Autonomous Equations and Population Dynamics (section 2.5)

In this lecture we study 4 examples of 1^{st} order autonomous differential equations:

$$\frac{dy}{dt} = f(y)$$

Note it is separable:

$$\frac{1}{f(y)} \frac{dy}{dt} = 1$$

If $H(y)$ denotes a primitive (i.e. anti-derivative) of $\frac{1}{f}$, then:

$$\frac{d}{dt}\left[H(y) - t\right] = 0$$

$$\boxed{H(y) = t + c}$$

which defines implicitly the solution.

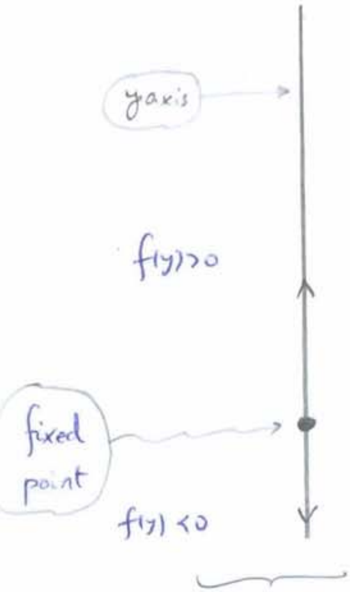
Model 1: Exponential Growth

$$\frac{dy}{dt} = r \cdot y \qquad (M1)$$

for some r, called the rate of growth (when $r>0$) or decline (when $r<0$).

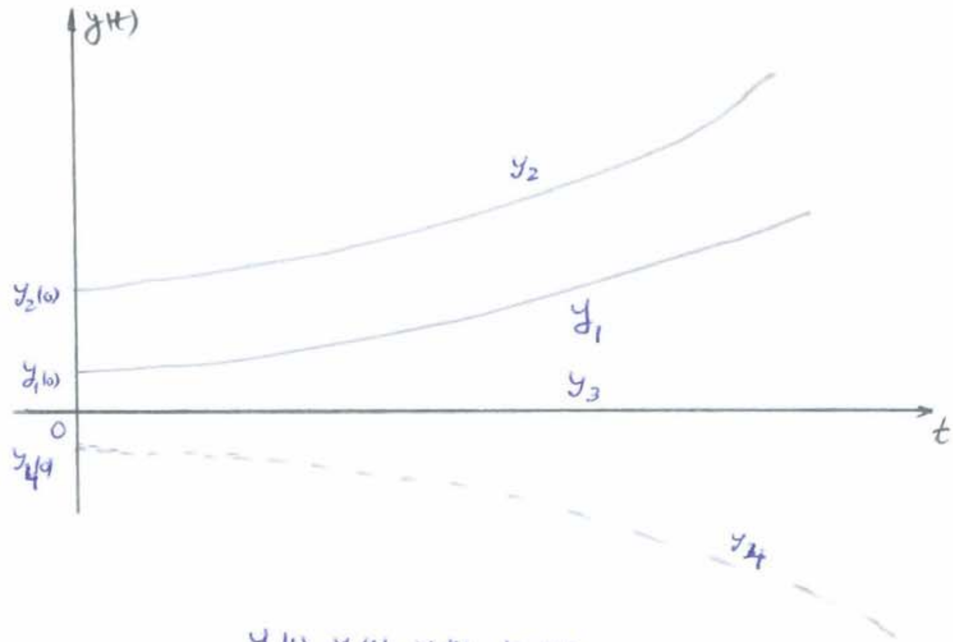
If $y(0) = y_0$, the unique solution is:

$$\boxed{y(t) = y_0 e^{rt}}$$



Phase line

(also known as the "phase space" in the case of higher-order differential equations)



$y_1(t), y_2(t), y_3(t), y_4(t)$ are different solutions of (M1) corresponding to different initial conditions: $y_1(0), y_2(0) > 0$, $y_3(0) = 0$, $y_4(0) < 0$.

To be physically meaningful (population), $y(t) \geq 0$.

Remark: $y = 0$ is a fixed point, or an equilibrium solution for (M1).

Arrows indicate how a solution evolves in time.

In this case $y=0$ is an unstable equilibrium (or a repellor).

If $y_0 > 0$ then $\lim_{t \to \infty} y(t) = \infty$, meaning that population would grow indefinitely. To alliavate this issue, consider:

Model 2: Logistic Growth

$$\frac{dy}{dt} = (r - ay)y = r\left(1 - \frac{y}{k}\right)y \quad , \; k = \frac{r}{a} \qquad \text{(M2)}$$

This is known as the logistic equation (or Verhulst equation).

k: intrinsic growth rate.

