Quiz 10, MATH 246, Professor Radu Balan, Tuesday, 24 November 2009

Your Name: ____________________________________

UMD SID:____________________________________

Discussion Instructor (circle one): **Jeremy Schwartz** **Geetanjali Kachari**

Discussion Time (circle one): **8:00** **9:00** **10:00** **11:00**

1. a. [3pts] Find the solution of the initial value problem:

$$x' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} x, x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   b. [1pt] Describe the behavior of the solution as t→∞.

<u>Solution</u>

a. The characteristic equation and spectrum are:

$$p_A(s) = \det(A - sI) = \begin{vmatrix} 1-s & -5 \\ 1 & -3-s \end{vmatrix} = (s-1)(s+3) + 5 = s^2 + 2s + 2 = 0 \Rightarrow \lambda_{1,2} = -1 \pm i$$

Eigenvector for λ1=-1+i:

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (2-i)a - 5b = 0 \Rightarrow v^1 = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

Hence a set of real valued solutions are:

$$x^1(t) = \begin{pmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix} e^{-t} \qquad x^2(t) = \begin{pmatrix} 5\sin(t) \\ 2\sin(t) - \cos(t) \end{pmatrix} e^{-t}$$

And the general solution is:
$$c_1 e^{-t} \begin{pmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 5\sin(t) \\ 2\sin(t) - \cos(t) \end{pmatrix}$$

Initial condition implies:
$$\begin{cases} 5c_1 = 1 \\ 2c_1 - c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_1 = 0.2 \\ c_2 = -0.6 \end{cases}$$

Hence the solution of this IVP is:

$$x(t) = \begin{pmatrix} (\cos(t) - 3\sin(t))e^{-t} \\ (\cos(t) - \sin(t))e^{-t} \end{pmatrix}$$

b) As t→∞ we obtain:

$$\lim_{t\to\infty} x(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Which reflects the fact that the origin is an asymptotically stable spiral point.

2. a) [3pts] Find the fundamental matrix Φ(t) satisfying Φ(0)=I for the system:

$$x' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} x$$

b) [3pts] Solve the initial value problem:

$$x' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} x$$

$$x(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Solution

a) The problem asks to compute exp(At).

Method 1: Using a fundamental system of solutions.

$$p_A(s) = \det(A - sI) = \begin{vmatrix} 1-s & -4 \\ 4 & -7-s \end{vmatrix} = (s-1)(s+7) + 16 = s^2 + 6s + 9 = (s+3)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = -3$$

First eigenvector: $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = b \Rightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The generalized eigenvector: $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow 4w_1 - 4w_2 = 1 \Rightarrow w = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Hence two independent solutions are: $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 0.25 \\ 0 \end{pmatrix} e^{-3t}$

A fundamental matrix is: $\begin{pmatrix} e^{-3t} & te^{-3t} + 0.25e^{-3t} \\ e^{-3t} & te^{-3t} \end{pmatrix}$ but the fundamental matrix Φ(t) is

$$e^{At} = \begin{pmatrix} e^{-3t} & te^{-3t} + 0.25e^{-3t} \\ e^{-3t} & te^{-3t} \end{pmatrix}\begin{pmatrix} 1 & 0.25 \\ 1 & 0 \end{pmatrix}^{-1} = -4\begin{pmatrix} e^{-3t} & te^{-3t} + 0.25e^{-3t} \\ e^{-3t} & te^{-3t} \end{pmatrix}\begin{pmatrix} 0 & -0.25 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} (4t+1)e^{-3t} & -4te^{-3t} \\ 4te^{-3t} & (1-4t)e^{-3t} \end{pmatrix}$$

Method 2. Use Laplace transform. First

$$(sI - A)^{-1} = \begin{pmatrix} s-1 & 4 \\ -4 & s+7 \end{pmatrix}^{-1} = \frac{1}{(s+3)^2}\begin{pmatrix} s+7 & -4 \\ 4 & s-1 \end{pmatrix} = \begin{pmatrix} \frac{1}{s+3} + \frac{4}{(s+3)^2} & -\frac{4}{(s+3)^2} \\ \frac{4}{(s+3)^2} & \frac{1}{s+3} - \frac{4}{(s+3)^2} \end{pmatrix}$$

Then:

$$e^{At} = L^{-1}\{(sI - A)^{-1}\} = \begin{pmatrix} e^{-3t} + 4te^{-3t} & -4te^{-3t} \\ 4te^{-3t} & e^{-3t} - 4te^{-3t} \end{pmatrix}$$

b) The solution of IVP: $x(t) = e^{At}x(0) = \begin{pmatrix} e^{-3t} + 4te^{-3t} & -4te^{-3t} \\ 4te^{-3t} & e^{-3t} - 4te^{-3t} \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} (3+4t)e^{-3t} \\ (2+4t)e^{-3t} \end{pmatrix}$