot y \\[2ex] \dfrac{dy}{dt} = -cy + \delta xy \end{cases}$$

Each $x = x(t)$, $y = y(t)$ is a function on one variable t.

This is a system of 1^{st} order differential equations.

Remark: This system is equivalent to a certain 2^{nd} order ODE.

Solution of a D.E.

Given:

$$F(t, y, y', \dots, y^{(n)}) = 0.$$

or:

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$$

a Solution is a function ϕ defined on an open interval $\alpha < t < \beta$ such that:

$$\phi^{(n)}(t) = f(t, \phi(t), \phi'(t), \dots, \phi^{(n-1)}(t))$$

for every t in $\alpha < t < \beta$.

Some times a solution can only be found implicitly:

$$G(t, \phi) = 0$$

First Order Differential Equations

The standard form:

$$\frac{dy}{dt} = f(t,y)$$

Linear Equations. Method of Integrating Factors (Section 2.1)

Our goal is to solve:

$$\frac{dy}{dt} + p(t)\cdot y = g(t)$$

Example

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}$$

Multiply by a function $\mu(t)$, as yet undetermined:

We want:

$$\underbrace{\mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t)\cdot y}_{\frac{d}{dt}[\mu(t)\cdot y]} = \frac{1}{2}\mu(t)e^{t/3} \qquad (*)$$

$$\frac{d}{dt}[\mu(t)\cdot y] = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}\cdot y$$

Hence:

$$\frac{d\mu(t)}{dt} = \frac{1}{2}\mu(t).$$

We solve as follows:

$$\frac{1}{\mu(t)} \frac{d\mu(t)}{dt} = \frac{1}{2}.$$

$$\frac{d}{dt}\left[\ln|\mu(t)|\right] = \frac{d}{dt}\left[\frac{t}{2}\right].$$

$$\ln|\mu(t)| = \frac{t}{2} + c_1.$$

Choose $c_1 = 0$. Thus, a solution is:

$$\boxed{\mu(t) = e^{t/2}.}$$

Then (*) becomes:

$$\frac{d}{dt}\left[e^{t/2}\cdot y\right] = \frac{1}{2}e^{t/2}\cdot e^{t/3} = \frac{1}{2}e^{\frac{5t}{6}}$$

Integrate:

$$e^{t/2}\cdot y = \frac{1}{2}\cdot\frac{6}{5}e^{\frac{5t}{6}} + \overbrace{c}^{\text{a real constant}} = \frac{3}{5}e^{\frac{5t}{6}} + c$$

$$\boxed{y(t) = \frac{3}{5}e^{t/3} + c\cdot e^{-t/2}.}$$

For the general linear 1^{st} order equation:

$$\frac{dy}{dt} + p(t) \cdot y = g(t)$$

Step 1. Multiply by an integrating factor $\mu(t)$:

$$\underbrace{\mu(t) \cdot \frac{dy}{dt} + \mu(t) p(t) \cdot y}_{} = \mu(t) \cdot g(t). \qquad (\square).$$

Want: $\frac{d}{dt}\left[\mu(t) \cdot y\right] = \frac{d\mu(t)}{dt} \cdot y + \mu(t) \cdot \frac{dy}{dt}$

Hence need to solve:

$$\frac{d\mu(t)}{dt} = \mu(t) \cdot p(t).$$

We do so by:

$$\frac{1}{\mu(t)} \frac{d\mu(t)}{dt} = p(t).$$

$$\frac{d}{dt}\left[\ln|\mu(t)|\right] = p(t)$$

$$\ln|\mu(t)| = \int p(t) dt.$$

$$\Rightarrow \quad \boxed{\mu(t) = \exp\left(\int p(t)\, dt\right)} \qquad (1)$$

(This is the integrating factor)

Step 2

Return to (□):

$$\frac{d}{dt}\left[\mu(t)\cdot y\right] = \mu(t)\cdot g(t)$$

Integrate again:

$$\mu(t)\cdot y = \int \mu(s)\, g(s)\, ds + c$$

The general solution:

$$\boxed{y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)\, g(s)\, ds + c\right]} \qquad (2)$$

(t_0 : some integration time , c : arbitrary constant)