Qualitative behavior of solutions:

i) First identify equilibrium solutions:

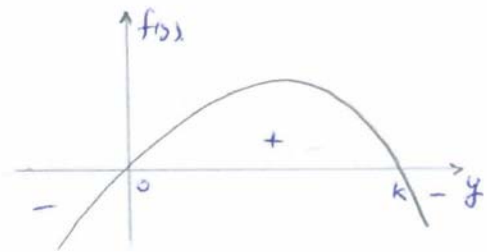
$$r\left(1-\frac{y}{K}\right)y = 0 \quad \Rightarrow \quad y=0 \quad \text{and} \quad y=K$$

with $r, K > 0$ parameters.

ii) Phase portrait:

phase line.

Need sign of $f(y) = r\left(1-\frac{y}{K}\right)y = -\frac{r}{K}y^2 + ry$



Hence:

$y=0$: (asymptotically) unstable equilibrium (repeller)

$y=K$: asymptotically stable equilibrium (attractor)

Quantitative analysis: Compute explicitly the solution

$$\underbrace{\frac{1}{\left(1-\frac{y}{K}\right)y}}_{\text{Need its primitive (anti-derivative)}} \frac{dy}{dt} = r$$

$$\frac{1}{\left(1-\frac{y}{K}\right)y} = \frac{\alpha}{1-\frac{y}{K}} + \frac{\beta}{y}$$

Find α by multiplying with $1-\frac{y}{K}$ and then set $y=K$:

$$\alpha = \frac{1}{K}$$

Find β by multiplying with y and then set $y=0$:

$$\int \frac{1}{\left(1-\frac{y}{k}\right)y}\,dy = \int \frac{\frac{1}{k}}{1-\frac{y}{k}}\,dy + \int \frac{1}{y}\,dy =$$

$$= -\ln\left|1-\frac{y}{k}\right| + \ln|y| = \ln\left|\frac{y}{1-\frac{y}{k}}\right|$$

Thus:

$$\frac{d}{dt}\left[\ln\left|\frac{y}{1-\frac{y}{k}}\right|\right] = r$$

Integrate

$$\ln\left|\frac{y}{1-\frac{y}{k}}\right| = rt + c \quad\Rightarrow\quad \left|\frac{y}{1-\frac{y}{k}}\right| = e^{c}\cdot e^{rt}$$

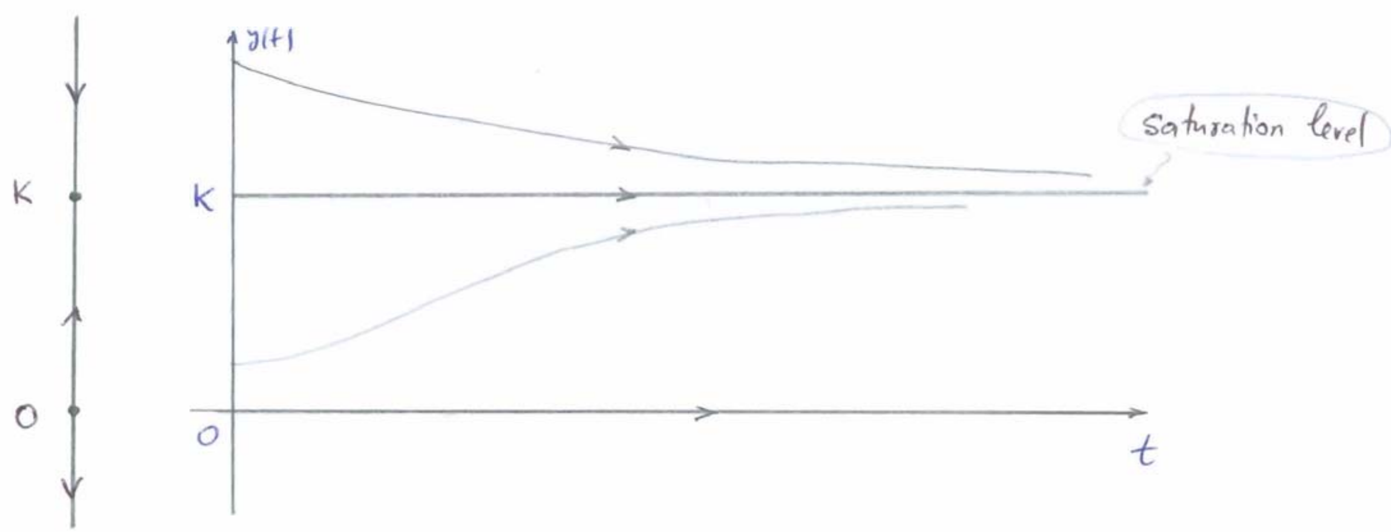
Initial condition $y(0)=y_0 \Rightarrow e^{c} = \left|\frac{y_0}{1-\frac{y_0}{k}}\right|$

Thus:

$$\frac{y(t)}{1-\frac{y(t)}{k}} = \frac{y_0}{1-\frac{y_0}{k}}\,e^{rt}$$

carries the signature

$$\Rightarrow \quad \boxed{y(t) = \frac{y_0\cdot K}{y_0 + (k-y_0)\,e^{-rt}}}$$



Model 3: A Critical Threshold

$$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)y \qquad (M3)$$

Note the sign!

