Separable Equations

(Section 2.2)

Consider:

$$\frac{dy}{dx} = f(x,y)$$

or:

$$\frac{dy}{dx} - f(x,y) = 0.$$

When: $\quad -f(x,y) = \dfrac{M(x)}{N(y)}$, for two functions $M(x)$, $N(y)$,

the equation is said **separable**.

$$\frac{dy}{dx} + \frac{M(x)}{N(y)} = 0.$$

or:

$$M(x) + N(y) \cdot \frac{dy}{dx} = 0 \qquad (\Delta).$$

Formally we can also write in differential form:

$$M(x)\,dx + N(y)\,dy = 0.$$

Example

Find integral curves for: $\quad \dfrac{dy}{dx} = \dfrac{x^2}{1-y^2}$

First note: $\quad \dfrac{dy}{dx} + \dfrac{-x^2}{1-y^2} = 0$

is separable, with $\quad M(x) = -x^2$, $N(y) = 1-y^2$.

Let H_1 be an antiderivative of M: $\quad H_1(x) = -\dfrac{x^3}{3}$

Let H_2 be an antiderivative of N: $\quad H_2(y) = y - \dfrac{y^3}{3}$

Then (Δ) turns into:

$$\frac{d}{dx}\left[-\frac{x^3}{3}\right] + \underbrace{\frac{d}{dy}\left[y - \frac{y^3}{3}\right]\cdot\frac{dy}{dx}}_{\frac{d}{dx}\left[y - \frac{y^3}{3}\right]} = 0.$$

(by chain rule)

Integrate:

$$\boxed{-\frac{x^3}{3} + y - \frac{y^3}{3} = C.}$$

an arbitrary constant

This equation defines the solution implicitly.

If we ask for solution that passes through (x_0, y_0), then:

$$C = -\frac{x_0^3}{3} + y_0 - \frac{y_0^3}{3}$$

In the general case (Δ):

$$M(x) + N(y)\frac{dy}{dx} = 0.$$

Step 1 Find antiderivatives for $M(x)$ and $N(y)$:

$$H_1(x) \longrightarrow H_1'(x) = M(x)$$

$$H_2(y) \longrightarrow H_2'(y) = N(y)$$

Step 2

The equation becomes:

$$\frac{d}{dx}\left[H_1(x)\right] + \frac{d}{dy}\left[H_2(y)\right]\cdot\frac{dy}{dx} = 0.$$

$$\frac{d}{dx}\left[H_1(x) + H_2(y)\right] = 0.$$

$$\boxed{H_1(x) + H_2(y) = C.} \qquad (\Delta\Delta)$$

This defines a family of solutions implicitly.

If we are given an initial condition: $y(x_0) = y_0$

then: $C = H_1(x_0) + H_2(y_0)$.

Differences between Linear and Nonlinear Equations

(section 2.4)

Recall the standard form of a 1^{st} order D.E.:

$$y' = f(t,y)$$

Typically you are asked to solve an Initial Value Problem (IVP):

$$\begin{cases} y' = f(t,y) & , \quad \alpha < t < \beta \\ y(t_0) = y_0 . \end{cases}$$

The second equation constraints the class of solutions ~~sa~~ that satisfy $y' = f(t,y)$

We plan to study the following properties:

(1) existence of solutions

(2) uniqueness of the solution

(3) domain (or interval) of definition for solutions.

We start with linear 1^{st} order differential equation.

Recall the standard form:

$$\frac{dy}{dt} + p(t) \cdot y = g(t) \quad , \quad \alpha < t < \beta$$

We showed that all solutions are obtained by:

$$\begin{cases} \mu(t) = \exp\left(\int_{t_0'}^{t} p(s)\, ds \right) \\ y(t) = \frac{1}{\mu(t)} \left[\int_{t_0'}^{t} \mu(s)\, g(s)\, ds + c \right] \end{cases} \qquad (*)$$

t_0' : is an arbitrary time in (α, β)

c : is an arbitrary real parameter (number).