Qualitative Analysis:

f: $+$ $\quad -$ $\quad +$

Now: $y=0$ is asymptotically stable equilibrium.

$y=T$ is unstable equilibrium

T: critical threshold: If $y_0 < T$ then population goes extinct.

Model 4: Logistic Growth with a Threshold

$$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y$$

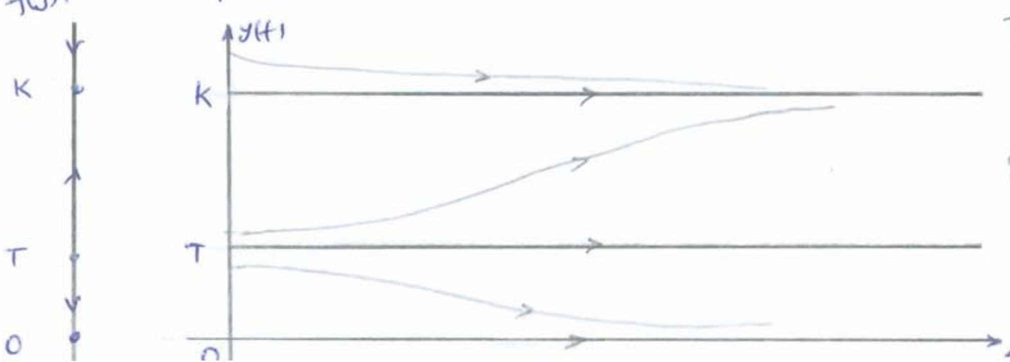
where: $r>0$, $\quad K>T>0$.

Qualitative analysis:

i) Equilibrium points: $y_1=0$, $y_2=T$, $y_3=K$.

ii) Phase line:

$f(y)$: $+$ $\quad -$ $\quad +$ $\quad -$



Thus: $y_1=0$, $y_3=K$ are asymptotically stable eq.

and: $y_2=T$ is an unstable equilibrium

(ADDITIONAL PAGE !)

A different type of equilibrium point: <u>Semistable</u>.

(See Problem 7).
Section 2.5

Consider the differential equation:

$$\frac{dy}{dt} = y^2$$

(i) Equilibrium points:

$y^2 = 0 \Rightarrow y = 0$, the only equilibrium point.

(ii) Signature of the rate function:

y		0	
$f(y) = y^2$	$+$	0	$+$

(iii) Phase Line:

Such an equilibrium is called <u>semistable</u>.

<u>Note</u>: The same would be for

$$\frac{dy}{dt} = -y^2$$

Phase line:

NUMERICAL METHODS (Chapter 8)

The Euler or Tangent Line Method (section 8.1)

Consider the first order initial value problem

$$(\text{IVP}) \quad \begin{cases} \dfrac{dy}{dt} = f(t, y) \\ y(t_0) = y_0. \end{cases}$$

We assume (see Theorem 2.4.2) a solution $y = \phi(t)$ exists uniquely in some interval about t_0.
We would like to have a numerical method to compute this solution.

<u>The Euler method</u> works as follows:

(1.) First we choose a time sequence $t_0, t_1, t_2, \ldots, t_n, \ldots$ that starts with the same t_0 as in (IVP).

Typically one chooses a uniform sequence characterized by a constant step size $h > 0$:

$$\begin{aligned} t_1 &= t_0 + h \\ t_2 &= t_1 + h = t_0 + 2h \\ t_3 &= t_2 + h = t_0 + 3h \\ &\vdots \\ t_n &= t_{n-1} + h = t_0 + n \cdot h \\ &\vdots \end{aligned}$$

(2) The iteration for y is as follows:

$$y_{n+1} = y_n + f(t_n, y_n)\cdot(t_{n+1} - t_n) \quad , \quad n = 0, 1, 2, \ldots$$

Explicitely:

$$y_1 = y_0 + f(t_0, y_0)\cdot(t_1 - t_0)$$
$$y_2 = y_1 + f(t_1, y_1)\cdot(t_2 - t_1)$$
$$\vdots$$

In the case of a uniform sequence, $t_n = t_0 + n\cdot h$:

$$\boxed{y_{n+1} = y_n + f(t_0 + nh, y_n)\cdot h.}$$

Why this formula?

A By approximating the derivative.

At $t = t_n$:

$$\frac{d\phi}{dt}(t_n) = f(t_n, \phi(t_n)) \qquad (*)$$

Two approximations:

(1) $\phi(t_n) = y_n$

(2) $\dfrac{d\phi}{dt}(t_n) = \dfrac{\phi(t_{n+1}) - \phi(t_n)}{t_{n+1} - t_n} = \dfrac{y_{n+1} - y_n}{t_{n+1} - t_n}.$

Then:

$$\frac{y_{n+1} - y_n}{t_{n+1} - t_n} = f(t_n, y_n) \implies y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)$$

B By approximating an integral.

Integrate $(*)$ over $[t_n, t_{n+1}]$:

$$\int_{t_n}^{t_{n+1}} \frac{d\phi}{dt}\, dt = \int_{t_n}^{t_{n+1}} f(t, \phi(t))\, dt$$

$$\phi(t_{n+1}) - \phi(t_n) = \int_{t_n}^{t_{n+1}} f(t, \phi(t))\, dt$$

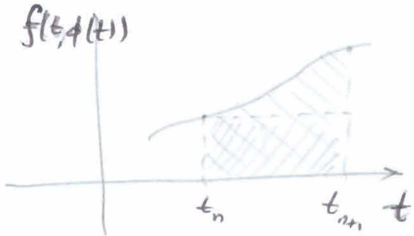
$$\phi(t_{n+1}) = \phi(t_n) + \int_{t_n}^{t_{n+1}} f(t, \phi(t))\, dt$$

Next we need to make two approximations:

(1) $\phi(t_n) = y_n, \quad \phi(t_{n+1}) = y_{n+1}$

(2) Approximate the integral:

$$\int_{t_n}^{t_{n+1}} f(t, \phi(t))\, dt = f(t_n, \phi(t_n)) \cdot (t_{n+1} - t_n)$$



This means:

Approximate the area of

by the are of rectangle

Putting together the two approximations:

$$y_{n+1} = y_n + f(t_n, y_n) \cdot (t_{n+1} - t_n).$$

An Alternative Method: The Backward Euler Formula

Approximate: $$\frac{d\phi}{dt}(t_n) = \frac{\phi(t_n) - \phi(t_{n-1})}{t_n - t_{n-1}} = \frac{y_n - y_{n-1}}{t_n - t_{n-1}}$$

Thus:

$$\frac{y_n - y_{n-1}}{t_n - t_{n-1}} = f(t_n, y_n) \Rightarrow y_n = y_{n-1} + f(t_n, y_n) \cdot (t_n - t_{n-1})$$

Increment $n \to n+1$:

$$y_{n+1} = y_n + f(t_{n+1}, y_{n+1}) \cdot (t_{n+1} - t_n)$$

<u>Next</u>: you need to solve for y_{n+1} as function of y_n, t_n, t_{n+1}.

Approximation Order. Error Analysis

Types of error:

1) Global Truncation Error:

$$E_n = \underset{\text{exact solution}}{\phi(t_n)} - \underset{\text{approximate solution}}{y_n}$$

2) Round-Off Error:

$$R_n = \underset{\substack{\text{approximate}\\ \text{solution computed with}\\ \text{infinite precision}}}{y_n} - \underset{\substack{\text{machine-precision}\\ \text{computed value}}}{Y_n}.$$

Total Error:

$$|\phi(t_n) - Y_n| \le |\phi(t_n) - y_n| + |y_n - Y_n| = |E_n| + |R_n|.$$

We will evaluate E_n. It has two causes:

(1) At $t = t_{n-1}$, y_{n-1} is only an approximation of $\phi(t_{n-1})$

(2) From t_{n-1} to t_n we approximate the integral by the area of a rectangle.

gives rise to e_n: local truncation error.

$e_n = \phi(t_n) - y_n$, under the assumption that $y_{n-1} = \phi(t_{n-1})$.

For Euler method:

$$y_n = y_{n-1} + f(t_{n-1}, y_{n-1})\, h = \phi(t_{n-1}) + f(t_{n-1}, \phi(t_{n-1}))\, h$$

$$\phi(t_n) = \phi(t_{n-1}) + \int_{t_{n-1}}^{t_n} f(t, \phi(t))\, dt.$$

$$e_n = \phi(t_n) - y_n = \int_{t_{n-1}}^{t_n} f(t, \phi(t))\, dt - f(t_{n-1}, \phi(t_{n-1}))\cdot h$$

It can be shown that: $|e_n| \le C \cdot h^2$ (also known as: $|e_n| = O(h^2)$)

C: independent of h, but dependent of f and on an interval $[a,b]$ about t_0 where we seek the solution.

Also: $|E_n| \le c' \cdot h$ (hence: $|E_n| = O(h)$)