Given an initial condition : $y(t_0) = y_0$

we choose: $t_0' = t_0$ and we solve for c :

$$\mu(t_0) = \exp\left(\int_{t_0}^{t_0} p(s)\, ds \right) = \exp(0) = 1$$

$$y_0 = y(t_0) = \frac{1}{\mu(t_0)} \left[\int_{t_0}^{t_0} \mu(s)\, g(s)\, ds + c \right] = 1 \cdot [0 + c] = c$$

Hence the solution to the IVP: $\begin{cases} y' + p(t) \cdot y = g(t) \\ y(t_0) = y_0 \end{cases}$

is given by

$$y(t) = \frac{1}{\mu(t)} \left[\int_{t_0}^{t} \mu(s)\, g(s)\, ds + y_0 \right]$$

Let $$A(t) = \int_{t_0}^{t} p(s)\,ds$$

Thus $$\mu(t) = e^{A(t)}$$

and:

$$y(t) = e^{-A(t)} \cdot \left[\int_{t_0}^{t} e^{A(s)} g(s)\,ds + y_0 \right] = \boxed{\int_{t_0}^{t} e^{A(s)-A(t)} g(s)\,ds + y_0 e^{-A(t)}}$$

<u>Remark:</u> $y(t)$ is defined over entire open interval (α, β).

<u>Theorem [2.4.1]</u> If functions p and g are continuous on the open interval (α, β), that is for $\alpha < t < \beta$, and t_0 is a point in this interval, $\alpha < t_0 < \beta$, then for every real number y_0, the Initial Value Problem $\begin{cases} \frac{dy}{dt} + p(t)y = g(t), & \alpha < t < \beta \\ y(t_0) = y_0. \end{cases}$

(a) has a solution $y = y(t)$, and

(b) this solution is unique, and

(c) this solution is defined over the entire open interval (α, β).

Example: Find an interval in which the IVP

$$\begin{cases} t y' + 2y = 4t^2 \\ y(1) = 2 \end{cases}$$

has a unique solution.

The standard form:

$$y' + \frac{2}{t} y = 4t$$

$\boxed{A}$ Without solving:

$$p(t) = \frac{2}{t} \quad , \quad g(t) = 4t$$

p is continuous on $(-\infty, 0)$ and on $(0, \infty)$

g is continuous on the entire real line $\mathbb{R} = (-\infty, \infty)$.

Hence p and g are continuous on $(-\infty, 0)$, and $(0, \infty)$.

The initial value problem has $t_0 = 1$ as data. Since 1 belongs to the interval $(0, \infty)$, it follows that an interval in which the IVP has unique solution is $(0, \infty)$.

$\boxed{B}$ By solving the IVP.

The explicit solution is given by:

$$\mu(t) = \exp\left(\int_1^t \frac{2}{s}\, ds\right) = \exp\left(2\cdot(\ln|t| - \ln 1)\right) =$$

$$= \exp(2\cdot \ln|t|) = \exp(\ln|t|^2) = |t|^2 = t^2$$

Hence:

$$y(t) = \frac{1}{t^2}\cdot\left[\int_1^t s^2\cdot 4s\, ds + 2\right] = \frac{1}{t^2}\left[s^4\Big|_1^t + 2\right] = \frac{1}{t^2}\left[t^4 - 1 + 2\right]$$

$$= t^2 + \frac{1}{t^2}$$

Therefore y is defined for all $t > 0$, hence on $(0,\infty)$. $(0,\infty)$ is an interval where the IVP has unique solution. ☒

Let us consider now the general (i.e. nonlinear) 1st order IVP in standard form:

$$(**)\quad \begin{cases} y' = f(t,y) & , \alpha < t < \beta \\ y(t_0) = y_0 \end{cases}$$

The situation is a bit more complicated, however the following is true:

Theorem [2.4.2] Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$ and $\gamma < y < \delta$ (or: $(t, y) \in (\alpha, \beta) \times (\gamma, \delta)$). Then, for some interval $(t_0 - h, t_0 + h)$ contained in (α, β) there exists a unique solution $y = \phi(t)$ of the IVP $(**)$.

Remarks: This theorem guarantees the following:

(a) the existence of a solution $y = \phi(t)$;

(b) this solution is unique ;

(c) this solution is defined only on some open interval around t_0 : $(t_0 - h, t_0 + h)$, for some $h > 0$.

Finding the maximal interval of definition is a hard problem! Usually it requires solving the IVP.

Example.

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases}$$

This is a nonlinear 1^{st} order differential equation (an example of a Riccati equation).

$$f(t,y) = y^2$$

$$\frac{\partial f}{\partial y}(t,y) = 2y.$$

Both f and $\frac{\partial f}{\partial y}$ are continuous for all $-\infty < t < \infty$ and $-\infty < y < \infty$ (that is on $\mathbb{R} \times \mathbb{R}$).

Based on Theorem 2.4.2. we can only say that <u>there exists</u> an open interval $(-h, h)$ where a solution exists and it is unique.

How big is this h ?

We need to solve the IVP to find out !

This is a separable equation. Hence:

$$\frac{dy}{dt} = y^2$$

$$\frac{1}{y^2}\frac{dy}{dt} = 1.$$

Integrate:

$$-\frac{1}{y} = t + c$$

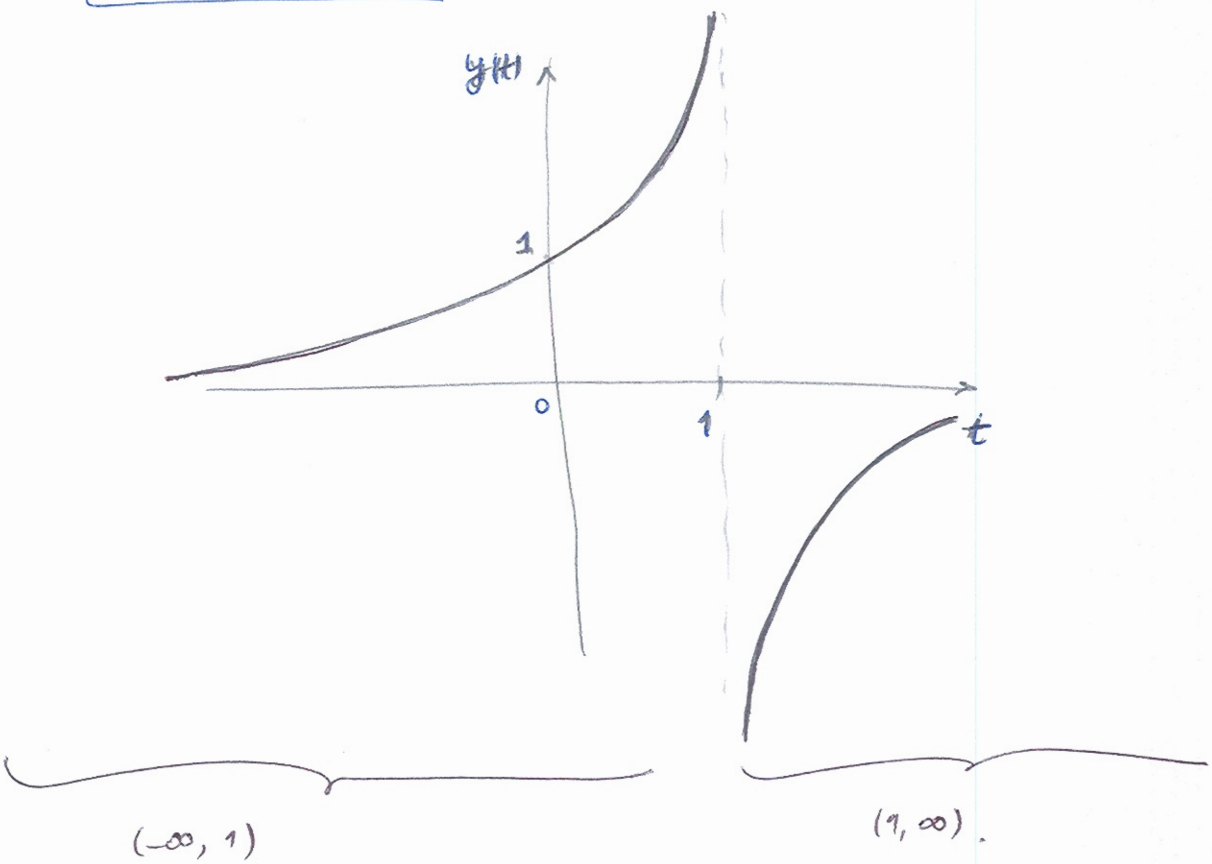
We find the constant c by:

$(t_0 = 0, y_0 = 1) \Rightarrow \quad -\frac{1}{1} = 0 + C \quad \Rightarrow \quad \underline{C = -1}$

Thus:

$$-\frac{1}{y(t)} = t - 1$$

$$\boxed{y(t) = \frac{1}{1-t}}$$



Hence: the largest open interval containing 0 is $(-\infty, 1)$.

On this interval the unique solution exists.

Example Consider the IVP:

$$\begin{cases} y' = y^{1/3} \\ y(0) = 0 \end{cases}, \quad 0 \le t < \infty.$$

It is a nonlinear 1^{st} order D.E. Note!

$$f(t,y) = y^{1/3} \longrightarrow f \text{ is continuous for all } (t,y)$$

$$\frac{\partial f}{\partial y}(t,y) = \frac{1}{3} y^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{y^2}} \longrightarrow \frac{\partial f}{\partial y} \text{ does not exist for } y=0.$$

Hence: $\frac{\partial f}{\partial y}$ is not continuous on any open interval around $y_0 = 0$.

and Theorem 2.4.2 does not apply: no conclusion can be drawn.

We need to solve it.

This is a separable equation:

$$\frac{dy}{dt} = y^{1/3}$$

$$y^{-1/3} \frac{dy}{dt} = 1$$

Integrate:

$$\frac{y^{2/3}}{2/3} = t + c$$

$$y^{2/3} = \frac{2}{3}(t + c)$$

Initial condition: $y(0) = 0 \Rightarrow 0 = \frac{2}{3}(0 + c) \Rightarrow \underline{c = 0}$

Thus:

$$y^{2/3}(t) = \frac{2}{3}t$$

So:

$$y = \phi(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}} \quad , \ t \geq 0$$

is a solution. But so is:

$$y = \phi_2(t) = -\left(\frac{2}{3}t\right)^{\frac{3}{2}} \quad , \ t \geq 0.$$

(check: $\frac{dy}{dt} = -\frac{3}{2}\left(\frac{2}{3}t\right)^{\frac{1}{2}} \cdot \frac{2}{3} = -\left(\frac{2}{3}t\right)^{1/2}$

$y^{1/3} = -\left(\frac{2}{3}t\right)^{1/2}$ & $\phi_2(0) = 0$. ok!)

Even more: $y = \phi_3(t) = 0$

is also a solution of the IVP.

To make things more complicated, we can combine:

$$\psi_+(t) = \begin{cases} 0 \ , & 0 \leq t < t_1 \\ \left[\frac{2}{3}(t - t_1)\right]^{3/2} , & t_1 < t \end{cases}$$

or:

$$\psi_-(t) = \begin{cases} 0 \ , & 0 \leq t < t_1 \\ -\left[\frac{2}{3}(t - t_1)\right]^{3/2} , & t_1 < t \end{cases}$$

are also solutions of the same IVP, for every real t_1.

Hence: There exists a solution, but this solution is not unique! In fact there are infinitely many solutions.